SOME IDEALS OF TERNARY SEMIGROUPS

BY

S. KAR and B.K. MAITY

Abstract. In this paper we characterize different types of ideals of ternary semigroup and study some interesting properties of these ideals of ternary semigroup.

Mathematics Subject Classification 2000: 20M17.

Key words: ternary semigroup, ternary group, regular ternary semigroup, prime ideal, semiprime ideal, strongly irreducible ideal, quasi-ideal, bi-ideal, weak bi-ideal.

1. Introduction

There is a large literature dealing with ternary operations. The notion of ternary semigroup is a natural generalization of ternary group. The notion of ideal play very important role to study the algebraic structures. In [5], SIOSON studied ideal theory in ternary semigroups. He introduced the notion of prime ideal, semiprime ideal, quasi-ideal and study regular ternary semigroup by using these ideals. In [1], DIXIT and DEWAN studied the notions of quasi-ideal and bi-ideal in ternary semigroups. In [4], SANTIAGO developed the theory of ternary semigroups and semiheaps. In [2,3], we study regular ternary semigroups, intra-regular ternary semigroups and congruences on ternary semigroup.

In this paper we study some interesting properties of various ideals of ternary semigroups.

2. Ideals of ternary semigroups

Definition 2.1. A non-empty set S together with a ternary operation, called ternary multiplication, denoted by juxtaposition, is said to be a ternary semigroup if $(abc)de = a(bcd)e = ab(cde)$, for all $a, b, c, d, e \in S$.

Definition 2.2. A ternary semigroup S is said to be commutative if $x_1x_2x_3 = x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$ for every permutation σ of $\{1, 2, 3\}$ and $x_1, x_2, x_3 \in S$.

Definition 2.3. A non-empty subset T of a ternary semigroup S is called a ternary subsemigroup if $t_1t_2t_3 \in T$, for all $t_1, t_2, t_3 \in T$.

Definition 2.4. A non-empty subset I of a ternary semigroup S is called

- (i) a left ideal of S if $SSI \subseteq I$;
- (ii) a lateral ideal of S if $SIS \subseteq I$;
- (iii) a right ideal of S if $ISS \subseteq I$;
- (iv) an ideal of S if I is a left, a right, a lateral ideal of S .

An ideal I of a ternary semigroup S is called a proper ideal if $I \neq S$.

In general, a lateral ideal of a ternary semigroup S is not an ideal of S . But in particular, we have the following result:

Proposition 2.5. *A minimal lateral ideal of a ternary semigroup S is a minimal ideal of S .*

Proof. Let M be a minimal lateral ideal of S . We shall show that M is a minimal ideal of S . Let $m \in M$. Then $SmS \cup SSmSS$ is a lateral ideal of S and $SmS \cup SSmSS \subseteq SMS \cup SSMSS \subseteq M$. Since M is minimal, we have $M = SmS \cup SSmSS$. Now $MSS = (SmS \cup SSmSS)SS = (SmS)SS \cup (SSmSS)SS \subseteq SmS \cup SSmSS \subseteq M$ and $SSM = SS(SmS \cup SSmSS) = SS(SmS) \cup SS(SSmSS) \subseteq SmS \cup SSmSS \subseteq M$. This implies that M is a right ideal and also a left ideal of S . Also M is a lateral ideal of S . Thus M is an ideal of S . Now it remains to show that M is a minimal ideal of S . If possible, let M' be an ideal of S such that $M' \subseteq M$. Since M' is an ideal of S , it is a lateral ideal of S . By hypothesis, we have $M' = M$. Consequently, M is a minimal ideal of S .

Proposition 2.6. *Let S be a ternary semigroup and $a \in S$. Then the principal*

- (i) *left ideal generated by 'a' is given by $\langle a \rangle_l = SSa \cup \{a\}$;*
- (ii) *right ideal generated by 'a' is given by $\langle a \rangle_r = aSS \cup \{a\}$;*

- (iii) lateral ideal generated by 'a' is given by $\langle a \rangle_m = SaS \cup SSaSS \cup \{a\}$;
 (iv) ideal generated by 'a' is given by $\langle a \rangle = SSa \cup aSS \cup SaS \cup SSaSS \cup \{a\}$.

Definition 2.7. A proper ideal P of a ternary semigroup S is called a prime ideal of S if for any three ideals A, B, C of S ; $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

The following theorem gives a useful characterization of a prime ideal in a ternary semigroup.

Theorem 2.8. In a ternary semigroup S , the following conditions are equivalent:

- (i) P is a prime ideal of S ;
 (ii) For $a, b, c \in S$; $aSbSc \subseteq P$, $aSSbSSc \subseteq P$, $aSSbScS \subseteq P$ and $SaSbSSc \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$;
 (iii) For $a, b, c \in S$; $\langle a \rangle \langle b \rangle \langle c \rangle \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$.

Theorem 2.9. An ideal P of a commutative ternary semigroup S is prime if and only if $abc \in P$ implies that $a \in P$ or $b \in P$ or $c \in P$, for all elements a, b, c of S .

Proof. Let S be a commutative ternary semigroup.

Suppose P is a prime ideal of S and $abc \in P$ for some $a, b, c \in S$. Then $(abc)SS \subseteq PSS \subseteq P$. This implies that $aSbSc \subseteq P$, since S is commutative. Similarly, $(abc)SSSS \subseteq PSS \subseteq P$ and hence by commutativity of S , we have $aSSbSSc \subseteq P$, $aSSbScS \subseteq P$, $SaSbSSc \subseteq P$. Since P is a prime ideal of S , by Theorem 2.8, we get $a \in P$ or $b \in P$ or $c \in P$.

Conversely, let $abc \in P$ implies that $a \in P$ or $b \in P$ or $c \in P$, for all a, b, c of S . We have to show that P is a prime ideal of S . Suppose $ABC \subseteq P$ for any three ideals A, B, C of S and $B \not\subseteq P$, $C \not\subseteq P$. Then there exist $b \in B$ such that $b \notin P$ and $c \in C$ such that $c \notin P$. Now for each $a \in A$, $abc \in ABC \subseteq P$ implies that $a \in P$ and hence $A \subseteq P$. Consequently, P is a prime ideal of S .

Example 2.10. In the commutative ternary semigroup $\mathbb{Z}^-$ of all negative integers, the ideal $P = \{3k : k \in \mathbb{Z}^-\}$ is a prime ideal.

For $xyz \in P$ ($x, y, z \in \mathbb{Z}^-$)
 $\iff xyz$ is divisible by 3
 $\iff x$ is divisible by 3 or y is divisible by 3 or z is divisible by 3
 $\iff x = 3k_1$ or $y = 3k_2$ or $z = 3k_3$ for $k_1, k_2, k_3 \in \mathbb{Z}^-$
 $\iff x \in P$ or $y \in P$ or $z \in P$.

But the ideal $Q = \{30k : k \in \mathbb{Z}^-\}$ is not a prime ideal of $\mathbb{Z}^-$, since $(-2)(-3)(-5) = -30 \in Q$ but $(-2) \notin Q$, $(-3) \notin Q$ and $(-5) \notin Q$.

Theorem 2.11. *If I is an ideal of a ternary semigroup S and P is a prime ideal of S , then $I \cap P$ is a prime ideal of I , considering I as a ternary semigroup.*

Proof. Clearly, $I \cap P$ is an ideal of I . Let $a, b, c \in I$ and $aSbSc \subseteq I \cap P$, $aSSbSSc \subseteq I \cap P$, $aSSbScS \subseteq I \cap P$ and $SaSbSSc \subseteq I \cap P$. Then $aSbSc \subseteq P$, $aSSbSSc \subseteq P$, $aSSbScS \subseteq P$ and $SaSbSSc \subseteq P$, since $I \cap P \subseteq P$. Since P is a prime ideal of S , we have $a \in P$ or $b \in P$ or $c \in P$. Thus $a \in I \cap P$ or $b \in I \cap P$ or $c \in I \cap P$. Consequently, by Theorem 2.8, $I \cap P$ is a prime ideal of I .

Note 2.12. Let $\{P_i\}$ be a collection of prime ideals of a ternary semigroup S . Then $\bigcup P_i$ and $\bigcap P_i$ are ideals of S but these are not prime ideals of S , in general.

However; in particular, we have the following result:

Proposition 2.13. *Let $\{P_i\}$ be a collection of prime ideals of a ternary semigroup S such that $\{P_i\}$ forms a chain. Then $\bigcup P_i$ and $\bigcap P_i$ are both prime ideals of S .*

Proof. Clearly, $\bigcap P_i$ is an ideal of S . Let $ABC \subseteq \bigcap P_i$ for any three ideals A, B, C of S . If either $A \subseteq P_i$, for all i or $B \subseteq P_i$, for all i or $C \subseteq P_i$, for all i , then either $A \subseteq \bigcap P_i$ or $B \subseteq \bigcap P_i$ or $C \subseteq \bigcap P_i$. If possible, let $A, B, C \not\subseteq \bigcap P_i$. Then there exist i, j and k such that $A \not\subseteq \bigcap P_i$, $B \not\subseteq \bigcap P_j$ and $C \not\subseteq \bigcap P_k$. Since $\{P_i\}$ is a chain, let $P_i \subseteq P_j \subseteq P_k$. This implies that $B, C \not\subseteq P_i$. Since $ABC \subseteq P_i$ and P_i is prime, we must have either $A \subseteq P_i$ or $B \subseteq P_i$ or $C \subseteq P_i$, a contradiction. Therefore, either $A \subseteq \bigcap P_i$ or $B \subseteq \bigcap P_i$ or $C \subseteq \bigcap P_i$. Consequently, $\bigcap P_i$ is a prime ideal of S .

Similarly, we can prove that $\bigcup P_i$ is a prime ideal of S .

Definition 2.14. A proper ideal Q of a ternary semigroup S is called a semiprime ideal of S if $I^3 \subseteq Q$ implies $I \subseteq Q$ for any ideal I of S .

Note 2.15. Every prime ideal of a ternary semigroup S is also a semi-prime ideal of S .

As in the case of prime ideals of a commutative ternary semigroup, we have the following result for semiprime ideals of a commutative ternary semigroup:

Theorem 2.16. *A proper ideal Q of a commutative ternary semigroup S is semiprime if and only if $x^3 \in Q$ implies that $x \in Q$ for any element x of S .*

Example 2.17. In the commutative ternary semigroup $\mathbb{Z}^-$ of all negative integers, the ideal $Q = \{6k : k \in \mathbb{Z}^-\}$ is a semiprime ideal.

$$\begin{aligned} \text{For } & x^3 \in Q \ (x \in \mathbb{Z}^-) \\ \iff & x^3 \text{ is divisible by } 6 \\ \iff & x \text{ is divisible by } 6 \\ \iff & x = 6k_1 \text{ for } k_1 \in \mathbb{Z}^- \\ \iff & x \in Q. \end{aligned}$$

Definition 2.18. A proper ideal I of a ternary semigroup S is said to be weakly irreducible if for ideals H and K of S , $H \cap K = I$ implies that $I = H$ or $I = K$.

We simply use the term irreducible to mean weakly irreducible.

Definition 2.19. A proper ideal I of a ternary semigroup S is said to be strongly irreducible if for ideals H and K of S , $H \cap K \subseteq I$ implies that $H \subseteq I$ or $K \subseteq I$.

Note 2.20. It is to be noted here that a strongly irreducible ideal of a ternary semigroup S is an irreducible ideal of S .

Definition 2.21. A non-empty subset A of a ternary semigroup S is called an i -system if $a, b \in A$ implies that $\langle a \rangle \cap \langle b \rangle \cap A \neq \emptyset$.

Theorem 2.22. *The following conditions in a ternary semigroup S are equivalent:*

- (i) I is a strongly irreducible ideal of S ;
- (ii) If for $a, b \in S$; $\langle a \rangle \cap \langle b \rangle \subseteq I$ then $a \in I$ or $b \in I$;

(iii) The complement of I i.e. I^c is an i -system.

Proof. (i) $\implies$ (ii).

This is an immediate consequence of the Definition 2.19.

(ii) $\implies$ (iii).

If possible let $a, b \in I^c$ and $\langle a \rangle \cap \langle b \rangle \cap I^c = \emptyset$.

Then $\langle a \rangle \cap \langle b \rangle \cap I^c = \emptyset$ implies that $\langle a \rangle \cap \langle b \rangle \subseteq I$ and hence by using (ii), we have $a \in I$ or $b \in I$, which is a contradiction.

Consequently, $\langle a \rangle \cap \langle b \rangle \cap I^c \neq \emptyset$ and hence I^c is an i -system.

(iii) $\implies$ (i).

Let H and K be two ideals of S such that $H \not\subseteq I$ and $K \not\subseteq I$.

Then there exist elements $a \in H - I$ and $b \in K - I$. Now from (iii), it follows that $\langle a \rangle \cap \langle b \rangle \cap I^c \neq \emptyset$ i.e. there exists an element $c \in (\langle a \rangle \cap \langle b \rangle) - I$. This implies that $c \in H \cap K$ and $c \notin I$. Hence $H \cap K \not\subseteq I$. This shows that I is strongly irreducible.

Theorem 2.23. A proper ideal P of a ternary semigroup S is prime if and only if it is semiprime and strongly irreducible.

Proof. Suppose P is a prime ideal of S . Then P is a semiprime ideal of S . Again, let H and K be two ideals of S such that $H \cap K \subseteq P$. Now $SHK \subseteq SSK \subseteq K$ and $SHK \subseteq SHS \subseteq H$. This implies that $SHK \subseteq H \cap K \subseteq P$. Since P is a prime ideal of S , it follows that $H \subseteq P$ or $K \subseteq P$ and hence P is strongly irreducible.

Conversely, suppose that a proper ideal P of S is both semiprime and strongly irreducible. Let $ABC \subseteq P$ for any three ideals A, B, C of S . Now $[(A \cap B) \cap C] [(A \cap B) \cap C] [(A \cap B) \cap C] \subseteq ABC \subseteq P$. Since P is a semiprime ideal of S , it follows that $[(A \cap B) \cap C] \subseteq P$. Again, since P is a strongly irreducible ideal, we have $A \cap B \subseteq P$ or $C \subseteq P$ which again implies that $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$. Consequently, P is a prime ideal of S .

Definition 2.24. A subset Q of a ternary semigroup S is called a quasi-ideal of S if $QSS \cap SQS \cap SSQ \subseteq Q$ and $QSS \cap SSQSS \cap SSQ \subseteq Q$.

Lemma 2.25. Every left, right, lateral ideal of a ternary semigroup S is a quasi-ideal of S .

Remark 2.26. The converse of Lemma 2.25 is not true, in general i.e. a quasi-ideal may not be a left, a right or a lateral ideal of S . This follows from the following example.

Example 2.27. Let $S = M_2(\mathbb{Z}_0^-)$ be the ternary semigroup of the set of all 2×2 square matrices over $\mathbb{Z}_0^-$, the set of all non-positive integers. Let $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Z}_0^- \right\}$. Then we can easily verify that Q is a quasi-ideal of S but Q is not a right ideal, a lateral ideal or a left ideal of S .

Proposition 2.28. *The intersection of arbitrary collection of quasi-ideals of a ternary semigroup S is a quasi-ideal of S .*

Theorem 2.29. *A subset Q of a ternary semigroup S is a quasi-ideal of S if and only if Q is the intersection of a right ideal, a lateral ideal and a left ideal of S .*

Corollary 2.30. *Every quasi-ideal of a ternary semigroup S is a ternary subsemigroup of S .*

Definition 2.31. A subsemigroup B of a ternary semigroup S is called a bi-ideal of S if $BSBSB \subseteq B$.

Lemma 2.32. *Every quasi-ideal of a ternary semigroup S is a bi-ideal of S .*

Note 2.33. The converse of Lemma 2.32 does not hold, in general i.e. a bi-ideal of a ternary semigroup S may not be a quasi-ideal of S .

Remark 2.34. Since every left, right and lateral ideal of S is a quasi-ideal of S , it follows that every left, right and lateral ideal of S is a bi-ideal of S but the converse is not true, in general.

In general, if B is a bi-ideal of a ternary semigroup S and C is a bi-ideal of B then C is not a bi-ideal of S . But in particular, we have the following result:

Theorem 2.35. *Let B be a bi-ideal of a ternary semigroup S and C a bi-ideal of B such that $C^3 = C$. Then C is a bi-ideal of S .*

Proof. Since B is a bi-ideal of S , $BSBSB \subseteq B$ and since C is a bi-ideal of B , $CBCBC \subseteq C$. Therefore, $CSCSC = (CCC)SCS(CCC) = CC(CSCSC)CC \subseteq CC(BSBSB)CC \subseteq CCBCC = CCBC(CCC) \subseteq C(CBCBC)C \subseteq CCC = C$. Thus C is a bi-ideal of S .

Definition 2.36. A ternary semigroup S is called a ternary group if for all $a, b, c \in S$, the equations $abx=c$, $ayb=c$ and $zab=c$ have solutions in S .

Remark 2.37. In a ternary group S , for all $a, b, c \in S$, the equations $abx=c$, $ayb=c$ and $zab=c$ have unique solutions in S .

Theorem 2.38. A ternary semigroup S has no proper bi-ideal if and only if S is a ternary group.

Proof. Let S be a ternary group. Let B be a bi-ideal of S . Suppose that $c \in S$ and $a, b \in B$. Since S is a ternary group, we have $xab=c$ has a solution in S . This implies that $c=yab$ for some $y \in S$. Consequently, $S=SBB$. Similarly, $S=BBS$ and $S=BSB$. Now $S=BBS=BBSBB=BBBSBB \subseteq BB(BSBSB) \subseteq BBB \subseteq B$. Consequently, $B=S$ and hence S has no proper bi-ideal.

Conversely, suppose that S contains no proper bi-ideals. Now for any $a, b, c \in S$; abS , aSb and Sab are all bi-ideals of S and hence $abS=aSb=Sab=S$. This implies that the equations $abx=c$, $ayb=c$ and $zab=c$ have solutions in S . Consequently, S is a ternary group.

Definition 2.39. An element a in a ternary semigroup S is called regular if there exists an element x in S such that $axa=a$.

A ternary semigroup S is called regular if all of its elements are regular.

Theorem 2.40. The following conditions in a ternary semigroup S are equivalent:

- (i) S is regular;
- (ii) For any right ideal R , lateral ideal M and left ideal L of S , $RML = R \cap M \cap L$;
- (iii) For $a, b, c \in S$; $\langle a \rangle_r \langle b \rangle_m \langle c \rangle_l = \langle a \rangle_r \cap \langle b \rangle_m \cap \langle c \rangle_l$;
- (iv) For $a \in S$; $\langle a \rangle_r \langle a \rangle_m \langle a \rangle_l = \langle a \rangle_r \cap \langle a \rangle_m \cap \langle a \rangle_l$.

Theorem 2.41. If for every quasi-ideal Q of S , $Q^3=Q$ then S is a regular ternary semigroup.

Proof. If R is a right ideal, M a lateral ideal and L a left ideal of S , then by Theorem 2.29, it follows that $R \cap M \cap L$ is a quasi-ideal of S . Now by hypothesis,

$$R \cap M \cap L = (R \cap M \cap L)^3 = (R \cap M \cap L)(R \cap M \cap L)(R \cap M \cap L) \subseteq RML.$$

Again, clearly we have $RML \subseteq R \cap M \cap L$. Consequently, $R \cap M \cap L = RML$ and hence by Theorem 2.40, S is a regular ternary semigroup.

The following theorem gives a characterization of a regular ternary semigroup S in terms of bi-ideal and quasi-ideal of S .

Theorem 2.42. *The following conditions in a ternary semigroup S are equivalent:*

- (i) S is regular;
- (ii) For every bi-ideal B of S , $BSBSB = B$;
- (iii) For every quasi-ideal Q of S , $QSQSQ = Q$.

Proof. (i) $\implies$ (ii)

Suppose S is regular. Let B be a bi-ideal of S . Let $b \in B$. Then there exists $x \in S$ such that $a = axa$. This implies that $a = axaxa \in BSBSB$. So we find that $B \subseteq BSBSB$. Again, since B is a bi-ideal of S , $BSBSB \subseteq B$. Consequently, we have $BSBSB = B$.

Clearly, (ii) $\implies$ (iii), by using Lemma 2.32.

(iii) $\implies$ (i)

Suppose (iii) holds. Let R be a right ideal, M a lateral ideal and L a left ideal of S . Then $Q = R \cap M \cap L$ is a quasi-ideal of S , by Theorem 2.29. By hypothesis, $QSQSQ = Q$. Now $R \cap M \cap L = Q = QSQSQ \subseteq RSMSL \subseteq RML$. Again, clearly $RML \subseteq R \cap M \cap L$. So $R \cap M \cap L = RML$ and hence by Theorem 2.40, S is a regular ternary semigroup.

Theorem 2.43. *A ternary subsemigroup B of a regular ternary semigroup S is a bi-ideal of S if and only if $B = BSB$.*

Proof. If $B = BSB$, then it is easy to see that B is a bi-ideal of S .

Conversely, suppose that B is a bi-ideal of a regular ternary semigroup S . Let $b \in B$. Then there exists $x \in S$ such that $b = bxb$. This implies that $b \in BSB$ and hence $B \subseteq BSB$. Again, $BSB \subseteq BSBSB \subseteq B$. Thus we find that $B = BSB$.

Theorem 2.44. *A subsemigroup B of a regular ternary semigroup S is a bi-ideal of S if and only if B is a quasi-ideal of S .*

Proof. Let S be a regular ternary semigroup. If B is a quasi-ideal of S , then from Lemma 2.32, it follows that B is a bi-ideal of S .

Conversely, let B be a bi-ideal of S . From Theorem 2.40, we find that if S is a regular ternary semigroup, then $R \cap M \cap L = RML$ for any right ideal R , any lateral ideal M and any left ideal L . Now

$$\begin{aligned}
 BSS \cap (SBS \cup SSBSS) \cap SSB &= BSS(SBS \cup SSBSS)SSB \\
 &= B(SSS)B(SSS)B \cup B(SSS)SB(SSS)SB \subseteq BSBSB \cup BSSBSSB \\
 &\subseteq B \cup BSB \quad (\text{since } B \text{ is a bi-ideal}) \\
 &= B \cup B \quad (\text{by Theorem 2.42}) \\
 &= B.
 \end{aligned}$$

Consequently, B is a quasi-ideal of S .

From Theorem 2.29 and Theorem 2.44, we have the following result:

Corollary 2.45. *A subsemigroup B of a regular ternary semigroup S is a bi-ideal of S if and only if B is the intersection of a right ideal, a lateral ideal and a left ideal of S .*

Definition 2.46. A subsemigroup B_w of a ternary semigroup S is called a weak bi-ideal of S if $bSbSb \subseteq B_w$, for all $b \in B_w$.

Proposition 2.47. *Every bi-ideal of a ternary semigroup S is a weak bi-ideal of S .*

Remark 2.48. Since every left ideal, right ideal and lateral ideal of S is a bi-ideal of S , it follows that every left ideal, right ideal and lateral ideal of S is a weak bi-ideal of S .

The converse of the above result is not true. This follows from the following example:

Example 2.49. Let $S = \mathbb{Z}^- \times \mathbb{Z}^- = \{(a, b) : a, b \in \mathbb{Z}^-, \text{ set of all negative integers}\}$.

Then S is a ternary semigroup w.r.t. the ternary multiplication defined as follows: $(a, b)(c, d)(e, f) = (a, f)$.

Let $B_w = \{(b, b) : (b, b) \in S\}$. Then B_w is a weak bi-ideal of S , since $(b, b)(u, v)(b, b)(x, y)(b, b) = (b, b) \in B_w$, for all $(u, v), (x, y) \in S$. It can be easily verified that B_w is neither a bi-ideal nor a left ideal, a right ideal and a lateral ideal of S .

Proposition 2.50. *The intersection of arbitrary collection of weak bi-ideals of a ternary semigroup S is a weak bi-ideal of S .*

Theorem 2.51. *A ternary semigroup S is regular if and only if $B_w = \bigcup_{b \in B_w} bSbSb$, where B_w is a weak bi-ideal of S .*

Proof. Let S be a regular ternary semigroup and B_w be a weak bi-ideal of S . Clearly, $\bigcup_{b \in B_w} bSbSb \subseteq B_w$. Now let $b \in B_w$. Since S is regular, we have $b = bxb$ for some $x \in S$. So $b = bxb = bxbxb \in \bigcup_{b \in B_w} bSbSb$. Thus $B_w \subseteq \bigcup_{b \in B_w} bSbSb$ and hence $B_w = \bigcup_{b \in B_w} bSbSb$.

Conversely, suppose that $B_w = \bigcup_{b \in B_w} bSbSb$ for any weak bi-ideal B_w of S . Let R be a right ideal, M be a lateral ideal and L be a left ideal of S . Then R, M, L are weak bi-ideals of S . Consequently, we find that $R \cap M \cap L$ is a weak bi-ideal of S , by Proposition 2.50. Clearly, $RML \subseteq R \cap M \cap L$. Let $a \in R \cap M \cap L$. Then $a = bsbtb$ for some $b \in R \cap M \cap L$ and $s, t \in S$. This implies that $a = bsbtb \in RML$. So $R \cap M \cap L \subseteq RML$ and hence $RML = R \cap M \cap L$. Consequently, S is a regular ternary semigroup, by Theorem 2.40.

Diagrammatic representation of different ideals of ternary semigroups

*The reverse implication does not always hold.

Prime Ideal $\iff$ Semiprime Ideal + Strongly Irreducible Ideal.

REFERENCES

1. CLIFFORD, A.H.; PRESTON, G.B. – *The Algebraic Theory of Semigroups*, Vol. I. Mathematical Surveys, No. 7, American Mathematical Society, Providence, R.I., 1961.
2. CLIFFORD, A.H.; PRESTON, G.B. – *The Algebraic Theory of Semigroups*, Vol. II. Mathematical Surveys, No. 7, American Mathematical Society, Providence, R.I., 1967.
3. DIXIT, V.N.; DEWAN, S. – *A note on quasi and bi-ideals in ternary semigroups*, Internat. J. Math. Math. Sci., 18 (1995), 501–508.
4. DUTTA, T.K.; KAR, S.; MAITY, B.K. – *On ideals in regular ternary semigroups*, Discuss. Math. Gen. Algebra Appl., 28 (2008), 147–159.
5. HOWIE, J.M. – *Fundamentals of semigroup theory*, London Mathematical Society Monographs, New Series, 12, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.
6. KAR, S.; MAITY, B.K. – *Congruences on ternary semigroups*, Journal of the Chungcheong Mathematics Society, 20 (2007), 191–201.
7. ŁOŚ, J. – *On the extending of models. I*, Fund. Math., 42 (1955), 38–54.
8. SANTIAGO, M.L. – *Some contribution to the study of ternary semigroups and semi-heaps*, Ph.D. Thesis, University of Madras, India, 1983.
9. SIOSON, F.M. – *Ideal theory in ternary semigroups*, Math. Japon., 10 (1965), 63–84.

Received: 16.III.2009

Department of Mathematics,
Jadavpur University,
Kolkata, 700032, West Bengal,
INDIA
karsukhendu@yahoo.co.in
bkmaity2002@yahoo.co.in