GRONWALL INEQUALITIES VIA PICARD OPERATORS

BY

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Abstract. In this paper we use some abstract Gronwall lemmas to study Volterra integral inequations in higher dimensions and Fredholm integral inequations.

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1. Introduction

Let $(X, \leq)$ be an ordered set and $A : X \rightarrow X$ an operator such that the equation

$$(1.1) \quad x = A(x)$$

has a unique solution, x_A^* . The operatorial inequality problem (see RUS [22]) is the following: Find conditions under which: (i) $x \leq A(x) \implies x \leq x_A^*$; (ii) $x \geq A(x) \implies x \geq x_A^*$.

To have a "concrete" result for this problem it is necessary to determine the solution x_A^* of the equation (1.1) or to find $y, z \in X$ such that $x_A^* \in [y, z]$. In the last case we ask: (i') $x \leq A(x) \implies x \leq z$; (ii') $x \geq A(x) \implies x \geq y$.

The aim of this paper is to study this problem in the case of integral operators using some abstract Gronwall lemmas.

2. Notions and notations

Throughout this paper we follow the terminology and notations in [21]. For the convenience of the reader we shall recall some of them.

Let X be a nonempty set and $A : X \rightarrow X$ an operator. Then $A^0 := 1_X$, $A^1 := A$, $A^{n+1} := A \circ A^n$, $n \in \mathbb{N}$ - the iterate operators of the operator A . $F_A := \{x \in X | A(x) = x\}$ - the fixed point set of the operators A . Let $s(X) := \{(x_n)_{n \in \mathbb{N}} | x_n \in X, n \in \mathbb{N}\}$. Let $c(x) \subset s(x)$ be a subset of $s(x)$ and $Lim : c(X) \rightarrow X$ an operator. By definition (Fréchet (1905); see [21]) the triple $(X, c(X), Lim)$ is called an L -space if the following conditions are satisfied:

- (i) If $x_n = x, \forall n \in \mathbb{N}$, then $(x_n)_{n \in \mathbb{N}} \in c(X)$ and $Lim(x_n)_{n \in \mathbb{N}} = x$;
- (ii) If $(x_n) \in c(X)$ and $Lim(x_n)_{n \in \mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i \in \mathbb{N}}$, of $(x_n)_{n \in \mathbb{N}}$ we have that $(x_{n_i})_{i \in \mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i \in \mathbb{N}} = x$.
By definition an element of $c(X)$ is a convergent sequence and $x := Lim(x_n)_{n \in \mathbb{N}}$ is the limit of this sequence; we write $x_n \rightarrow x$ as $n \rightarrow \infty$.

In what follows we denote an L -space by $(X, \rightarrow)$. For some examples of L -spaces see [21].

3. Abstract Gronwall lemmas

Let $(X, \rightarrow)$ be an L -space.

Definition 3.1. An operator $A : X \rightarrow X$ is a Picard operator (PO) if

- (i) $F_A = \{x_A^*\}$
- (ii) $A^n(x) \rightarrow x_A^*$ as $n \rightarrow \infty, \forall x \in X$.

For PO s on ordered L -spaces we have the following results (see RUS [18], [19], [20] and [21]).

Lemma 3.1 (Abstract Gronwall lemma). *Let $(X, \rightarrow, \leq)$ be an ordered L -space and $A : X \rightarrow X$ an operator. We suppose that*

- (i) A is PO ;
- (ii) A is increasing.

If we denote by x_A^* the unique fixed point of A , then

- (a) $x \leq A(x) \implies x \leq x_A^*$;
- (b) $x \geq A(x) \implies x \geq x_A^*$.

Lemma 3.2 (Abstract Gronwall-comparison lemma; [22]). *Let $(X, \rightarrow, \leq)$ be an ordered L -space and $A, B : X \rightarrow X$ two operators. We suppose that:*

- (i) A is increasing;
- (ii) A and B are POs;
- (iii) $A \leq B$.

Then $x \leq A(x) \implies x \leq x_B^$.*

Lemma 3.3 ([22]). *Let $(X, \rightarrow)$ be an L -space and $A, B : X \rightarrow X$ two operators. We suppose that*

- (i) A and B are increasing operators;
- (ii) A and B are POs;
- (iii) $x = A(x) \implies x \leq B(x)$.

Then $x \leq A(x) \implies x \leq x_B^$.*

In what follows we shall apply the above lemmas to integral inequalities. For other applications see BUICA [3], CRACIUN [6], LUNGU [11], LUNGU and RUS [12], RUS [18]-[21]. For other integral inequalities see AGARWAL and THANDAPANI [1], BAINOV and SIMEONOV [2], CORDUNEANU [4], CORDUNEANU [5], DRAGOMIR and IONESCU [7], FLETT [8], LAKSHMIKANTHAM and LEELA [10], LAKSHMIKANTHAM, LEELA and MARTYNYUK [11], MITRINOVIĆ, PEČARIĆ and FINK [14], PACHPATTE [15], [16], POLYNIAN and MANZHIROV [17], VER ECKE [23].

4. Volterra integral inequations in higher dimensions

4.1. Volterra integral equations

In what follows we consider the following integral equation

$$\begin{aligned}
 (4.1) \quad & u(x_1, x_2, \dots, x_n) = \alpha + \int_0^{x_1} K_1(s_1, x_2, \dots, x_m) u(s_1, x_2, \dots, x_m) ds_1 \\
 & + \int_0^{x_1} \int_0^{x_2} K_2(s_1, s_2, x_3, \dots, x_m) u(s_1, s_2, x_3, \dots, x_m) ds_1 ds_2 + \dots \\
 & + \int_0^{x_1} \dots \int_0^{x_m} K_m(s_1, \dots, s_m) u(s_1, \dots, s_m) ds_1 \dots ds_m,
 \end{aligned}$$

where $\alpha > 0, a_i > 0, i = \overline{1, m}, D = \prod_{i=1}^m [0, a_i], K_i \in C(D), i = \overline{1, m}$ and $M_{k_i} > 0$ is such that $|K_i(x)| \leq M_{k_i}, \forall x \in D, i = \overline{1, m}$.

Let $A : C(D) \rightarrow C(D)$ be the operator defined by $A(u)(x_1, \dots, x_m) :=$ second part of (4.1). If we consider the Banach space $(C(D), \|\cdot\|_B)$ where $\|\cdot\|$ is a Bielecki norm $\|u\|_B := \max_{x \in D} (|u(x)|e^{-\tau x_1}), \tau > 0$; then the operator A is Lipschitz with the constant $L_A = \frac{1}{\tau}(M_{k_1} + M_{k_2}a_2 + \dots + M_{k_m}a_2 \dots a_m)$. Thus A is a contraction with respect to $\|\cdot\|_B$, for $\tau > 0$ suitable chosen. So, A is PO in $(C(D), \xrightarrow{\|\cdot\|_B})$. From the above considerations we have

Theorem 4.1. *If $K_i \in C(D), i = \overline{1, m}$, then the equation (4.1) has in $C(D)$ a unique solution u^* , and $A^n(u)$ converge uniformly to u^* as $n \rightarrow \infty$, for all $u \in C(D)$.*

Theorem 4.2. *We suppose that $\alpha > 0, K_i \in C(D, \mathbb{R}_+), i = \overline{1, m}$. Then*

- (a) $u^*(x) > 0, \forall x \in D$;
- (b) *If $K_i(x_1, \dots, x_m)$ is increasing with respect to $x_{i+1}, \dots, x_m, i = \overline{1, m-1}$, then u^* is increasing.*

Proof. (a) Let $u_0 \in C(D, \mathbb{R}_+)$ and $u_n := A^n(u_0), n \in \mathbb{N}^*$. Then from (4.1) we have that $u_n(x) \geq \alpha > 0, \forall x \in D$ and u_n is increasing. By Theorem 4.1, $(u_n)_{n \in \mathbb{N}}$ converges uniformly to $u^*(u_n \xrightarrow{unif} u^*)$. So, we have (a).

(b) Follows from (a) and from (4.1). $\square$

4.2. Lower solutions of (4.1)

Now we consider the inequation

$$(4.2) \quad u \leq A(u).$$

By definition a solution of (4.2) is a lower solution of (4.1). We have

Theorem 4.3. *We suppose that the conditions of Theorem 4.2 are satisfied. If $u \in C(D, \mathbb{R}_+)$ is a lower solution of (4.1), then*

$$\begin{aligned} u(x_1, \dots, x_m) &\leq \alpha \exp \left(\int_0^{x_1} K_1(s_1, x_2, \dots, x_m) ds_1 \right. \\ &\quad \left. + \dots + \int_0^{x_1} \dots \int_0^{x_m} K_m(s_1, \dots, s_m) ds_1 \dots ds_m \right). \end{aligned}$$

Proof. First, we remark that under the conditions of Theorem 4.3 the operator A is increasing. Let $u \in C(D, \mathbb{R}_+)$ be a lower solution of (4.1). Then by Lemma 3.1, $u \leq u^*$. From the Theorem 4.2 we have that

$$\begin{aligned} u^*(s_1, s_2, x_3, \dots, x_m) &\leq u^*(s_1, x_2, \dots, x_m), \dots, u^*(s_1, \dots, s_m) \\ &\leq u^*(s_1, x_2, \dots, x_m), \end{aligned}$$

for all $s, x \in D, s \leq x$. Hence we have

$$\begin{aligned} (4.3) \quad u^*(x) &= A(u^*)(x) \leq \alpha + \int_0^{x_1} K_1(s_1, x_2, \dots, x_m) u(s_1, x_2, \dots, x_m) ds_1 \\ &+ \int_0^{x_1} \int_0^{x_2} K_2(s_1, s_2, x_3, \dots, x_m) u(s_1, x_2, \dots, x_m) ds_1 ds_2 + \dots \\ &+ \int_0^{x_1} \dots \int_0^{x_m} K_m(s_1, \dots, s_m) u(s_1, x_2, \dots, x_m) ds_1 \dots ds_m. \end{aligned}$$

Consider the operator $B : C(D) \rightarrow C(D)$ defined by $B(u)(x) :=$ last part of (4.3). It is clear that the operator B is PO on $(C(D), \xrightarrow{unif})$ and is increasing. Let u_B^* be the unique fixed point of B . Thus we have (see 4.3), $u^* = A(u^*) \leq B(u^*)$. From Lemma 3.3, we have that $u^* \leq u_B^*$. From the definition of B we have that

$$(4.4) \quad \begin{aligned} \frac{\partial u_B^*(x)}{\partial x_1} &= \left[K_1(x) + \int_0^{x_2} K_2(x_1, s_2, x_3, \dots, x_m) ds_2 + \dots + \right. \\ &\left. \int_0^{x_2} \dots \int_0^{x_m} K_m(x_1, s_2, \dots, s_m) ds_2 \dots ds_m \right] u_B^*(x) \end{aligned}$$

and $u^*(0, x_2, \dots, x_m) = \alpha$. From (4.4) we have the Theorem 4.3. $\square$

Remark 4.1. For $m = 2$ we have a result of KIM [9].

Remark 4.2. Theorem 4.3 remains true if (4.1) is replaced by

$$\begin{aligned} u(x_1, x_2, \dots, x_m) &= \alpha + \int_0^{x_1} K_1(s_1, x_2, \dots, x_m) f_1(u(s_1, x_2, \dots, x_m)) ds_1 + \dots \\ &+ \int_0^{x_1} \dots \int_0^{x_m} K_m(s_1, \dots, s_m) f_m(u(s_1, \dots, s_m)) ds_1 \dots ds_m, \end{aligned}$$

where $f_i(u) \leq u$, f_i increasing and Lipschitz.

Remark 4.3. As in [8] and [13] we can use the above results to study the solutions and lower solutions of Darboux problem $\frac{\partial^m u}{\partial x_1 \dots \partial x_m} = f(x, u(x))$, $u(x_1, \dots, x_{m-1}, 0) = \varphi_1(x_1, \dots, x_{m-1}), \dots, u(0, x_2, \dots, x_m) = \varphi_m(x_2, \dots, x_m)$, where $f \in C(D \times \mathbb{R})$, $f(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is L_f -Lipschitz and increasing and $\varphi_i, i = \overline{1, m}$ are continuous.

5. Volterra-Fredholm integral inequations

5.1. Volterra-Fredholm integral equations

In what follows we consider the following integral equation

$$(5.1) \quad u(x_1, x_2) = \alpha + \int_0^{x_1} \int_0^b F(x_1, x_2, s_1, s_2, u(s_1, s_2)) ds_1 ds_2$$

where $F \in C(D \times D \times \mathbb{R})$, $D = [0, a] \times [0, b]$, $a > 0, b > 0$.

Let us consider the operator $A : C(D) \rightarrow C(D)$ defined by $A(u)(x_1, x_2) :=$ second part of (5.1). We consider the Banach space $(C(D), \|\cdot\|_B)$ where $\|\cdot\|_B$ is a Bielecki norm $\|\cdot\|_B := \max_{x \in D} \{|u(x_1, x_2)|e^{-\tau x_1}\}$, $\tau > 0$. If we suppose that $F(x, s, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz with a constant L , for all $(x, s) \in D \times D$, then the operator A is Lipschitz with the constant $L_A = \frac{bL}{\tau}$. Thus A is a contraction with respect to $\|\cdot\|_B$, for $\tau > L \cdot b$. So, A is PO in $(C(D), \xrightarrow{\text{unif.}})$.

From the above considerations we have

Theorem 5.1. *We suppose that*

- (i) $F \in C(D \times D \times \mathbb{R})$,
- (ii) *there exists $L > 0$ such that $|F(x, s, v) - F(x, s, w)| \leq L|v - w|$, for all $x, s \in D, v, w \in \mathbb{R}$.*

Then the equation (5.1) has in $C(D)$ a unique solution u^ and $A^n(u) \xrightarrow{\text{unif.}} u^*$ as $n \rightarrow \infty$, for all $u \in C(D)$.*

In addition we suppose that

- (iii) $F(x, s, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is increasing, for all $x, s \in D$,
- (iv) *there exists $K \in C(D \times D, \mathbb{R}_+)$ such that $F(x, s, v) \leq K(x, s)v$, for all $x, s \in D, v \in \mathbb{R}$.*

We consider the equation

$$(5.2) \quad u(x_1, x_2) = \alpha + \int_0^{x_1} \int_0^b K(x_1, x_2, s_1, s_2) u(s_1, s_2) ds_1 ds_2.$$

Let $B : C(D) \rightarrow C(D)$ be defined by $B(u)(x_1, x_2) :=$ second part of (5.2).

It is clear that the operator B is PO on $(C(D), \xrightarrow{unif})$.

Let u_B^* the unique fixed point of B . Let $u \in C(D)$ be a lower solution of (5.1). Then by Lemma 3.1, $u \leq u^*$. Moreover we have $u^* = A(u^*) \leq B(u^*)$.

From Lemma 3.3, we have that $u^* \leq u_B^*$. So, we have

Theorem 5.2. *We suppose that the conditions (i)-(iv) are satisfied. If $u \in C(D)$ is a lower solution of (5.1), then*

$$u(x_1, x_2) \leq \alpha + \int_0^{x_1} \int_0^b \Gamma(x_1, x_2, s_1, s_2) ds_1 ds_2,$$

where Γ is the resolvent kernel of the equation (5.2).

Example 5.1. We consider the following integral equation

$$(5.3) \quad u(x_1, x_2) = \alpha + \int_0^{x_1} \int_0^b e^{x_1 - x_2 - s_1 + s_2} u^p(s_1, s_2) ds_1 ds_2, \quad \alpha > 1$$

Evidently,

$$(5.4) \quad u(x_1, x_2) \leq \alpha + \int_0^{x_1} \int_0^b e^{x_1 - x_2 - s_1 + s_2} u(s_1, s_2) ds_1 ds_2$$

and the operators A and B are:

$$(5.5) \quad A := \alpha + \int_0^{x_1} \int_0^b (e^{x_1 - x_2 - s_1 + s_2}, u^p(s_1, s_2)) ds_1 ds_2,$$

$$(5.6) \quad B := \alpha + \int_0^{x_1} \int_0^b e^{x_1 - x_2 - s_1 + s_2} u(s_1, s_2) ds_1 ds_2.$$

Let $u \in C(D)$ be a lower solution of (5.3), then $u \leq u^*$, where u^* is the unique solution of (5.3). If u_B^* is the unique fixed point of B , then $u^* \leq u_B^*$.

But, u_B^* can be obtained by successive approximation. So,

$$\begin{aligned}
 K_1(x_1, x_2, s_1, s_2) &= K(x_1, x_2, s_1, s_2) = e^{x_1 - x_2 - s_1 + s_2} \\
 K_2(x_1, x_2, s_1, s_2) &= \int_{s_1}^{x_1} \int_0^b e^{x_1 - x_2 - z_1 + z_2} \cdot e^{z_1 - z_2 - s_1 + s_2} dz_1 dz_2 \\
 &= b \frac{(x_1 - s_1)}{1!} e^{x_1 - x_2 - s_1 + s_2} \\
 K_3(x_1, x_2, s_1, s_2) &= b^2 \frac{(x_1 - s_1)^2}{2!} e^{x_1 - x_2 - s_1 + s_2} \\
 &\dots\dots\dots \\
 K_n(x_1, x_2, s_1, s_2) &= \int_{s_1}^{x_1} \int_0^b K(x_1, x_2, z_1, z_2) K_{n-1}(z_1, z_2, s_1, s_2) dz_1 dz_2 \\
 K_n(x_1, x_2, s_1, s_2) &= b^{n-1} \frac{(x_1 - s_1)^{n-1}}{(n-1)!} e^{x_1 - x_2 - s_1 + s_2}.
 \end{aligned}$$

In this case the resolvent kernel is

$$\Gamma(x_1, x_2, s_1, s_2) = \sum_{n=0}^{\infty} b^n \frac{(x_1 - s_1)^n}{n!} e^{x_1 - x_2 - s_1 + s_2} = e^{(b+1)(x_1 - s_1) - s_1 + s_2}.$$

Then $u_B^*(x_1, x_2) = \alpha + \frac{\alpha}{b+1}(e^{b-x_2} - e^{-x_2})(e^{(b+1)x_1} - 1)$ and $u^*(x_1, x_2) \leq \alpha + \frac{\alpha}{b+1}(e^{b-x_2} - e^{-x_2})(e^{(b+1)x_1} - 1)$.

Remark 5.1. Let X be an ordered Banach space, $\Omega \subset \mathbb{R}^m$ a bounded domain, $a > 0$, $g \in C(D, X)$, $F \in C(D \times D \times X, X)$, where $D := [0, a] \times \bar{\Omega}$. We consider the following integral equation

$$(5.7) \quad u(t, x) = g(t, x) + \int_0^t \int_{\Omega} F(t, x, \xi, s) d\xi ds.$$

In a similar way we have

Theorem 5.3. *We suppose that*

- (i) *There exist $L > 0$ such that $\|F(t, x, \xi, s, v) - F(t, x, \xi, s, w)\|_X \leq L\|v - w\|_X$, for all $t, \xi \in [0, \alpha]$, $x, s \in \bar{\Omega}$, $v, w \in X$;*
- (ii) *$F(t, x, \xi, s, \cdot) : X \longrightarrow X$ is increasing, for all $t, \xi \in [0, \alpha]$, $x, s \in \bar{\Omega}$;*

- (iii) *there exists a continuous and linear positive operator $K(t, x, \xi, s) : X \rightarrow X$ for each $t, \xi \in [0, a], x, s \in \bar{\Omega}$, such that $F(t, x, \xi, s, v) \leq K(t, x, \xi, s)v$, for all $t, \xi \in [0, a], x, s \in \bar{\Omega}$.*

Then the equation (5.7) has in $C(D, X)$ a unique solution and if $u \in C(D, X)$ is a lower solution of (5.7), then

$$u(t, x) \leq g(t, x) + \int_0^t \int_{\Omega} \Gamma(t, x, \xi, s) g(\xi, s) d\xi ds,$$

where Γ is the resolvent kernel of the equation

$$u(t, x) = g(t, x) + \int_0^t \int_{\Omega} K(t, x, \xi, s) g(\xi, s) d\xi ds.$$

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