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VIABILITY FOR A NONLINEAR MULTI-VALUED SYSTEM ON LOCALLY CLOSED GRAPH

BY

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Abstract. The purpose of this paper is to prove some necessary and sufficient conditions in order that the graph $\mathcal{K}$ of the multi-function $K : I \rightsquigarrow \overline{D(A)} \times \overline{D(B)}$ be C^0-viable with respect to the nonlinear system of the form

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t), v(t)), & t \geq \tau \\ v'(t) \in Bv(t) + G(t, u(t), v(t)), & t \geq \tau \\ u(\tau) = \xi, v(\tau) = \eta, \end{cases}$$

where $I \subseteq \mathbb{R}$ is an open from the right interval, X and Y are real Banach spaces, $A : D(A) \subseteq X \rightsquigarrow X$ and $B : D(B) \subseteq Y \rightsquigarrow Y$ are m-dissipative operators generating nonlinear semigroups of contractions, $F : \mathcal{K} \to X$ is a given function and $G : \mathcal{K} \rightsquigarrow Y$ is a nonempty valued multi-function. We provide a necessary and sufficient condition in order that the system has at least one C^0-solution (u, v) satisfying time-dependent constrains $(u(t), v(t)) \in K(t)$ for each t. We include a comparison result referring to a nonlinear system with multi-valued perturbations of subdifferentials in a Hilbert space.

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1. Introduction

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces and let $A : D(A) \subseteq X \rightsquigarrow X$ and $B : D(B) \subseteq Y \rightsquigarrow Y$ be m-dissipative operators generating the nonlinear semigroups of contractions, $\{S_A(t) : \overline{D(A)} \to \overline{D(A)};\ t \geq 0\}$ and $\{S_B(t) : \overline{D(B)} \to \overline{D(B)};\ t \geq 0\}$ respectively. Let $I \subseteq \mathbb{R}$ be a nonempty and open from the right interval, let $K : I \rightsquigarrow \overline{D(A)} \times \overline{D(B)}$ be a given nonempty valued multi-function and $\mathcal{K} :=$ graph (K).

We consider the system

$$(1.1)\qquad \begin{cases} u'(t) \in Au(t) + F(t, u(t), v(t)), & t \geq \tau \\ v'(t) \in Bv(t) + G(t, u(t), v(t)), & t \geq \tau \\ u(\tau) = \xi, \quad v(\tau) = \eta, \end{cases}$$

where $(\tau, \xi, \eta) \in \mathcal{K}$, $F : \mathcal{K} \to X$ is a given function and $G : \mathcal{K} \rightsquigarrow Y$ is a multi-function with nonempty values.

We are interested in finding necessary and sufficient conditions in order that $\mathcal{K}$ be C^0-viable with respect to $(A + F, B + G)$.

Definition 1.1. The continuous function $(u, v) : [\,\tau, T\,] \to \overline{D(A)} \times \overline{D(B)}$ is a C^0-*solution* of (1.1) on $[\,\tau, T\,]$ if $(t, u(t), v(t)) \in \mathcal{K}$ for each $t \in [\,\tau, T\,]$, the function $f(t) = F(t, u(t), v(t))$ for each $t \in [\,\tau, T\,]$, belongs to $L^1(\,\tau, T; X\,)$ and there exists $g \in L^1(\,\tau, T; Y\,)$ such that $g(t) \in G(t, u(t), v(t))$ a.e. for $t \in [\,\tau, T\,]$ and (u, v) is a C^0-solution on $[\,\tau, T\,]$ of the problem

$$(1.2)\qquad \begin{cases} u'(t) \in Au(t) + f(t), \\ v'(t) \in Bv(t) + g(t), \\ u(\tau) = \xi, \quad v(\tau) = \eta. \end{cases}$$

Definition 1.2. The graph, $\mathcal{K}$, of $K : I \rightsquigarrow \overline{D(A)} \times \overline{D(B)}$, is C^0-*viable with respect to* $(A + F, B + G)$ if for each $(\tau, \xi, \eta) \in \mathcal{K}$ there exists $T > \tau$, $[\,\tau, T\,] \subseteq I$ such that the problem (1.1) has at least one C^0-solution on $[\,\tau, T\,]$.

A growing literature is devoted to the abstract theory of viability referring to the differential inclusion $u'(t) \in Au(t) + F(t, u(t))$. We mention the starting paper of Pavel [15] for the semilinear single-valued case and Vrabie [19] for nonlinear case. Referring to the semilinear multi-valued case when A is linear unbounded, $K \subseteq I \times X$ is locally closed set and $F : K \rightsquigarrow X$ is multi-valued, we notice the works of Pavel-Vrabie [16], [17], Cârjă-Vrabie [11], [10], Cârjă-Necula-Vrabie [8]. As concerns the fully nonlinear inclusion see Bothe [2], [3], Cârjă-Necula-Vrabie [7],[9]. For the case in which F is defined on the graph $\mathcal{K}$, of the multi-function $K : I \rightsquigarrow \overline{D(A)}$ see Necula-Popescu-Vrabie [13], [14]. They introduced the concept of A-quasi-tangent set to the graph of K at a given point $(\tau, \xi) \in \mathcal{K}$ in order to give a necessary and sufficient condition for

C^0-viability referring to the nonlinear inclusion with multi-valued and t-discontinuous perturbations defined on graphs. We recall this concept and the necessary condition for C^0-viability in section 2.

The C^0-viability problem of a locally closed set with respect to the multi-valued reaction-diffusion system of the form (1.1) has been studied by BURLICĂ [4] in the semilinear case and by ROŞU [18] in the nonlinear case. The semilinear system with multi-valued perturbations defined on graphs has been considered by BURLICĂ [5].

Our paper is divided into 5 sections. In Section 2 we recall some results referring to evolutions governed by m-dissipative operators, the concept of tangent sets and some concepts about multi-functions. The third section contains the main results for C^0-viability concerning the system (1.1), while in Section 4 we prove the main sufficient condition. In Section 5, as an application of our viability results, we give a comparison result referring to a nonlinear system with multi-valued perturbations of subdifferentials in a Hilbert space.

2. Preliminaries

In that follows, $(X, \|\cdot\|)$ denotes a real Banach space. For $\xi \in X$ and $\rho > 0$, $D(\xi, \rho)$ denotes the closed ball in X of radius ρ centered in ξ and $S(\xi, \rho)$ denotes the corresponding open ball. For $x \in X, C \subseteq X$ and $E \subseteq X$, we denote

$$\|C\| = \sup_{x \in C} \|x\|, \ \ \mathrm{dist}(x, C) = \inf_{y \in C} \|x-y\| \ \text{and} \ \mathrm{dist}(E, C) = \inf_{(x,y) \in E \times C} \|x-y\|.$$

Let $A : D(A) \subseteq X \rightsquigarrow X$ be an m-dissipative operator, $\xi \in \overline{D(A)}$ and $f \in L^1(\tau, T; X)$ and let us consider the Cauchy problem:

$$\begin{cases} u'(t) \in Au(t) + f(t), \quad t \geq \tau \\ u(\tau) = \xi. \end{cases} \tag{2.1}$$

Definition 2.1. A C^0-*solution* of the problem (2.1) is a continuous function $u \in C([\tau, T]; X)$ satisfying $u(\tau) = \xi$ and, for each $\tau < c < T$ and $\varepsilon > 0$ there exist

(i) $\tau = t_0 < t_1 < \cdots < c \leq t_n < T, \ t_k - t_{k-1} \leq \varepsilon$ for $k = 1, 2, \ldots, n$;

(ii) $f_1, \ldots, f_n \in X$ with $\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|f(t) - f_k\| dt \leq \varepsilon$;

(iii) $v_0, \ldots, v_n \in X$ satisfying:

$$\frac{v_k - v_{k-1}}{t_k - t_{k-1}} \in Av_k + f_k \quad \text{for} \quad k = 1, 2, \ldots, n$$
$$\|u(t) - v_k\| \leq \varepsilon \quad \text{for} \quad t \in [t_{k-1}, t_k), \; k = 1, 2, \ldots, n.$$

Theorem 2.1. *Let $A : D(A) \subseteq X \rightsquigarrow X$ be m-dissipative. Then, for each $\xi \in \overline{D(A)}$ and $f \in L^1(\tau, T; X)$, there exists a unique C^0-solution $u : [\tau, T] \to \overline{D(A)}$ of the problem* (2.1).

See Lakshmikantham-Leela [12, Theorem 3.6.1, p. 116].

We denote by $u(\cdot, \tau, \xi, f) : [\tau, T] \to \overline{D(A)}$ the unique C^0-solution of (2.1) satisfying $u(\tau, \tau, \xi, f) = \xi$. We recall that, if $\widetilde{u} = u(\cdot, \tau, \xi, f)$ and $\widetilde{v} = u(\cdot, \tau, \eta, g)$ we have

$$\|\widetilde{u}(t) - \widetilde{v}(t)\| \leq \|\xi - \eta\| + \int_\tau^t \|f(s) - g(s)\| ds, \tag{2.2}$$

for each $t \in [\tau, T]$. Moreover, for each $\tau \leq a \leq \nu \leq t \leq \nu + \delta$, we have the following evolution property $u(t, a, \xi, f) = u(t, \nu, u(\nu, a, \xi, f), f\,|_{[\nu, \nu+\delta]})$. See Vrabie [20, Corollary 1.7.1, p. 26].

If $A : D(A) \subseteq X \rightsquigarrow X$ is m-dissipative, $\{S(t) : \overline{D(A)} \to \overline{D(A)};\, t \geq 0\}$ denotes the nonlinear semigroup of nonexpansive mappings generated by A on $\overline{D(A)}$, i.e. $S(t)\xi = u(t, 0, \xi, 0)$ for each $t \geq 0$ and $\xi \in \overline{D(A)}$.

For more details referring to evolutions governed by m-dissipative operators, see Barbu [1], Lakshmikantham-Leela [12] or Vrabie [20].

Now, let us consider the Cauchy problem

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t)), \\ u(\tau) = \xi, \end{cases} \tag{2.3}$$

where $A : D(A) \subseteq X \rightsquigarrow X$ is m-dissipative, $I \subseteq \mathbb{R}$ is a nonempty and open from the right interval, $K : I \rightsquigarrow \overline{D(A)}$ and $F : \mathcal{K} \rightsquigarrow X$ are two multifunctions with nonempty values and $\mathcal{K} \subseteq I \times \overline{D(A)}$ is the graph of K, i.e. $\mathcal{K} = \{(t, u);\, u \in K(t)\}$.

Definition 2.2. The continuous function $u : [\tau, T] \to X$ is a C^0-*solution* of the problem (2.3) if $(t, u(t)) \in \mathcal{K}$ for each $t \in [\tau, T]$, there exists $f \in L^1(\tau, T; X)$, with $f(s) \in F(s, u(s))$ a.e. for $s \in [\tau, T]$ and u is a C^0-solution on $[\tau, T]$ of the problem (2.1) in the sense of Definition 2.1.

Definition 2.3. The graph $\mathcal{K}$ is C^0-*viable with respect to* $A+F$ if for each $(\tau,\xi)\in\mathcal{K}$ there exists $T>\tau$, $[\,\tau,T\,]\subseteq I$ such that (2.3) has at least one C^0-solution $u:[\,\tau,T\,]\to X$.

We recall the concepts of A-tangent set and A-quasi-tangent set introduced by CÂRJĂ-NECULA-VRABIE [6] in the constant case, i.e. $K(t)=K$ for each $t\in I$, and by NECULA-POPESCU-VRABIE [14] in the general case. Let $(\tau,\xi)\in\mathcal{K}$ and let E be a nonempty and bounded subset in X.

Definition 2.4. We say that the set $E\subseteq X$ is

(i) A-*tangent* to $\mathcal{K}$ at the point (τ,ξ) if

$$\liminf_{h\downarrow 0}\frac{1}{h}\mathrm{dist}\,(u(\tau+h,\tau,\xi,E);K(\tau+h))=0, \tag{2.4}$$

where

$$u(\tau+h,\tau,\xi,E)=\{u(\tau+h,\tau,\xi,\eta);\ \eta\in E\};$$

(ii) A-*quasi-tangent* to $\mathcal{K}$ at the point (τ,ξ) if

$$\liminf_{h\downarrow 0}\frac{1}{h}\mathrm{dist}\,(u(\tau+h,\tau,\xi,\mathcal{F}_E);K(\tau+h))=0, \tag{2.5}$$

where $u(\tau+h,\tau,\xi,\mathcal{F}_E)=\{u(\tau+h,\tau,\xi,f); f\in L^1_{\mathrm{loc}}(\mathbb{R};X),\ f(s)\in E$ a.e. for $s\in\mathbb{R}\}$.

We denote by $\mathcal{TS}^A_{\mathcal{K}}(\tau,\xi)$ the class of all A-tangent sets to $\mathcal{K}$ at $(\tau,\xi)\in\mathcal{K}$ and by $\mathcal{QTS}^A_{\mathcal{K}}(\tau,\xi)$ the class of all A-quasi-tangent sets to $\mathcal{K}$ at $(\tau,\xi)\in\mathcal{K}$. Obviously, we have $\mathcal{TS}^A_{\mathcal{K}}(\tau,\xi)\subseteq\mathcal{QTS}^A_{\mathcal{K}}(\tau,\xi)$.

Next, we recall some basic concepts and results from NECULA-POPESCU-VRABIE [14], we need in the sequel. Throughout, λ is the Lebesgue measure on $\mathbb{R}$.

Definition 2.5. The multi-function $F:\mathcal{K}\rightsquigarrow X$ is called (*strongly-weakly*) *almost u.s.c.* if, for each $\varepsilon>0$, there exists an open set $\mathcal{O}_\varepsilon\subseteq I$ such that $\lambda(\mathcal{O}_\varepsilon)\le\varepsilon$ and $F\,|_{[(I\setminus\mathcal{O}_\varepsilon)\times X]\cap\mathcal{K}}$ is (strongly-weakly) u.s.c.

Definition 2.6. The multi-function $F:\mathcal{K}\rightsquigarrow X$ is called *essentially locally bounded* if, for each $(\tau,\xi)\in\mathcal{K}$, there exist a negligible set $N_1\subset I$, $\rho>0$ and $l_1\in L^\infty_{\mathrm{loc}}(I;\mathbb{R})$ such that for all $(t,u)\in((I\setminus N_1)\times D(\xi,\rho))\cap\mathcal{K}$, we have $\|F(t,u)\|\le l_1(t)$. If the same condition is satisfied with $l_1\in L^1_{\mathrm{loc}}(I;\mathbb{R})$, we say that F is *locally integrally bounded*.

Remark 2.1. If $D(A)$ is separable, we can choose N_1 in Definition 2.6 independent of $(\tau,\xi)\in\mathcal{K}$.

Definition 2.7. The graph $\mathcal{K}$ is A-C^0-*viable by itself* if for each $(\tau,\xi)\in\mathcal{K}$ there exist $T>\tau$, $\rho>0$ and $l_2\in L^1_{\rm loc}(I;\mathbb{R})$, so that for each $(\widetilde{\tau},\widetilde{\xi})\in([\,\tau,T\,)\times S(\xi,\rho))\cap\mathcal{K}$, there exist $\widetilde{T}\in(\widetilde{\tau},T\,]$ and a pair of functions, $(g,v)\in L^1([\widetilde{\tau},\widetilde{T}];X)\times C([\,\widetilde{\tau},\widetilde{T}\,];X)$, satisfying:

(i) $v(t)=u(t,\widetilde{\tau},\widetilde{\xi},g)$ for each $t\in[\widetilde{\tau},\widetilde{T}]$;

(ii) $(t,v(t))\in([\widetilde{\tau},\widetilde{T}]\times S(\xi,\rho))\cap\mathcal{K}$, for each $t\in[\widetilde{\tau},\widetilde{T}]$;

(iii) $\|g(s)\|\leq l_2(s)$ a.e. for $s\in[\widetilde{\tau},\widetilde{T}]$.

A pair (g,v) satisfying (i)-(iii) is called *simple solution issuing from* $(\widetilde{\tau},\widetilde{\xi})\in([\,\tau,T\,)\times S(\xi,\rho))\cap\mathcal{K}$.

Remark 2.2. (i) If $K:I\rightsquigarrow\overline{D(A)}$ is constant and $S(t)K\subseteq K$ for each $t\geq 0$, then $\mathcal{K}$ is A-C^0-viable by itself.

(ii) If $\mathcal{K}$ is C^0-viable with respect to $A+F$, where $F:\mathcal{K}\rightsquigarrow X$ is essentially locally bounded multi-function, then $\mathcal{K}$ is A-C^0-viable by itself.

Definition 2.8. (i) The graph $\mathcal{K}$ is *locally closed from the left* if for each $(\tau,\xi)\in\mathcal{K}$ there exist $T>\tau$ and $\rho>0$ such that, for each $(\tau_n,\xi_n)\in([\,\tau,T\,]\times D(\xi,\rho))\cap\mathcal{K}$, with $(\tau_n)_n$ nondecreasing, $\lim_n\tau_n=\widetilde{\tau}$ and $\lim_n\xi_n=\widetilde{\xi}$, we have $(\widetilde{\tau},\widetilde{\xi})\in\mathcal{K}$;

(ii) The graph $\mathcal{K}$ is *closed from the left* if for each $(\tau_n,\xi_n)\in\mathcal{K}$, with $(\tau_n)_n$ nondecreasing, $\lim_n\tau_n=\widetilde{\tau}$ and $\lim_n\xi_n=\widetilde{\xi}$, we have $(\widetilde{\tau},\widetilde{\xi})\in\mathcal{K}$.

Definition 2.9. An m-dissipative operator $A:D(A)\subseteq X\rightsquigarrow X$ is of *compact type* if for each sequence $((f_n,u_n))_n$ in $L^1(\tau,T;X)\times C([\,\tau,T\,];X)$ with u_n a C^0-solution of the problem $u_n'(t)\in Au_n(t)+f_n(t)$ on $[\,\tau,T\,]$ for $n=1,2,\ldots$, $\lim_n f_n=f$ weakly in $L^1(\tau,T;X)$ and $\lim_n u_n=u$ strongly in $C([\,\tau,T\,];X)$, it follows that u is a C^0-solution of the problem $u'(t)\in Au(t)+f(t)$ on $[\,\tau,T\,]$.

We conclude this section with a necessary condition for C^0-viability referring to the inclusion (2.3).

Theorem 2.2. *Let X be a Banach space, let $A:D(A)\subseteq X\rightsquigarrow X$ be an m-dissipative operator which generates a nonlinear semigroup of contractions on $\overline{D(A)}$ and let $F:\mathcal{K}\rightsquigarrow X$ be a multi-function with nonempty* *and closed values, almost u.s.c. and locally integrally bounded. If $D(A)$ is separable and $\mathcal{K}$ is C^0-viable with respect to $A+F$, then $\mathcal{K}$ is A-C^0-viable by itself and there exists a set $N \subseteq I$, with $\lambda(N) = 0$, such that, for each $(\tau,\xi) \in [\,(I \setminus N) \times X] \cap \mathcal{K}$, we have $F(\tau,\xi) \in \mathfrak{QTS}^{A}_{\mathcal{K}}(\tau,\xi)$.*

See Necula-Popescu-Vrabie [14, Theorem 4.1].

3. The main results

The aim of this section is to present the necessary and sufficient conditions for C^0-viability referring to the systems of the form (1.1).

Remark 3.1. The system (1.1) can be rewritten as a multi-valued nonlinear Cauchy problem in the product space $\mathcal{Z} = X \times Y$ endowed with the norm $\|\cdot\|_{\mathcal{Z}}$, defined by $\|(x,y)\|_{\mathcal{Z}} = \|x\|_X + \|y\|_Y$ for each $(x,y) \in \mathcal{Z}$. Let $\mathcal{A} = (A,B) : D(\mathcal{A}) \subseteq \mathcal{Z} \to \mathcal{Z}$ be defined by $D(\mathcal{A}) = D(A) \times D(B)$, $\mathcal{A}(z) = (\,Ax, By\,)$ for each $z = (x,y) \in D(\mathcal{A})$ and let $\mathcal{F} : \mathcal{K} \rightsquigarrow \mathcal{Z}$ be the multi-function defined by $\mathcal{F}(t,z) = (\,F(t,z), G(t,z)\,) = \{(F(t,z),w);\ w \in G(t,z)\}$ for each $(t,z) \in \mathcal{K}$. So, the system (1.1) can be rewritten as

$$\begin{cases} z'(t) \in \mathcal{A}z(t) + \mathcal{F}(t,z(t)), & t \geq \tau \\ z(\tau) = \zeta, \end{cases} \tag{3.1}$$

where $\zeta = (\xi,\eta)$. Let us remark that, if $\{S_A : \overline{D(A)} \to \overline{D(A)};\ t \geq 0\}$ and $\{S_B(t) : \overline{D(B)} \to \overline{D(B)};\ t \geq 0\}$ are the semigroups of contractions generated by A and B, then $\mathcal{A}$ generates the semigroup $\{\,\mathcal{S}(t) : \mathcal{Z} \to \mathcal{Z};\ t \geq 0\}$, given by $\mathcal{S}(t)(x,y) = (S_A(t)x, S_B(t)y)$, for each $t \geq 0$ and $(x,y) \in \mathcal{Z}$.

We also remark that the graph $\mathcal{K}$ is C^0-viable with respect to $(A+F, B+G)$ in sense of Definition 1.2 if and only if $\mathcal{K}$ is C^0-viable with respect to $\mathcal{A}+\mathcal{F}$ in sense of Definition 2.3, which means that for each $(\tau,\zeta) \in \mathcal{K}$, there exists $T > \tau$, $[\,\tau,T\,] \subseteq I$ such that (3.1) has at least one C^0-solution $z : [\,\tau,T\,] \to \mathcal{Z}$ with $(t,z(t)) \in \mathcal{K}$ for each $t \in [\,\tau,T\,]$.

We introduce the following hypotheses:

(H_1) $A : D(A) \subseteq X \rightsquigarrow X$ and $B : D(B) \subseteq Y \rightsquigarrow Y$ are m-dissipative operators and $\{S_A(t) : \overline{D(A)} \to \overline{D(A)};\ t \geq 0\}$ and $\{S_B(t) : \overline{D(B)} \to \overline{D(B)};\ t \geq 0\}$ are the nonlinear semigroups of contractions generated by A and B respectively;

(H$_2$) the semigroup $\{S_B(t) : \overline{D(B)} \to \overline{D(B)};\ t \geq 0\}$ is compact, i.e., $S_B(t)$ is compact for each $t > 0$;

(H$_3$) $B : D(B) \subseteq Y \rightsquigarrow Y$ is of compact type;

(H$_4$) the graph $\mathcal{K}$ is $\mathcal{A}$-C^0-viable by itself;

(H$_5$) the graph $\mathcal{K}$ is locally closed from the left;

(H$_6$) $F : I \times X \times Y \to X$ is continuous on $I \times X \times Y$ and locally Lipschitz with respect to its second argument, i.e., for each $(\tau, \xi, \eta) \in I \times X \times Y$ there exist $T > \tau, \rho > 0$ and $L = L_{\tau,\xi,\eta} > 0$ such that $\|F(t, u, v) - F(t, \widetilde{u}, v)\|_X \leq L\|u - \widetilde{u}\|_X$ for each $(t, u, v), (t, \widetilde{u}, v) \in [\tau, T] \times D_X(\xi, \rho) \times D_Y(\eta, \rho)$;

(H$_7$) $G : \mathcal{K} \rightsquigarrow Y$ is essentially locally bounded;

(H$_8$) $G : \mathcal{K} \rightsquigarrow Y$ is almost strongly-weakly u.s.c.;

(H$_9$) $G : \mathcal{K} \rightsquigarrow Y$ is almost u.s.c.;

(H$_{10}$) there exists a set $N \subseteq I$, with $\lambda(N) = 0$ such that for each $(\tau, \xi, \eta) \in [(I \setminus N) \times X \times Y] \cap \mathcal{K}$, we have $(F(\tau, \xi, \eta), G(\tau, \xi, \eta)) \in \mathfrak{QTS}^{\mathcal{A}}_{\mathcal{K}}(\tau, \xi, \eta)$.

From Theorem 2.2 and Remark 3.1 we deduce:

Theorem 3.1. *Let us assume that* (H$_1$) *is satisfied and* $D(A)$ *and* $D(B)$ *are separable. Let* $I \subseteq \mathbb{R}$ *be a nonempty and open from the right interval, let* $K : I \rightsquigarrow \overline{D(A)} \times \overline{D(B)}$ *be a multi-function with nonempty values and* $\mathcal{K} = \mathrm{graph(K)}$. *Let* $F : \mathcal{K} \to X$ *be a continuous function and let* $G : \mathcal{K} \rightsquigarrow Y$ *be a multi-function with nonempty and closed values, almost u.s.c. and locally integrally bounded. If* $\mathcal{K}$ *is* C^0*-viable with respect to* $(A + F, B + G)$, *then* (H$_4$) *and* (H$_{10}$) *are satisfied.*

The next result is the main sufficient condition of C^0-viability.

Theorem 3.2. *Let* X *and* Y *be Banach spaces,* $I \subseteq \mathbb{R}$ *be a nonempty and open from the right interval, let* $K : I \rightsquigarrow \overline{D(A)} \times \overline{D(B)}$ *be a multi-function with nonempty values and* $\mathcal{K} = \mathrm{graph(K)}$. *Let us assume that* (H$_1$), (H$_2$), (H$_3$), (H$_4$), (H$_5$) *and* (H$_6$) *are satisfied. Let* $G : \mathcal{K} \rightsquigarrow Y$ *be a nonempty, convex and weakly compact valued multi-function which satisfies* (H$_7$) *and* (H$_8$). *If the tangency condition* (H$_{10}$) *is satisfied, then* $\mathcal{K}$ *is* C^0*-viable with respect to* $(A + F, B + G)$. *If* $D(A)$ *is separable and, instead* *of* (H_8), *we assume that* (H_9) *is satisfied, then a necessary and sufficient condition in order that* $\mathcal{K}$ *be* C^0*-viable with respect to* $(A+F, B+G)$ *is* (H_{10}).

The necessity follows from Theorem 3.1 by observing that (H_2) implies the separability of $D(B)$, while the sufficiency will be proved in the next section.

4. Proof of Theorem 3.2

Proof. From Remark 3.1, it suffices to show that the set $\mathcal{K}$ is C^0-viable with respect to $\mathcal{A}+\mathcal{F}$. Let $(\tau,\zeta)=(\tau,\xi,\eta)\in\mathcal{K}$ be arbitrary. We will prove that the problem (3.1) has at least one C^0-solution on an interval $[\,\tau,T_0\,]$. Let $Z\subseteq I$ be a negligible set including the negligible sets from (H_7) and (H_{10}). Let $\rho>0$, $\widetilde{T}>\tau$, $[\,\tau,\widetilde{T}\,]\subseteq I$, $M_0>0$, $L>0$ and $l_1\in L^\infty_{\mathrm{loc}}(I;\mathbb{R})$ be such that $([\,\tau,\widetilde{T}\,]\times D_{\mathcal{Z}}(\zeta,\rho))\cap\mathcal{K}$ is closed from the left and the next condition are satisfied:

$$\|F(t,u,v)\|_X\le M_0 \tag{4.1}$$

for each $(t,u,v)\in[\,\tau,\widetilde{T}\,]\times D_X(\xi,\rho)\times D_Y(\eta,\rho)$,

$$\|F(t,u,v)-F(t,\widetilde{u},v)\|_X\le L\|u-\widetilde{u}\|_X \tag{4.2}$$

for each $(t,u,v),(t,\widetilde{u},v)\in[\,\tau,\widetilde{T}\,]\times D_X(\xi,\rho)\times D_Y(\eta,\rho)$,

$$\|G(t,u,v)\|_Y\le l_1(t) \tag{4.3}$$

for each $(t,u,v)\in(([\,\tau,\widetilde{T}\,]\setminus Z)\times D_X(\xi,\rho)\times D_Y(\eta,\rho))\cap\mathcal{K}$.

Let $(\varepsilon_n)_n\downarrow 0$ be a sequence in $(0,1)$ and let $(\mathcal{O}_n)_{n\ge1}\subseteq\mathbb{R}$ be a sequence of open sets such that:

(a) $Z\subseteq\mathcal{O}_n$ for each $n\ge1$;

(b) $\mathcal{O}_{n+1}\subseteq\mathcal{O}_n$ and $\lambda([\,\tau,T\,]\cap\mathcal{O}_n)\le\varepsilon_n$ for each $n\ge1$;

(c) $G_{|[(I\setminus\mathcal{O}_n)\times\mathcal{Z}]\cap\mathcal{K}}$ is strongly-weakly u.s.c., for each $n\ge1$.

Now, from Necula-Popescu-Vrabie [14, Lemma 5.1], we deduce that there exist $T\in(\tau,\widetilde{T}]$, independent of n, and a sequence of $(\varepsilon_n,\mathcal{O}_n)$-approximate C^0-solution of (3.1) on $[\,\tau,T\,]$, $((\alpha_n,\widetilde{f}_n,z_n))_n$. This means that, for each $n\ge1$, $\alpha_n:[\,\tau,T\,]\to[\,\tau,T\,]$ is nondecreasing and right continuous, $\widetilde{f}_n:[\,\tau,T\,]\to\mathcal{Z}$ is measurable, $z_n\in C([\,\tau,T\,];\mathcal{Z})$ and the following conditions are satisfied.

(i) $t-\varepsilon_n \le \alpha_n(t) \le t$ for all $t \in [\,\tau, T\,]$, $\alpha_n(T)=T$;

(ii) for each $t \in [\,\tau, T\,]$ for which $\alpha_n(t) \in \mathcal{O}_n$ it follows that $[\alpha_n(t), t] \subseteq \mathcal{O}_n$;

(iii) $z_n(\alpha_n(t)) \in D_{\mathcal{Z}}(\zeta, \rho) \cap K(\alpha_n(t))$ for all $t \in [\,\tau, T\,]$;

(iv) $\widetilde{f}_n(t) \in \mathcal{F}(\alpha_n(t), z_n(\alpha_n(t)))$ for each $t \in [\,\tau, T\,] \setminus \mathcal{O}_n$;

(v) $\|\widetilde{f}_n(t)\|_{\mathcal{Z}} \le l(t)$ a.e. for $t \in [\,\tau, T\,]$, with $l(t) = \max\{M_0 + l_1(t), l_2(t)\}$ where M_0 is from (4.1), $l_1 \in L^\infty_{\mathrm{loc}}(I;\mathbb{R})$ is from (4.3) and $l_2 \in L^1_{\mathrm{loc}}(I;\mathbb{R})$ is as in Definition 2.7;

(vi) $z_n(\tau) = \zeta$ and $\|z_n(t) - u(t, \alpha_n(s), z_n(\alpha_n(s)), \widetilde{f}_n)\|_{\mathcal{Z}} \le (t-\alpha_n(s))\varepsilon_n$ for all $t, s \in [\,\tau, T\,]$, $\tau \le s \le t \le T$;

(vii) $\|z_n(t) - z_n(\alpha_n(t))\|_{\mathcal{Z}} \le \varepsilon_n$ for all $t \in [\,\tau, T\,]$;

(viii) $\sup_{t\in[\,\tau,T\,]} \|\mathcal{S}(t)\zeta - \zeta\|_{\mathcal{Z}} + \int_\tau^T l(s)\,ds + T - \tau < \rho$.

Let us denote by $\widetilde{f}_n := (f_n, g_n)$ and by $z_n := (u_n, v_n)$. By (vi), for $s=\tau$, we have

$$\|z_n(t) - u(t, \tau, \zeta, \widetilde{f}_n)\|_{\mathcal{Z}} \le (t-\tau)\varepsilon_n \tag{4.4}$$

for each $t \in [\,\tau, T\,]$, where $u(\cdot, \tau, \zeta, \widetilde{f}_n)$ is the C^0-solution of the problem

$$\begin{cases} z'(t) \in \mathcal{A}z(t) + \widetilde{f}_n(t) \\ z(\tau) = \zeta. \end{cases} \tag{4.5}$$

This means that $u(\cdot, \tau, \zeta, \widetilde{f}_n) = (x(\cdot, \tau, \xi, f_n), y(\cdot, \tau, \eta, g_n))$ where $x(\cdot) = x(\cdot, \tau, \xi, f_n)$ is the C^0-solution of the problem

$$\begin{cases} x'(t) \in Ax(t) + f_n(t) \\ x(\tau) = \xi, \end{cases} \tag{4.6}$$

and $y(\cdot) = y(\cdot, \tau, \eta, g_n)$ is the C^0-solution of the problem

$$\begin{cases} y'(t) \in By(t) + g_n(t) \\ y(\tau) = \eta. \end{cases} \tag{4.7}$$

We will prove that, on a subsequence at least, $(z_n)_n$ is uniformly convergent on an interval $[\,\tau, T_0\,]$ to some function z which is a C^0-solution of (3.1).

From (i), we obtain

$$\lim_{n\to\infty} \alpha_n(t) = t \tag{4.8}$$

uniformly for $t \in [\,\tau, T\,]$.

By (v), we have $\|g_n(t)\|_Y \le l(t)$ for each $n = 1, 2, \ldots$ and for all $t \in [\,\tau, T\,]$ with $l \in L^1_{\text{loc}}(I; \mathbb{R})$. Thus $\{g_n;\, n = 1, 2, \ldots\}$ is uniformly integrable in $L^1(\tau, T; Y)$. Since B generates a compact semigroup, from (4.7), thanks to Vrabie [20, Baras' Theorem 2.3.3, p. 47], we conclude that there exists $\widetilde{y} \in C([\,\tau, T\,]; Y)$ such that, on a subsequence at least,

$$\lim_{n\to\infty} y(t, \tau, \eta, g_n) = \widetilde{y}(t) \tag{4.9}$$

uniformly for $t \in [\,\tau, T\,]$. From (4.4) and (4.8), we deduce

$$\lim_{n\to\infty} v_n(t) = \lim_{n\to\infty} v_n(\alpha_n(t)) = \widetilde{y}(t) \tag{4.10}$$

uniformly for $t \in [\,\tau, T\,]$. Since, by (iii), we have $v_n(\alpha_n(t)) \in D_Y(\eta, \rho)$ we deduce $\widetilde{y}(t) \in D_Y(\eta, \rho)$ for all $t \in [\,\tau, T\,]$.

Now, let us consider the Cauchy problem:

$$\begin{cases} x'(t) = Ax(t) + F(t, x(t), \widetilde{y}(t)), & t \in [\,\tau, T\,] \\ x(\tau) = \xi. \end{cases} \tag{4.11}$$

Since A is m-dissipative operator, $\xi \in \overline{D(A)}$ and the function $f_0 : [\,\tau, T\,] \times X \to X$, $f_0(t, x) = F(t, x, \widetilde{y}(t))$ for each $(t, x) \in [\,\tau, T\,] \times X$, is continuous on $[\,\tau, T\,] \times X$ and locally Lipschitz with respect to $x \in X$, it follows that there exists $T_0 \in (\,\tau, T\,]$ such that the problem (4.11) has an unique C^0-solution $\widetilde{x} : [\,\tau, T_0] \to \overline{D(A)}$.

Let us prove now that $\lim_n u_n(t) = \widetilde{x}(t)$ uniformly for $t \in [\,\tau, T_0\,]$. From (iv) we have $f_n(s) \in F(\alpha_n(s), u_n(\alpha_n(s)), v_n(\alpha_n(s)))$ for all $s \in [\,\tau, T_0\,] \setminus \mathcal{O}_n$. From this and (2.2), we deduce

$$\begin{aligned} &\|x(t, \tau, \xi, f_n) - \widetilde{x}(t)\|_X \le \int_\tau^t \|f_n(s) - F(s, \widetilde{x}(s), \widetilde{y}(s))\|_X \, ds \\ &\le \int_{[\,\tau, t]\setminus\mathcal{O}_n} \|F(\alpha_n(s), z_n(\alpha_n(s))) - F(s, \widetilde{x}(s), \widetilde{y}(s))\|_X \, ds \end{aligned}$$

$$
\begin{aligned}
&+\int_{[\tau,t]\cap\mathcal{O}_n}\|f_n(s)-F(s,\widetilde{x}(s),\widetilde{y}(s))\|_X\,ds\\
&\leq\int_{[\tau,t]}\|F(\alpha_n(s),u_n(\alpha_n(s)),v_n(\alpha_n(s)))-F(\alpha_n(s),\widetilde{x}(s),v_n(\alpha_n(s)))\|_X\,ds\\
&+\int_{[\tau,t]}\|F(\alpha_n(s),\widetilde{x}(s),v_n(\alpha_n(s)))-F(s,\widetilde{x}(s),\widetilde{y}(s))\|_X\,ds\\
&+\int_{[\tau,t]\cap\mathcal{O}_n}(\|f_n(s)\|_X+\|F(s,\widetilde{x}(s),\widetilde{y}(s))\|_X)\,ds
\end{aligned}
$$

for each $t\in[\tau,T_0]$. Now, using (4.1), (4.2) and (v), we get

$$
\begin{aligned}
&\|x(t,\tau,\xi,f_n)-\widetilde{x}(t)\|_X\\
&\leq L\int_\tau^t\|u_n(\alpha_n(s))-\widetilde{x}(s)\|_X\,ds+2\int_{[\tau,t]\cap\mathcal{O}_n}l(s)\,ds\\
&+\int_\tau^{T_0}\|F(\alpha_n(s),\widetilde{x}(s),v_n(\alpha_n(s)))-F(s,\widetilde{x}(s),\widetilde{y}(s))\|_X\,ds.
\end{aligned}
\tag{4.12}
$$

By (b), (4.8), (4.10) and the continuity of F on $I\times X\times Y$, we deduce that, there exists $\gamma_n\downarrow 0$ such that $\int_{[\tau,T_0]\cap\mathcal{O}_n}l(s)ds\leq\gamma_n$ for $n=1,2,\dots$ and $\|F(\alpha_n(s),\widetilde{x}(s),v_n(\alpha_n(s)))-F((s),\widetilde{x}(s),\widetilde{y}(s))\|_X\leq\gamma_n$ for $n=1,2,\dots$ and each $s\in[\tau,T_0]$.

On the other hand, we have $\|u_n(\alpha_n(t))-\widetilde{x}(t)\|_X\leq\|u_n(\alpha_n(t))-u_n(t)\|_X+\|u_n(t)-x(t,\tau,\xi,f_n)\|_X+\|x(t,\tau,\xi,f_n)-\widetilde{x}(t)\|_X$. From (vii), (4.4) and (4.12), we obtain

$$
\|u_n(\alpha_n(t))-\widetilde{x}(t)\|_X\leq\delta_n+L\int_\tau^t\|u_n(\alpha_n(s))-\widetilde{x}(s)\|_Xds,
$$

for each $t\in[\tau,T_0]$, where $\delta_n:=(1+T_0-\tau)\varepsilon_n+(2+T_0-\tau)\gamma_n$. By Gronwall's Lemma, we conclude $\|u_n(\alpha_n(t))-\widetilde{x}(t)\|_X\leq\delta_ne^{L(T_0-\tau)}$, for each $n\geq 1$ and each $t\in[\tau,T_0]$. Since $\delta_n\to 0$ and $\alpha_n(t)\to t$ uniformly for $t\in[\tau,T_0]$, we get

$$
\lim_{n\to\infty}u_n(\alpha_n(t))=\lim_{n\to\infty}u_n(t)=\widetilde{x}(t),
\tag{4.13}
$$

uniformly for $t\in[\tau,T_0]$ and by (4.4),

$$
\lim_{n\to\infty}x(t,\tau,\xi,f_n)=\widetilde{x}(t),
\tag{4.14}
$$

uniformly for $t \in [\tau, T_0]$. Thus

$$\lim_{n\to\infty} z_n(\alpha_n(t)) = \lim_{n\to\infty} z_n(t) = \lim_{n\to\infty} u(t,\tau,\zeta,\widetilde{f}_n) = z(t) \tag{4.15}$$

uniformly for $t \in [\tau, T_0]$, where $z := (\widetilde{x}, \widetilde{y}) \in C([\tau, T_0]; \mathcal{Z})$.

Since $z_n(\alpha_n(t)) \in K(\alpha_n(t))$, $\alpha_n(t) \uparrow t$ uniformly for $t \in [\tau, T_0]$ and $([\tau, T_0] \times D_{\mathcal{Z}}(\zeta,\rho)) \cap \mathcal{K}$ is closed from the left, from (4.15), we deduce that

$$(t, z(t)) \in \mathcal{K}, \tag{4.16}$$

for $t \in [\tau, T_0]$.

Let $k \in \mathbb{N}, k \geq 1$ be arbitrary. From (4.8) and (4.15) we deduce that, the set

$$\overline{\{(\alpha_n(t), z_n(\alpha_n(t)));\, t \in [\tau, T_0],\, n \geq k\}}$$

is compact. Since $\mathcal{F}$ has weakly compact values and $\mathcal{F}_{|[([\tau,T_0]\setminus\mathcal{O}_k)\times\mathcal{Z}]\cap\mathcal{K}}$ is strongly-weakly u.s.c., by Cârjă-Necula-Vrabie [6, Lemma 2.6.1, p. 47], it follows that, for each $k \geq 1$, the set

$$C_k := \overline{\text{conv}}\Big(\bigcup_{n\geq k}\ \bigcup_{t\in[\tau,T_0]\setminus\mathcal{O}_k} \mathcal{F}(\alpha_n(t), z_n(\alpha_n(t)))\Big)$$

is weakly compact. From (v) we deduce that $\{\widetilde{f}_n;\, n \geq 1\}$ is uniformly integrable in $L^1(\tau, T_0; \mathcal{Z})$ and from (iv) we deduce that, for each $k \geq 1$ and each $n \geq k$, $\widetilde{f}_n(t) \in C_k$ for each $t \in [\tau, T_0] \setminus \mathcal{O}_k$. Since C_k is weakly compact in $\mathcal{Z}$ and $\lambda([\tau, T_0] \cap \mathcal{O}_k) \leq \varepsilon_k$, by Cârjă-Necula-Vrabie [6, Diestel's Theorem 1.3.8, p. 10], it follows that $\{\widetilde{f}_n, n \geq 1\}$ is weakly relatively compact in $L^1(\tau, T_0; \mathcal{Z})$. So, there exists $\widetilde{f} \in L^1(\tau, T_0; \mathcal{Z})$ such that, on a subsequence at least, $\lim_{n\to\infty} \widetilde{f}_n = \widetilde{f}$ weakly in $L^1(\tau, T_0; \mathcal{Z})$. Let us denote by $\widetilde{f} := (f, g)$.

Since $f_n(s) = F(\alpha_n(s), z_n(s))$ for each $s \in [\tau, T_0] \setminus \mathcal{O}_n$, F is a continuous function, $\lim_n f_n = f$ weakly in $L^1(\tau, T_0; X)$, $\lim_n(\alpha_n(s), z_n(s)) = (s, z(s))$ uniformly for $s \in [\tau, T_0]$ and $\lim_n \lambda(\mathcal{O}_n) = 0$, it follows that

$$f(s) = F(s, \widetilde{x}(s), \widetilde{y}(s)) \tag{4.17}$$

a.e. for $s \in [\tau, T_0]$. We notice that, in fact, $\lim_n f_n = f$ strongly in $L^1(\tau, T_0; X)$. Since $g_n(s) \in G(\alpha_n(s), z_n(\alpha_n(s)))$ for each $s \in [\tau, T_0] \setminus \mathcal{O}_k$, $G_{|[(I\setminus\mathcal{O}_k)\times\mathcal{Z}]\cap\mathcal{K}}$ is strongly-weakly u.s.c., $\lim_n g_n = g$ weakly in $L^1(\tau, T_0; Y)$, $\lim_n(\alpha_n(s), z_n(s)) = (s, z(s))$ uniformly for $s \in [\tau, T_0]$ and, for each $n \geq k$, we have $\alpha_n(s) \in [\tau, T_0] \setminus \mathcal{O}_k$, from VRABIE [20, Theorem 3.1.2, p. 88], we conclude that $g(s) \in G(s, z(s))$ for each $k \geq 1$ and a.e. for $s \in [\tau, T_0] \setminus \mathcal{O}_k$. Since $\lim_k \lambda(\mathcal{O}_k) = 0$, we conclude that

$$g(s) \in G(s, \widetilde{x}(s), \widetilde{y}(s)) \tag{4.18}$$

a.e. for $s \in [\tau, T_0]$. Since B is m-dissipative of compact type, $\lim_n g_n = g$ weakly in $L^1(\tau, T_0; Y)$ and $\lim_n y(\cdot, \tau, \eta, g_n) = \widetilde{y}$ strongly in $C([\tau, T_0]; Y)$, we get $\widetilde{y} = y(\cdot, \tau, \eta, g)$. Finally, taking into account that $\widetilde{x}$ is the C^0-solution of (4.11), from (4.16), (4.17) and (4.18), we conclude that $z = (\widetilde{x}, \widetilde{y})$ is a C^0-solution of (3.1). The proof is complete. □

5. An example

Let H be a real Hilbert space, let $C \subseteq H$ be a proper pointed closed convex cone and let "$\preceq$" be the partial order on H defined by C, i.e., $x \preceq y$ if and only if $y - x \in C$. Let $\varphi : H \to \mathbb{R}_+ \cup \{\infty\}$ and $\psi : H \to \mathbb{R}_+ \cup \{\infty\}$ be two proper, convex and l.s.c. functions and let $\partial\varphi : D(\partial\varphi) \subseteq H \rightsquigarrow H$ and $\partial\psi : D(\partial\psi) \subseteq H \rightsquigarrow H$ be the subdifferentials of φ and ψ respectively. Let us denote by $\{S_\varphi(t) : \overline{D(\partial\varphi)} \to \overline{D(\partial\varphi)};\, t \geq 0\}$ and by $\{S_\psi(t) : \overline{D(\partial\psi)} \to \overline{D(\partial\psi)};\, t \geq 0\}$ the nonlinear semigroups of contractions generated by the m-dissipative operators $-\partial\varphi$ and $-\partial\psi$ respectively. Let $a : I \to D(\partial\varphi)$ and $b : I \to D(\partial\psi)$ be two continuous functions. Let $K : I \rightsquigarrow H \times H$ be defined $K(t) := \{(x, y);\, a(t) \preceq x,\ y \preceq b(t)\}$ for each $t \in I$ and let $\mathcal{K}$ be the graph of K. Let $F : \mathcal{K} \to H$ be a given function and $G : \mathcal{K} \rightsquigarrow H$ be a multi-function with nonempty values. Let us consider the problem:

$$\begin{cases} u'(t) \in -\partial\varphi(u(t)) + F(t, u(t), v(t)) \\ v'(t) \in -\partial\psi(v(t)) + G(t, u(t), v(t)) \\ u(\tau) = \xi,\ v(\tau) = \eta \\ a(t) \preceq u(t),\ v(t) \preceq b(t). \end{cases} \tag{5.1}$$

Definition 5.1. A continuous function $(u, v) : [\tau, T] \to D(\partial\varphi) \times D(\partial\psi)$ is a *strong solution* on $[\tau, T]$ of (5.1) if $(u, v) \in W^{1,2}(\tau, T; H \times H)$, the function $t \mapsto f(t) = F(t, u(t), v(t))$ belongs to $L^2(\tau, T; H)$ and there exists

$g \in L^2(\tau, T; H)$, $g(t) \in G(t, u(t), v(t))$ a.e. for $t \in [\,\tau, T\,]$ such that:

$$\begin{cases} u'(t) \in -\partial\varphi(u(t)) + f(t), \text{ a.e. for } t \in [\,\tau, T\,], \\ v'(t) \in -\partial\psi(v(t)) + g(t), \text{ a.e. for } t \in [\,\tau, T\,], \\ u(\tau) = \xi, \;\; v(\tau) = \eta, \\ a(t) \preceq u(t), \;\; v(t) \preceq b(t) \text{ for each } t \in [\,\tau, T\,]. \end{cases} \tag{5.2}$$

Using Theorem 3.2, we obtain a sufficient condition in order that $\mathcal{K}$ be *strongly-viable* with respect to $(-\partial\varphi + F, -\partial\psi + G)$, i.e., in order that, for each $(\tau, \xi, \eta) \in I \times H \times H$ with $a(\tau) \preceq \xi,\ \eta \preceq b(\tau)$, the problem (5.1) has at least one strong solution on an interval $[\,\tau, T\,]$.

Definition 5.2. A convex function $\psi : H \to \mathbb{R}_+ \cup \{\infty\}$ is called *of compact type* if, for each $k > 0$, the level set $\mathcal{L}_k = \{u \in H;\ \|u\|^2 + \psi(u) \leq k\}$ is relatively compact in H.

Remark 5.1. If $\psi : H \to \mathbb{R}_+ \cup \{\infty\}$ is a proper, convex, l.s.c. function of compact type, then $-\partial\psi$ generates a compact semigroup − see Vrabie [20, Proposition 2.2.2, p. 42],− and it is an m-dissipative operator of compact type in the sense of Definition 2.9 − see Vrabie [20, Corollary 2.3.2, p. 50].

Theorem 5.1. *Let H be a real Hilbert space, let $\varphi : H \to \mathbb{R}_+ \cup \{\infty\}$ and $\psi : H \to \mathbb{R}_+ \cup \{\infty\}$ be two proper, convex and l.s.c. functions, with $\partial\varphi$ and $\partial\psi$ single-valued, let $a : I \to D(\partial\varphi)$, $b : I \to D(\partial\psi)$, $a, b \in W^{1,1}_{\mathrm{loc}}(I; H)$, let $C \subseteq \overline{D(\partial\varphi)} \cap \overline{D(\partial\psi)}$ be a closed convex cone with $C \cap (-C) = \{0\}$ and $\overline{D(\partial\varphi) \cap C} = \overline{D(\partial\psi) \cap C} = C$. Let $\mathcal{K}$ be the graph of multi-function $K : I \rightsquigarrow \overline{D(\partial\varphi)} \times \overline{D(\partial\psi)}$, $K(t) = (a(t) + C, b(t) - C)$ for each $t \in I$. Let us assume that ψ is of compact type, $S_\varphi(t)C \subseteq C$, $S_\psi(t)C \subseteq C$ for each $t \in I$ and $\mathcal{K}$ is $(-\partial\varphi, -\partial\psi) - C^0$-viable by itself. Let $F : I \times H \times H \to H$ be a continuous function which is locally Lipschitz with respect to its second argument and $G : \mathcal{K} \rightsquigarrow H$ be a nonempty, convex and weakly compact valued multi-function which is essentially locally bounded and almost strongly-weakly u.s.c. Then, a sufficient condition in order that $\mathcal{K}$ be C^0-viable with respect to $(-\partial\varphi + F, -\partial\psi + G)$ is to exists a neglijable set $N \subseteq I$ such that, for each $\tau \in I \setminus N$ and each $(\xi, \eta) \in \partial(C \times C) \cap (D(\partial\varphi) \times D(\partial\psi))$, we have*

$$\begin{cases} \mathrm{dist}(\partial\varphi(\xi) - \partial\varphi(a(\tau) + \xi) - a'(\tau) + F(\tau, a(\tau) + \xi, b(\tau) - \eta); C) = 0 \\ \mathrm{dist}(\partial\psi(\eta) + \partial\psi(b(\tau) - \eta) + b'(\tau) - G(\tau, a(\tau) + \xi, b(\tau) - \eta); C) = 0. \end{cases} \tag{5.3}$$

Proof. We will prove that (5.3) implies the next tangency condition $(F(\tau, a(\tau)+\xi, b(\tau)-\eta), G(\tau, a(\tau)+\xi, b(\tau)-\eta)) \in \mathcal{TS}_{\mathcal{K}}^{(-\partial\varphi,-\partial\psi)}(\tau, a(\tau)+\xi, b(\tau)-\eta)$ or, equivalently

$$\liminf_{h\downarrow 0} \frac{1}{h}\mathrm{dist}(z(\tau+h,\tau,a(\tau)+\xi,b(\tau)-\eta,E);\mathcal{K}(\tau+h)) = 0 \tag{5.4}$$

for each $\tau \in I \setminus N$ and each $(\xi,\eta) \in (C\times C)\cap(D(\partial\varphi)\times D(\partial\psi))$, where

$$\begin{aligned} E &= (F(\tau,a(\tau)+\xi,b(\tau)-\eta), G(\tau,a(\tau)+\xi,b(\tau)-\eta)) \\ &= \{(x,y);\ x = F(\tau,a(\tau)+\xi,b(\tau)-\eta),\ y \in G(\tau,a(\tau)+\xi,b(\tau)-\eta)\}, \\ & z(\tau+h,\tau,a(\tau)+\xi,b(\tau)-\eta,E) \\ &= \{z(\tau+h,\tau,a(\tau)+\xi,b(\tau)-\eta,(x,y));\ \ (x,y)\in E\}, \end{aligned}$$

and $z(\cdot,\tau,a(\tau)+\xi,b(\tau)-\eta,(x,y))$ is the C^0-solution of the problem

$$\begin{cases} z'(t) \in (-\partial\varphi,-\partial\psi)z(t)+(x,y) \\ z(\tau) = (a(\tau)+\xi, b(\tau)-\eta). \end{cases} \tag{5.5}$$

Let us denote by $z(\cdot,\tau,a(\tau)+\xi,b(\tau)-\eta,(x,y)) := (u(\cdot),v(\cdot))$. Then $u(\cdot) := u(\cdot,\tau,a(\tau)+\xi,x)$, where $x = F(\tau,a(\tau)+\xi,b(\tau)-\eta)$, is a C^0-solution of

$$\begin{cases} u'(t) \in -\partial\varphi(u(t))+x \\ u(\tau) = a(\tau)+\xi, \end{cases} \tag{5.6}$$

and $v(\cdot) := v(\cdot,\tau,b(\tau)-\eta,y)$, where $y \in G(\tau,a(\tau)+\xi,b(\tau)-\eta)$, is a C^0-solution of

$$\begin{cases} v'(t) \in -\partial\psi(v(t))+y \\ v(\tau) = b(\tau)-\eta. \end{cases} \tag{5.7}$$

Let us denote by O_j, $j=1,2,\dots$ some functions defined on $(0,1)$ with values in H, with $\lim_{h\downarrow 0} O_j(h) = 0$.

Let $h \in (0,1)$, $\xi \in C\cap D(\partial\varphi)$, $\eta \in C\cap D(\partial\psi)$ such that $a(\tau)+\xi \in D(\partial\varphi)$ and $b(\tau)-\eta \in D(\partial\psi)$. Let us denote by $x = F(\tau,a(\tau)+\xi,b(\tau)-\eta)$ and let $y \in G(\tau,a(\tau)+\xi,b(\tau)-\eta)$ be arbitrary but fixed. We have

$$\begin{cases} a(\tau+h) = a(\tau)+ha'(\tau)+hO_1(h) \\ b(\tau+h) = b(\tau)+hb'(\tau)+hO_1(h), \end{cases} \tag{5.8}$$

$$\begin{cases} u(\tau+h,\tau,a(\tau)+\xi,x) = a(\tau)+\xi-h\partial\varphi(a(\tau)+\xi)+hx+hO_2(h) \\ v(\tau+h,\tau,b(\tau)-\eta,y) = b(\tau)-\eta-h\partial\psi(b(\tau)-\eta)+hy+hO_2(h), \end{cases} \tag{5.9}$$

and

$$\begin{cases} S_\varphi(h)\xi = \xi - h\partial\varphi(\xi) + hO_3(h) \\ S_\psi(h)\eta = \eta - h\partial\psi(\eta) + hO_3(h), \end{cases} \tag{5.10}$$

where $\{S_\varphi(t) : \overline{D(\partial\varphi)} \to \overline{D(\partial\varphi)},\ t \geq 0\}$ and $\{S_\psi(t) : \overline{D(\partial\psi)} \to \overline{D(\partial\psi)},\ t \geq 0\}$ are the nonlinear semigroups generated by $-\partial\varphi$ and $-\partial\psi$ respectively.

Since $\xi \in C \cap D(\partial\varphi)$, $\eta \in C \cap D(\partial\psi)$, $S_\varphi(h)C \subseteq C$, $S_\psi(h)C \subseteq C$ and C is a convex cone, it follows that

$$C \subseteq -S_\varphi(h)\xi + C,\quad C \subseteq -S_\psi(h)\eta + C \quad \text{and} \quad h\,C = C. \tag{5.11}$$

In view of (5.8), (5.9), (5.10) and (5.11) we get

$$\begin{aligned}
&\operatorname{dist}(u(\tau+h,\tau,a(\tau)+\xi,x); a(\tau+h)+C) \\
&= \operatorname{dist}(a(\tau)+\xi-h\partial\varphi(a(\tau)+\xi)+hx+hO_2(h); a(\tau)+ha'(\tau)+hO_1(h)+C) \\
&= \operatorname{dist}(\xi - S_\varphi(h)\xi - h\partial\varphi(a(\tau)+\xi) - ha'(\tau) + hx + hO_4(h); -S_\varphi(h)\xi + C) \\
&= \operatorname{dist}(h\partial\varphi(\xi) - h\partial\varphi(a(\tau)+\xi) - ha'(\tau) + hx + hO_5(h); -S_\varphi(h)\xi + C) \\
&\leq \operatorname{dist}(h\partial\varphi(\xi) - h\partial\varphi(a(\tau)+\xi) - ha'(\tau) + hx + hO_5(h); C) \\
&\leq \operatorname{dist}(h\partial\varphi(\xi) - h\partial\varphi(a(\tau)+\xi) - ha'(\tau) + hx + hO_5(h); h\,C).
\end{aligned}$$

So, we have

$$\begin{aligned}
&\operatorname{dist}(u(\tau+h,\tau,a(\tau)+\xi,x); a(\tau+h)+C) \\
&\quad \leq h\operatorname{dist}(\partial\varphi(\xi) - \partial\varphi(a(\tau)+\xi) - a'(\tau) + x; C) + h\|O_5(h)\|.
\end{aligned} \tag{5.12}$$

Similarly we deduce

$$\begin{aligned}
&\operatorname{dist}(v(\tau+h,\tau,b(\tau)-\eta,y); b(\tau+h)-C) \\
&= \operatorname{dist}(b(\tau)-\eta-h\partial\psi(b(\tau)-\eta)+hy+hO_2(h); b(\tau)+hb'(\tau)+hO_1(h)-C) \\
&= \operatorname{dist}(\eta - S_\psi(h)\eta + h\partial\psi(b(\tau)-\eta) + hb'(\tau) - hy + hO_6(h); -S_\psi(h)\eta + C) \\
&= \operatorname{dist}(h\partial\psi(\eta) + h\partial\psi(b(\tau)-\eta) + hb'(\tau) - hy + hO_7(h); -S_\psi(h)\eta + C) \\
&\leq \operatorname{dist}(h\partial\psi(\eta) + h\partial\psi(b(\tau)-\eta) + hb'(\tau) - hy + hO_7(h); C) \\
&\leq \operatorname{dist}(h\partial\psi(\eta) + h\partial\psi(b(\tau)-\eta) + hb'(\tau) - hy + hO_7(h); h\,C).
\end{aligned}$$

So, we have

$$
\begin{aligned}
(5.13)\qquad & \operatorname{dist}(v(\tau+h,\tau,b(\tau)-\eta,y);b(\tau+h)-C)\\
&\leq h\operatorname{dist}(\partial\psi(\eta)+\partial\psi(b(\tau)-\eta)+b'(\tau)-y;C)+h\|O_7(h)\|.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\operatorname{dist}(z(\tau+h,\tau,a(\tau)+\xi,b(\tau)-\eta,E);\mathcal{K}(\tau+h))\\
(5.14)\qquad &\leq \operatorname{dist}(u(\tau+h,\tau,a(\tau)+\xi,x);a(\tau+h)+C)\\
&+\operatorname{dist}(v(\tau+h,\tau,b(\tau)-\eta,y);b(\tau+h)-C).
\end{aligned}
$$

Dividing by h and passing to the limit for $h\downarrow 0$ in (5.14) and using (5.12) and (5.13), we get

$$
\begin{aligned}
&\liminf_{h\downarrow 0}\frac{1}{h}\operatorname{dist}(z(\tau+h,\tau,a(\tau)+\xi,b(\tau)-\eta,E);\mathcal{K}(\tau+h))\\
(5.15)\qquad &\leq \operatorname{dist}(\partial\varphi(\xi)-\partial\varphi(a(\tau)+\xi)-a'(\tau)+x;C)\\
&+\operatorname{dist}(\partial\psi(\eta)+\partial\psi(b(\tau)-\eta)+b'(\tau)-y;C)
\end{aligned}
$$

for each $y\in G(\tau,a(\tau)+\xi,b(\tau)-\eta)$, where $x=F(\tau,a(\tau)+\xi,b(\tau)-\eta)$.

If $(\xi,\eta)\in\partial(C\times C)\cap(D(\partial\varphi)\times D(\partial\psi))$, we have

$$
\begin{aligned}
&\operatorname{dist}(\partial\psi(\eta)+\partial\psi(b(\tau)-\eta)+b'(\tau)-G(\tau,a(\tau)+\xi,b(\tau)-\eta);C)\\
&=\inf\{\operatorname{dist}(\partial\psi(\eta)+\partial\psi(b(\tau)-\eta)+b'(\tau)-y;C);\\
&\qquad y\in G(\tau,a(\tau)+\xi,b(\tau)-\eta)\}
\end{aligned}
$$

and, by (5.3) and (5.15) we conclude that (5.4) holds true.

If $(\xi,\eta)\in((C\times C)\setminus\partial(C\times C))\cap(D(\partial\varphi)\times D(\partial\psi))$ then, for $h>0$ small enough, we have $\operatorname{dist}(z(\tau+h,\tau,a(\tau)+\xi,b(\tau)-\eta,E);\mathcal{K}(\tau+h))=0$. So (5.4) holds true for each $(\xi,\eta)\in(C\times C)\cap(D(\partial\varphi)\times D(\partial\psi))$ and this completes the proof. $\square$

Remark 5.2. Since F is single-valued, G is a convex and weakly compact valued multi-function and C is convex and closed, (5.3) is equivalent to: for each $\tau\in I\setminus N$ and each $(\xi,\eta)\in\partial(C\times C)\cap(D(\partial\varphi)\times D(\partial\psi))$, there exists $y\in G(\tau,a(\tau)+\xi,b(\tau)-\eta)$ such that

$$
(5.16)\qquad \begin{cases}\partial\varphi(\xi)-\partial\varphi(a(\tau)+\xi)-a'(\tau)+x\in C\\ \partial\psi(\eta)+\partial\psi(b(\tau)-\eta)+b'(\tau)-y\in C,\end{cases}
$$

where $x=F(\tau,a(\tau)+\xi,b(\tau)-\eta)$.

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