THE COMPARISON OF SOME HYPERTOPOLOGIES

BY

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Abstract. If (X, d) is a metric space and $\mathcal{Cl}(X)$ is the family of closed subsets of X , we search the "largest" family $\mathcal{A} \subset \mathcal{Cl}(X)$ such that Vietoris topology τ_V and locally finite topology τ_{lf} coincide on $\mathcal{A}$. We also compare the convergences in Vietoris and bounded-Vietoris sense and in other hyperconvergences.

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1. Introduction

In this paper we compare the locally finite and Vietoris topology on some classes of nonempty closed subset of a metric space. We find the "better" class with this property. Then we obtain another results in the problem of comparison of hypertopologies.

So consider (X, d) a metric space. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X . If $\mathcal{A} \subset \mathcal{P}(X)$, we define some topologies on $\mathcal{A}$ which extend the initial topology (see, for e.g., [3], [4], [7], [8], [12]). This means that, if we restrict these topologies to the family $\mathcal{S}(X)$ of singleton subsets of X , the topology of induced subspace agrees with the initial topology on X . These topologies are called hypertopologies or hyperspacial topologies. In this category one includes some well-known topologies, like Hausdorff topology τ_H , Attouch-Wets topology τ_{AW} , Vietoris and bounded-Vietoris topology (τ_v and τ_{bV} respectively), proximal and bounded-proximal topology (τ_P and τ_{bP} respectively), locally finite topology τ_{lf} and Wijsman topology τ_W .

On the family of the closed nonempty subsets $\mathcal{Cl}(X)$ of X , there exist some relations between these hypertopologies, as synthesized by SONNTAG and ZĂLINESCU in [10] and [11]. We select below and adjust a fragment from the schema of this papers:

The relations described in schema above are generally strict. So many efforts have been done in order to find some conditions on the space (X, d) which can assure the coincidence of these hypertopologies on $\mathcal{Cl}(X)$. We give some of these results (which are specified in [5], [6] or [9]):

$\tau_V = \tau_H \iff X$ is compact; $\tau_V = \tau_P \iff (X, d)$ is UC space ([6], Lemma 5.2); $\tau_V = \tau_{lf} \iff X$ is compact ([5], p.169); $\tau_H = \tau_W \iff (X, d)$ is a totally bounded space ([6], Corollary 5.7); $\tau_H = \tau_P \iff (X, d)$ is a totally bounded space ([6], Lemma 5.1); $\tau_H = \tau_{lf} \iff (X, d)$ is UC space ([5], Theorem 2.2); $\tau_P = \tau_W \iff (X, d)$ is a totally bounded space ([6], Theorem 5.5); $\tau_P = \tau_{lf} \iff X$ is compact ([6], Section 5); $\tau_W = \tau_{lf} \iff X$ is compact ([6], Corollary 5.8); $\tau_V = \tau_{bV} \iff (X, d)$ is d -bounded ([9], Proposition 3.2); $\tau_H = \tau_{bV} \iff X$ is compact ([9], Proposition 3.5); $\tau_{bV} = \tau_{bP} \iff (X, d)$ is BUC space ([9], Section 3).

Recall that (X, d) is a *UC space* if every continuous real function on X is uniformly continuous.

We say that (X, d) is a *BUC space* if for any $A, B \in \mathcal{Cl}(X)$ and B a bounded set, $A \cap B = \emptyset$ implies $d(A, B) > 0$.

These results give an answer to the following question: to search special classes of metric spaces (X, d) which make two specified hypertopologies to coincide on the fixed family $\mathcal{A} = \mathcal{Cl}(X)$.

Another problem, initiated in [2], is to specify some family $\mathcal{A} \subset \mathcal{Cl}(X)$ (even the largest family) on which two precise hypertopologies coincide, when the space (X, d) is fixed. In this paper we focus especially on locally finite and Vietoris topologies.

In Section 2 we recall some notations, notions and results concerning topologies from schema above.

Section 3 is dedicated to the comparison between locally finite and Vietoris topologies on $\mathcal{A} \subset \mathcal{Cl}(X)$. We find the largest family on which they coincide.

In Section 4 we obtain a sufficient condition for coincidence between the Vietoris and bounded Vietoris convergences of nets and finally we give some new conclusions on comparison between the hypertopologies defined in Section 2.

2. Notations and preliminaries

We consider a metric space (X, d) . Define the following families of non-empty subsets of X :

$$\begin{aligned}\mathcal{P}(X) &= \{A \subset X; A \neq \emptyset\}; \\ \mathcal{Cl}(X) &= \{A \in \mathcal{P}(X); A \text{ is a closed subset}\}; \\ \mathcal{K}(X) &= \{A \in \mathcal{P}(X); A \text{ is a compact subset}\}; \\ \mathcal{F}(X) &= \{A \in \mathcal{P}(X); A \text{ is a finite subset}\}; \\ \mathcal{S}(X) &= \{A \in \mathcal{P}(X); A \text{ is a singleton}\}; \\ \mathcal{B}(X) &= \{A \in \mathcal{Cl}(X); A \text{ is a } d\text{-bounded subset}\}; \\ \mathcal{Pk}(X) &= \{A \in \mathcal{P}(X); A \text{ is a } d\text{-totally bounded subset}\}.\end{aligned}$$

We denote by $S(a, \varepsilon) = \{x \in X; d(a, x) < \varepsilon\}$ with $a \in X, \varepsilon > 0$ the ball of center a and radius ε and by $B(a, \varepsilon) = \{x \in X; d(a, x) \leq \varepsilon\}$ with $a \in X, \varepsilon > 0$ the closed ball of center a and radius ε .

$S_\varepsilon(A)$ is the notation of ε -enlargement of A : $S_\varepsilon(A) = \{x \in X; \exists a \in A \text{ such that } d(x, a) < \varepsilon\}$ with $A \subset X, \varepsilon > 0$.

The topologies which we consider in this paper must be written like a supremum of two topologies, namely a lower topology τ^- and an upper topology τ^+ .

The Hausdorff topology τ_H is defined on $\mathcal{A} \subset \mathcal{Cl}(X)$ by $\tau_H = \tau_H^- \vee \tau_H^+$, where a basic neighbourhoods of a set $A_0 \in \mathcal{A}$ is, respectively:

$$\text{in } \tau_H^- \quad U_-(A_0, \varepsilon) = \{A \in \mathcal{A}; A_0 \subset S_\varepsilon(A)\}, \text{ with } \varepsilon > 0,$$

$$\text{and in } \tau_H^+ \quad U_+(A_0, \varepsilon) = \{A \in \mathcal{A}; A \subset S_\varepsilon(A_0)\}, \text{ with } \varepsilon > 0.$$

The Attouch-Wets topology τ_{AW} on $\mathcal{A} \subset \mathcal{Cl}(X)$ is $\tau_{AW} = \tau_{AW}^- \vee \tau_{AW}^+$, where a basic neighbourhoods of a set $A_0 \in \mathcal{A}$ is given by:

in τ_{AW}^-

$$U_-(A_0; x_0, \varepsilon) = \{A \in \mathcal{A}; A_0 \cap S(x_0, \frac{1}{\varepsilon}) \subset S_\varepsilon(A)\}$$

and in τ_{AW}^+

$$U_+(A_0; x_0, \varepsilon) = \{A \in \mathcal{A}; A \cap S(x_0, \frac{1}{\varepsilon}) \subset S_\varepsilon(A_0)\},$$

where $\varepsilon > 0$ and x_0 is fixed arbitrarily in X .

The Vietoris topology τ_V on $\mathcal{A} \subset \mathcal{Cl}(X)$ is $\tau_V = \tau_V^- \vee \tau_V^+$; a subbase for τ_V^- is given by all the sets $V^- = \{A \in \mathcal{A}; A \cap V \neq \emptyset\}$, where V is an open subset of X , and a subbase for τ_V^+ is given by all the sets $E^+ = \{A \in \mathcal{A}; A \cap E = \emptyset\}$, where E is a closed subset of X .

The bounded-Vietoris topology ([9]) τ_{bV} on $\mathcal{A} \subset \mathcal{Cl}(X)$ is $\tau_{bV} = \tau_V^- \vee \tau_{bV}^+$; a subbase for τ_{bV}^+ is given by all the sets $E^+ = \{A \in \mathcal{A}; A \cap E = \emptyset\}$, where E is a closed and bounded subset of X .

The proximal topology τ_P on $\mathcal{A} \subset \mathcal{Cl}(X)$ is $\tau_P = \tau_V^- \vee \tau_P^+$, where τ_P^+ is generated by all the sets $E^{++} = \{A \in \mathcal{A}; \exists \varepsilon > 0 \text{ such that } S_\varepsilon(A) \subset E\}$ with E open in X .

In fact, $\tau_P^+ = \tau_H^+$.

The bounded-proximal topology τ_{bP} on $\mathcal{A} \subset \mathcal{Cl}(X)$ is $\tau_{bP} = \tau_V^- \vee \tau_{AW}^+$.

The Wijsman topology τ_W on $\mathcal{A} \subset \mathcal{Cl}(X)$ is $\tau_W = \tau_V^- \vee \tau_W^+$, where the basic neighbourhoods of A_0 in upper Wijsman topology are the sets

$$\{A \in \mathcal{A}; d(x_i, A_0) < d(x_i, A) + \varepsilon \text{ for } i = \overline{1, n}\},$$

with $\{x_1, x_2, \dots, x_n\}$ a finite subset of X and $\varepsilon > 0$.

The locally finite topology τ_{lf} on $\mathcal{A} \subset \mathcal{Cl}(X)$ is $\tau_{lf} = \tau_{lf}^- \vee \tau_V^+$, where τ_{lf}^- is generated by all the sets $\mathcal{L}^- = \{A \in \mathcal{A}; A \cap V \neq \emptyset, \text{ for every } V \in \mathcal{L}\}$, for any locally finite family $\mathcal{L} \subset \mathcal{P}(X)$ of open subsets of X .

(A family of subsets $\mathcal{L} \subset \mathcal{P}(X)$ is called *locally finite* if for every $x \in X$ there exists a neighbourhood $V(x)$ of x such that $V(x)$ has nonempty intersection with a finite number of elements of $\mathcal{L}$.)

We note that Vietoris and locally finite topologies do not depend on the metric of the space (like Hausdorff, Attouch-Wets, proximal, bounded-proximal, bounded-Vietoris and Wijsman topologies), but they depend on the topology of the space.

3. The comparison between the locally finite and Vietoris topologies

Now we search the family $\mathcal{A} \subset \mathcal{Cl}(X)$ such that τ_V and τ_{lf} coincide on $\mathcal{A}$. The upper topology for τ_V and τ_{lf} is the same, so we are interested by the hypertopologies τ_{lf}^- and τ_V^- . First we give the following lemma:

Lemma 3.1. *Let (X, d) be an arbitrary metric space and $\mathcal{A} \subset \mathcal{Cl}(X)$ be a class of sets such that for every locally finite family $\mathcal{L}$ of open subsets of X , the condition $\mathcal{L}^- \cap \mathcal{A} \neq \emptyset$ implies that $\mathcal{L}$ is finite. Then $\tau_V^- = \tau_{lf}^-$ on $\mathcal{A}$ (so $\tau_V = \tau_{lf}$ on $\mathcal{A}$).*

Proof. We suppose that $\mathcal{L}$ is a locally finite family of open subsets of X such that $\mathcal{L}^- \cap \mathcal{A} \neq \emptyset$. Let $V_1, V_2, \dots, V_n$ be the open sets such that $\mathcal{L} = \{V_1, V_2, \dots, V_n\}$. Then $\mathcal{L}^- = V_1^- \cap V_2^- \cap \dots \cap V_n^-$ is a finite intersection of open sets from the subbase of τ_V^- , so $\mathcal{L}^-$ is a τ_V^- -open set. We deduce that $\tau_{lf}^- \subset \tau_V^-$, so $\tau_{lf} \subset \tau_V$. The reverse inclusion is valid on every class $\mathcal{A} \subset \mathcal{Cl}(X)$ (see the schema of section 1). $\square$

Remark 3.1. There exist some classes $\mathcal{A} \subset \mathcal{Cl}(X)$ such that for every locally finite family $\mathcal{L}$ the condition $\mathcal{L}^- \cap \mathcal{A} \neq \emptyset$ implies that $\mathcal{L}$ is finite.

For example, in the case of the class $\mathcal{S}(X)$ we have: if $\mathcal{L}$ is an arbitrary locally finite family for which $\mathcal{L}^- \cap \mathcal{S}(X) \neq \emptyset$, then there exists $x_0 \in X$ such that $\{x_0\} \in \mathcal{L}^-$. So $\{x_0\} \cap V \neq \emptyset$ for all $V \in \mathcal{L}$. But $\mathcal{L}$ is a locally finite family; hence there exists $U(x_0)$ a neighbourhood of x_0 which has a nonempty intersection with a finite number of elements of $\mathcal{L}$. Let they be $V_1, V_2, \dots, V_n$. Then $U(x_0) \cap V \neq \emptyset$ for all $V \in \mathcal{L}$, so $\mathcal{L}$ contains only $V_1, V_2, \dots, V_n$.

Another family $\mathcal{A} \subset \mathcal{Cl}(X)$ which has the property from Lemma 3.1 is $\mathcal{A} = \mathcal{K}(X)$:

Lemma 3.2. *Let (X, d) be a metric space. If $\mathcal{L}$ is a locally finite family of open subsets of X and $\mathcal{L}^- \cap \mathcal{K}(X) \neq \emptyset$, then $\mathcal{L}$ contains a finite number of elements.*

Proof. We suppose by contradiction that $\mathcal{L}^- \cap \mathcal{K}(X) \neq \emptyset$ and $\mathcal{L}$ has an infinite number of elements. Let A be a compact subset of X such that $A \in \mathcal{L}^-$. Then $A \cap V \neq \emptyset$ for all $V \in \mathcal{L}$. We choose $(V_n)_{n \in \mathbb{N}^*} \subset \mathcal{L}$ and let $x_n \in V_n \cap A$, $n \in \mathbb{N}^*$. (We denote by $\mathbb{N}^*$ the set of positive integer $\mathbb{N} \setminus \{0\}$.)

Because A is a compact set, hence a sequential compact set, there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}^*}$ and $x_0 \in A$ with $x_{n_k} \rightarrow x_0$. For x_0 there exists $U(x_0)$ a neighbourhood which has a nonempty intersection with a finite number p of elements of $\mathcal{L}$. Eventually by renumbering the sets of $\mathcal{L}$, let them be $V_1, V_2, \dots, V_p$. Now we denote again the sequence $(V_{n_k})_{k \in \mathbb{N}^*} \setminus \{V_1, V_2, \dots, V_p\}$ by $\tilde{V}_{n_1}, \tilde{V}_{n_2}, \dots, \tilde{V}_{n_k}, \dots$ and by $\tilde{x}_{n_k}$ the elements of the sequence $(x_{n_k})_{k \in \mathbb{N}^*}$ from $\tilde{V}_{n_k} \cap A$. So $\tilde{x}_{n_k} \rightarrow x_0$. For the neighbourhood $U(x_0)$ there exists $k_0 \in \mathbb{N}^*$ such that for every $n_k \geq n_{k_0}$ we have $\tilde{x}_{n_k} \in U(x_0)$. Hence $U(x_0)$ has a nonempty intersection with an infinite number of subsets of $\mathcal{L}$. This contradiction proves the lemma. $\square$

Theorem 3.3. *Let (X, d) be a metric space. On the class $\mathcal{K}(X)$ of compact subsets of X we have $\tau_V^- = \tau_{lf}^-$ or, equivalently, $\tau_V = \tau_{lf}$.*

Proof. This is an immediate consequence of Lemma 3.1 and Lemma 3.2. $\square$

Remark 3.2. If (X, d) is a compact metric space, then $\mathcal{Cl}(X) = \mathcal{K}(X)$ and hence $\tau_V = \tau_{lf}$ on $\mathcal{Cl}(X)$. So we obtain from Theorem 3.3 the coincidence between τ_V and τ_{lf} on $\mathcal{Cl}(X)$, a known result (see, for example, [5], p.169).

Now we intend to find the largest class $\mathcal{A} \subset \mathcal{Cl}(X)$ on which $\tau_V^- = \tau_{lf}^-$. When we study the same problem for another pair of hypertopologies (τ_H^- and τ_{AW}^- , τ_H^+ and τ_V^+ , τ_H^- and τ_V^- etc.) we obtain (in [2]) the classes $\mathcal{B}(X)$, $\mathcal{Pk}(X)$ and $\mathcal{K}(X)$. So one requires that the class $\mathcal{A}$ contains only bounded subsets. To give an answer to this question, we first prove the following lemma:

Lemma 3.4. *Let (X, d) be a metric space and $\mathcal{A} \subset \mathcal{Cl}(X)$ be such that $\mathcal{A} \setminus \mathcal{B}(X) \neq \emptyset$. Then we can construct a locally finite family $\mathcal{L}$ of open unbounded subsets of X such that $\mathcal{A} \setminus \mathcal{B}(X) = \mathcal{L}^-$.*

Proof. Let $A_0 \in \mathcal{A} \setminus \mathcal{B}(X)$. We fix $a \in X$. Since A_0 is unbounded subset of X , for every $n \in \mathbb{N}^*$ there exists $a_n \in A_0$ such that $d(a_n, a) > n$. So $a_n \in B(a, n)^c$, where $B(a, n)^c$ denotes the complement of the closed ball $B(a, n)$. The set $V_n = B(a, n)^c$ is open and unbounded in X and $A_0 \cap V_n \neq \emptyset$. Now we put $\mathcal{L} = (V_n)_{n \in \mathbb{N}^*}$ and we prove that $\mathcal{L}$ is a locally finite family:

Let $x \in X$ be an arbitrary point. Denote by $n_0 = [d(a, x)]$ the integer with the property $n_0 \leq d(a, x) < n_0 + 1$. Let $\varepsilon = n_0 + 1 - d(a, x)$ be

a positive number. Then $S(x, \varepsilon)$ is for $\mathcal{L}$ the desired neighbourhood of x from the definition of locally finite family: if $y \in S(x, \varepsilon)$ then $d(a, y) \leq \varepsilon + d(a, x) = n_0 + 1$, so $y \notin V_{n_0+1}$. Also $y \notin V_k$, where $k \geq n_0 + 1$. Hence y is contained at most in $V_1, V_2, \dots, V_{n_0}$. So the neighbourhood $S(x, \varepsilon)$ of x intersects a finite number of elements of $\mathcal{L}$. $\square$

Now we can give the result:

Proposition 3.5. *Let (X, d) be a metric space and $\mathcal{A} \subset Cl(X)$ be a nonvoid class such that $\tau_V^- = \tau_{lf}^-$ on $\mathcal{A}$. Then $\mathcal{A} \subset \mathcal{B}(X)$.*

Proof. We suppose by contradiction that $\mathcal{A} \not\subset \mathcal{B}(X)$. Then $\mathcal{A} \setminus \mathcal{B}(X) \neq \emptyset$. From the proof of Lemma 3.4 we have a locally finite family $\mathcal{L} = (V_n)_{n \in \mathbb{N}}$ of open and unbounded subsets of X , where $V_n = B(a, n)^c$ and a is fixed in X . Observe that $\mathcal{L}^-$ is open in $(\mathcal{A}, \tau_{lf}^-)$. Now if $A \in \mathcal{L}^-$, then $A \cap V_n \neq \emptyset$, for every $n \in \mathbb{N}^*$, so A is unbounded and $\mathcal{L}^-$ contains only unbounded closed subsets of X .

Since $\tau_V^- = \tau_{lf}^-$ on $\mathcal{A}$, it follows that $\mathcal{L}^-$ is τ_V^- -open; we have $\mathcal{L}^- = \bigcup_{t \in T} \mathcal{W}_t$, where $\mathcal{W}_t = (W_1^t) \cap (W_2^t) \cap \dots \cap (W_k^t)$, $k \in \mathbb{N}^*$ and $W_1^t, W_2^t, \dots, W_k^t$ are open subsets of X . We choose $a_1^t \in W_1^t, \dots, a_k^t \in W_k^t$. The set $A = \{a_1^t, \dots, a_k^t\} \in \mathcal{B}(X)$ has the property that $A \cap W_j^t \neq \emptyset$ for every $j \in \{1, 2, \dots, k\}$. This condition implies that $A \in \mathcal{W}_t$, so $A \in \mathcal{L}^-$. This is a contradiction, because $\mathcal{L}^-$ contains only unbounded closed subsets of X . $\square$

Remark 3.3. If there exists the largest family $\mathcal{A} \subset Cl(X)$ for which $\tau_V^- = \tau_{lf}^-$, then by Lemma 3.2 and Proposition 3.5, we have $\mathcal{K}(X) \subset \mathcal{A} \subset \mathcal{B}(X)$. In fact $\mathcal{A} \subsetneq \mathcal{B}(X)$. Indeed, assume by contrary that $\mathcal{A} = \mathcal{B}(X)$. Since τ_V and τ_{lf} are not dependent on the metric of the space (but they are depended on the topology of the space), replacing d by $\frac{d}{1+d}$, it follows that the two topologies coincide on every metric space. This contradiction proves the above assertion.

Definition 3.1. A metric space (X, d) is called *boundedly compact* if every closed and d -bounded subset of X is compact.

Corollary 3.6. *If (X, d) is a boundedly compact space, then the largest class $\mathcal{A} \subset Cl(X)$ for which $\tau_V^- = \tau_{lf}^-$ (i.e. $\tau_V = \tau_{lf}$) on $\mathcal{A}$ is $\mathcal{K}(X)$.*

In order to prove a similar result on an arbitrary metric space, we recall the following definition introduced in [2]:

Definition 3.2. A family $\mathcal{A} \subset \mathcal{Cl}(X)$ of parts of a metric space X is called *stable with respect to closed subsets* if for any set $A \in \mathcal{A}$ and $B \subset A$, with $B \in \mathcal{Cl}(X)$ it follows that $B \in \mathcal{A}$.

Remark 3.4. All the families $\mathcal{B}(X)$, $\mathcal{Pk}(X)$, $\mathcal{K}(X)$, $\mathcal{F}(X)$ are stable with respect to closed subsets.

Theorem 3.7. Let (X, d) be a metric space and

$$\mathcal{E} = \{\mathcal{A} \subset \mathcal{Cl}(X); \mathcal{A} \neq \emptyset \text{ and } \tau_V^- = \tau_{lf}^- \text{ on } \mathcal{A}, \\ \mathcal{A} \text{ stable with respect to closed subsets}\}.$$

Then $\mathcal{K}(X)$ is the largest element of $\mathcal{E}$.

Proof. From Theorem 3.3 it follows that $\mathcal{K}(X) \in \mathcal{E}$.

Now we show that $\mathcal{K}(X)$ is upper bound element for class $\mathcal{E}$.

We suppose by contradiction that $\mathcal{A} \subsetneq \mathcal{K}(X)$ for some $\mathcal{A} \in \mathcal{E}$ and let $A \in \mathcal{A} \setminus \mathcal{K}(X)$. So there exists a sequence $(a_n)_{n \in \mathbb{N}^*} \subset A$ such that $(a_n)_{n \in \mathbb{N}^*}$ does not have cluster points. We consider the subsets $A_0 = \{a_1, a_2, \dots, a_k, \dots\}$ and $A_k = \{a_1, a_2, \dots, a_k\}$. Since A_k is finite, A_k is closed. The set of cluster points of A_0 is $\emptyset$, so A_0 is closed too. Obviously, $A_0, A_k \subset A \in \mathcal{A}$ which is stable with respect to closed sets, hence $A_0, A_k \in \mathcal{A}$.

Now we observe that $A_0 \in \tau_V^- - \lim_{k \rightarrow \infty} A_k$: for any open set V with $V \cap A_0 \neq \emptyset$, there exists an $a_{k_V} \in V \cap A_0$. One deduces that for every $k \geq k_V$ we have $a_{k_V} \in V \cap A_k$.

We show that $A_0 \notin \tau_{lf}^- - \lim_{k \rightarrow \infty} A_k$.

First we can construct a family $\mathcal{L}_0$ composed by all the open spheres $V_k = S(a_k, 1/k)$; it comes out that $\mathcal{L}_0$ is a locally finite family. Indeed, we suppose that there exists an $a_0 \in X$ such that for any $n \in \mathbb{N}^*$, $S(a_0, 1/n)$ intersects an infinite number of V_k , say V_{k_p} . So there exists $b_n^p \in S(a_0, 1/n) \cap S(a_{k_p}, 1/k_p)$. Then $d(a_{k_p}, a_0) \leq d(a_{k_p}, b_n^p) + d(b_n^p, a_0) < (1/k_p) + (1/n)$. Passing to the limit as $n \rightarrow \infty$, one derives that $d(a_{k_p}, a_0) \leq 1/k_p$, hence $a_{k_p} \rightarrow a_0$. This is a contradiction, because $(a_k)_{k \in \mathbb{N}^*}$ has not cluster points.

It is obviously that $A_0 \in \mathcal{L}_0^- : a_k \in A_0 \cap V_k$ for every $k \in \mathbb{N}^*$.

Now we prove that for any $n_0 \in \mathbb{N}^*$, there exists $n \geq n_0$ with $A_n \notin \mathcal{L}_0$; more exactly if we fix n_0 , we can find k_0 such that $\{a_1, a_2, \dots, a_{n_0}\} \cap S(a_{k_0}, 1/k_0) = \emptyset$. If we suppose by contradiction that for every $k \in \mathbb{N}^*$, $\{a_1, a_2, \dots, a_{n_0}\} \cap S(a_k, 1/k) \neq \emptyset$, there exists $j_0 \in \{1, \dots, n_0\}$ and a infinite number of terms $k_p \in \mathbb{N}^*$ such that $d(a_{k_p}, a_{j_0}) < 1/k_p$, so $a_{k_p} \rightarrow a_{j_0}$. The contradiction we arrived at, proves the theorem. $\square$

4. Some consequences

In this section, we compare Vietoris and bounded Vietoris convergences on $\mathcal{A} \subset \mathcal{P}(X)$. In fact we want to find sufficient conditions on a class $\mathcal{A} \subset \mathcal{P}(X)$, such that τ_V^+ -convergence and τ_{bV}^+ -convergence coincide. We recall that $\tau_V^+ = \tau_{bV}^+$ if and only if (X, d) is d -bounded (see Proposition 3.2 from [9]). But we have not the coincidence of these topologies on $\mathcal{B}(X)$. For our study we first give the following definition.

Definition 4.1. A net $(A_i)_{i \in I} \in \mathcal{A} \subset \mathcal{P}(X)$ is said to be *uniformly bounded net* if there exists a closed ball $B(a, r) \subset X$ such that $A_i \subset B(a, r)$ for all $i \in I$, where $a \in X$ and $r > 0$.

Remark 4.1. If $(A_i)_{i \in I} \in \mathcal{P}(X)$ is a uniformly bounded net, then A_i is bounded for any $i \in I$.

Proposition 4.1. Let (X, d) be a metric space and $A \in \mathcal{B}(X)$. If $(A_i)_{i \in I}$ is a uniformly bounded net τ_{bV}^+ -convergent to A , then $(A_i)_{i \in I}$ is also τ_V^+ -convergent to A (and thus the τ_{bV} -convergence of a uniformly bounded net with bounded limit coincides with τ_V -convergence of them).

Proof. Let $B(a, r)$ be a closed ball such that $A, A_i \subset B(a, r)$ for any $i \in I$ and G be a closed set of X such that $A \cap G = \emptyset$. But $A \cap G = (A \cap B(a, r)) \cap G = A \cap (B(a, r) \cap G)$. Since $A \in \tau_{bV}^+ - \lim A_i$, we use the closed and d -bounded set $E = B(a, r) \cap G$ in the definition of τ_{bV}^+ -neighbourhoods of A . Then there exists $i_E \in I$ such that $A_i \cap (B(a, r) \cap G) = \emptyset$ for every $i \geq i_E$. We put $i_G = i_E$ and we obtain that $A \in \tau_V^+ - \lim A_i$. $\square$

Corollary 4.2. If (X, d) is a metric space without isolated points and $(A_i)_{i \in I} \in \mathcal{K}(X)$ is a uniformly bounded net, then for a given $A \in \mathcal{K}(X)$, the following convergences are equivalent:

- i) $A \in \tau_{bV}^+ - \lim A_i$;
- ii) $A \in \tau_V^+ - \lim A_i$;
- iii) $A \in \tau_H^+ - \lim A_i$;
- iv) $A \in \tau_{AW}^+ - \lim A_i$.

Proof. $i) \iff ii)$ is Proposition 4.1 (applied on the subclass $\mathcal{K}(X)$).

$ii) \iff iii)$ follows from [2], Theorem 6.2: on $\mathcal{K}(X)$ we have $\tau_V^+ = \tau_H^+$ when (X, d) is a metric space without isolated points. (Therefore the uniformly bounded condition for the net $(A_i)_{i \in I}$ is not necessary.)

$iii) \iff iv)$ is immediately from [2], Proposition 4.1, (ii): if $(A_i)_{i \in I} \subset \mathcal{Cl}(X)$ is uniformly bounded net, the τ_{AW}^+ -convergence coincides with τ_H^+ -convergence on $\mathcal{Cl}(X)$ (also on $\mathcal{K}(X)$). $\square$

Corollary 4.3. *Let (X, d) be a metric space and $A, A_i \in \mathcal{K}(X)$, for any $i \in I$. Then we have the equivalence for the following convergences:*

$$i) \ A \in \tau_V^- - \lim A_i;$$

$$ii) \ A \in \tau_{lf}^- - \lim A_i;$$

$$iii) \ A \in \tau_{AW}^- - \lim A_i;$$

$$iv) \ A \in \tau_H^- - \lim A_i.$$

Proof. The assertion $i) \Leftrightarrow ii)$ results from Theorem 3.3, for an arbitrary net.

The τ_V^- -convergence coincides with the τ_{AW}^- -convergence on the class $\mathcal{Pk}(X) \supset \mathcal{K}(X)$.

The τ_{AW}^- -convergence and the τ_H^- -convergence are the same on $\mathcal{B}(X) \supset \mathcal{K}(X)$. $\square$

Corollary 4.4. *If $A \in \mathcal{K}(X)$ and $(A_i)_{i \in I} \subset \mathcal{K}(X)$ is a uniformly bounded net from a metric space (X, d) without isolated points, then the following assertions are equivalent:*

$$i) \ A = \tau_{bV} - \lim A_i;$$

$$ii) \ A = \tau_V - \lim A_i;$$

$$iii) \ A = \tau_H - \lim A_i;$$

$$iv) \ A = \tau_{AW} - \lim A_i;$$

$$v) \ A = \tau_P - \lim A_i;$$

$$vi) \ A = \tau_{bP} - \lim A_i.$$

(The proof follows from Corollaries 4.2 and 4.3.)

Corollary 4.5. *On the class $\mathcal{K}(X)$ of compact subsets of a metric space (X, d) without isolated points, the following relations between hyperconvergences are valid:*

$$\begin{array}{c} \tau_{lf} = \tau_H = \tau_V = \tau_P \\ \downarrow \\ \tau_{AW} = \tau_{bP} \end{array}$$

For the monotone sequences of subsets of $\mathcal{K}(X)$ all this convergences coincide with τ_W .

Proof. The results are consequences of Theorem 3.3 above and of Corollary 5.5 and Theorem 6.2 from [2]. Also we use Theorem 3.1 from [1]. $\square$

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