CYCLIC GROUPS OBTAINED AS QUOTIENT HYPERGROUPS

BY

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Abstract. We introduce a strongly regular equivalence relation $\rho_{\mathcal{A}}^*$ on the hypergroup H , such that in a particular case the quotient $\frac{H}{\rho_{\mathcal{A}}^*}$ is a cyclic group. Then by using the notion of $\rho_{\mathcal{A}}^*$ -parts, we investigate the transitivity condition of $\rho_{\mathcal{A}}$. Finally, a characterization of the derived hypergroup $D_c(H)$ has been considered.

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1. Introduction

Hyperstructure theory was born in 1934 when MARTY [16] introduced hypergroups as a generalization of groups at the 8th congress of Scandinavian Mathematicians. After that, he proved the utility of hypergroups in solving some problems of groups, algebraic functions and rational fractions. Complete enough surveys of the algebraic hyperstructure theory are the books by CORSINI [2], CORSINI and LEOREANU [3], VOUGIOUKLIS [19]. Some relations in groupoids and hyperstructures which have been considered since the beginning are $\mathcal{A}$, $\mathcal{A}_n$, β , β_n , γ and γ_n , see CORSINI [1, 4], DAVVAZ and KARIMIAN [5, 6, 12], FRENI [8, 9], KOSKAS [10, 11], LEOREANU [13, 14], ROMEO [17], VOUGIOUKLIS [18], ZHAN [20] and others. Using the relation β , complete parts are introduced by KOSKAS [11] and studied by CORSINI, MIGLIORATO and SUREAU [2, 3, 15].

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We present now some basic definitions and results about hypergroups (see [3]).

Let H be a non-empty set and $P^*(H)$ be the set of all non-empty subsets of H . Let $\star$ be a hyperoperation or a join operation on H , that is, $\star$ is a function from $H \times H$ into $P^*(H)$. If $(a, b) \in H \times H$, then its image under $\star$ in $P^*(H)$ is denoted by $a \star b$ or ab . The join operation is extended to subsets of H in a natural way, that is $A \star B = \bigcup \{ab \mid a \in A, b \in B\}$. The notation aA is used for $\{a\}A$ and Aa for $A\{a\}$. Generally, the singleton $\{a\}$ is identified with its member a . The structure $(H, \star)$ is called a *semi-hypergroup* if $a(bc) = (ab)c$ for all $a, b, c \in H$ and is called a *hypergroup* if it is semi-hypergroup and $aH = Ha = H$ for all $a \in H$.

Let H be a hypergroup and K be a non-empty subset that is closed under the hyperoperation in H . If K is itself a hypergroup under the hyperoperation in H , then K is said to be a *subhypergroup* of H .

Suppose $(H, \star)$ and $(H', \star')$ are two semi-hypergroups. A function $f : H \rightarrow H'$ is called a *homomorphism* if $f(a \star b) \subseteq f(a) \star' f(b)$ for all a and b in H . We say f is a *good homomorphism* if for all a and b in H , $f(a \star b) = f(a) \star' f(b)$.

If $(H, \star)$ is a hypergroup and $R \subseteq H \times H$ is an equivalence relation, we set

$$A \bar{R} B \Leftrightarrow a R b, \quad \forall a \in A, \forall b \in B,$$

for all pairs (A, B) of non-empty subsets of H . The relation R is said to be *left (right) strongly regular* if $x R y \Rightarrow a \star x \bar{R} a \star y$ (resp. $x R y \Rightarrow x \star a \bar{R} y \star a$), for all $(x, y, a) \in H^3$. Moreover, R is called *strongly regular* if it is both left and right strongly regular.

If $(H, \star)$ is a semi-hypergroup (hypergroup) and R is strongly regular, then the quotient $\frac{H}{R}$ is a semigroup (group respectively) under the operation:

$$R(x) \otimes R(y) = R(z), \quad \forall z \in x \star y.$$

For all $n > 1$ define the relation β_n on a semi-hypergroup H , as follows:

$$a \beta_n b \Leftrightarrow \exists (x_1, \dots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^n x_i$$

and $\beta = \bigcup_{i=1}^n \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation H . If β^* is the transitive closure of β then β^* is an equivalence relation (see [2]).

If H is a hypergroup and we consider the canonical projection $\pi : H \rightarrow H/\beta^*$ then the *heart* of H is the inverse image through π of the identity of the group H/β^* and it is denoted by ω_H . Hence $\omega_H = \pi^{-1}(e_{H/\beta^*})$.

We call the kernel of a homomorphism f , the set $K(f) = f^{-1}(\omega_{H'})$, where $\omega_{H'}$ is the heart of H' .

The following result was shown by FRENİ [7]; another proof can be found in [2].

Theorem 1.1. *If H is hypergroup then $\beta = \beta^*$.*

A non-empty subset A of H is called a *complete part* of H if for all $n \geq 2$ and for all $(x_1, x_2, \dots, x_n) \in H^n$ we have the following implication:

$$\prod_{i=1}^n x_i \cap A \neq \emptyset \Rightarrow \prod_{i=1}^n x_i \subseteq A.$$

A subhypergroup K of a hypergroup $(H, \cdot)$ is called

- *closed* if from $K \cap aK \neq \emptyset$ or $K \cap Ka \neq \emptyset$, where $a \in H$, it follows that $a \in K$;
- *invertible* if from $a \in bK$ it follows that $b \in aK$ and from $a \in Kb$ it follows that $b \in Ka$, where $a, b \in H$;
- *normal* if for all $a \in H$, $aK = Ka$.

Remark 1.2. The kernel of every homomorphism is a complete part and a normal subhypergroup.

Proposition 1.3. *Let K be a subhypergroup of a hypergroup H .*

- (i) *If K is a complete part, then it is invertible;*
- (ii) *If K is a invertible, then it is closed.*

In [8], FRENİ introduced the relation $\gamma = \bigcup_{n \geq 1} \gamma_n$, where γ_1 is the diagonal relation and for every integer $n > 1$, γ_n is the relation defined as follows:

$$x \gamma_n y \Leftrightarrow \exists (z_1, \dots, z_n) \in H^n, \exists \sigma \in S_n, : x \in \prod_{i=1}^n z_i, y \in \prod_{i=1}^n z_{\sigma(i)},$$

where S_n is the symmetric group of order n .

In [5], DAVVAZ and KARIMIAN introduced the concept of γ -parts and studied some properties of $C_\gamma(A)$, where $C_\gamma(A)$ is the intersection of all γ -parts of a semi-hypergroup H . The transitive closure γ^* of γ is the smallest strongly regular equivalence such that the quotient H/γ^* is commutative semigroup. Moreover, if H is hypergroup then γ is transitive (see [8]).

Throughout this paper, H is a hypergroup and $\mathcal{C} = \bigcup_{a \in \mathcal{A}} H_a$, where $\emptyset \neq \mathcal{A} \subseteq H$, $H_a = \bigcup a^n$, $n \in \mathbb{N}$ and for all $a, b \in \mathcal{A}$ if $a \neq b$, then $H_a \cap H_b = \emptyset$.

2. The relation $\rho_{\mathcal{A}}$

In this section we introduce the relation $\rho_{\mathcal{A}}$ on a hypergroup H , which we use in order to obtain a cyclic group as a quotient structure of H . For all $n \geq 1$ define $\mathfrak{R}_n^{\mathcal{A}}$ as follows $\mathfrak{R}_n^{\mathcal{A}} \stackrel{\text{def}}{=} \mathfrak{S}_n^{\mathcal{A}} \cup \wp_n^{\mathcal{A}} \cup \mathcal{J}_n^{\mathcal{A}}$, where

$$\begin{aligned} \mathfrak{S}_n^{\mathcal{A}} &\stackrel{\text{def}}{=} \left\{ \left(\prod_{i=1}^n x_i, \prod_{i=1}^n y_i \right) \mid \exists \sigma \in S_n, \prod_{i=1}^n y_i = \prod_{i=1}^n x_{\sigma(i)} \right\}, \\ \wp_n^{\mathcal{A}} &\stackrel{\text{def}}{=} \left\{ \left(\prod_{i=1}^n x_i, \prod_{i=1}^n y_i \right) \mid \exists a \in \mathcal{A}, \{x_1, \dots, x_n\} \cap H_a = \{y_1, \dots, y_n\} \cap H_a \neq \emptyset, \right. \\ &\quad \left. \exists \sigma_a \in S_n, y_j \in \{y_1, \dots, y_n\} \cap H_a \Rightarrow y_j = x_{\sigma_a(j)} \right\} \end{aligned}$$

and

$$\mathcal{J}_n^{\mathcal{A}} \stackrel{\text{def}}{=} \left\{ \left(\prod_{i=1}^n x_i, \prod_{i=1}^n y_i \right) \mid \{x_1, \dots, x_n\} \cap \mathcal{C} = \{y_1, \dots, y_n\} \cap \mathcal{C} = \emptyset \right\}.$$

Notice that $\mathfrak{R}_1^{\mathcal{A}} \stackrel{\text{def}}{=} \{(\{x\}, \{y\}) \mid \{x, y\} \cap \mathcal{C} = \emptyset \text{ or } x = y\}$ and

$$\mathfrak{R}_2^{\mathcal{A}} \stackrel{\text{def}}{=} \left\{ (xz, yz), (xz, zy), (zx, zy), (zx, yz) \mid (\{x\}, \{y\}) \in \mathfrak{R}_1^{\mathcal{A}}, z \in H \right\}.$$

For every $n \geq 1$, define the relation $\rho_{\mathcal{A}}^n$ on H by $x \rho_{\mathcal{A}}^n y \Leftrightarrow \exists (A, B) \in \mathfrak{R}_n^{\mathcal{A}}, x \in A, y \in B$. Notice that for $n = 1$ we obtain $x \rho_{\mathcal{A}}^1 y \Leftrightarrow (\{x\}, \{y\}) \in \mathfrak{R}_1^{\mathcal{A}}$ or $x = y \in \mathcal{C}$. Obviously, for every $n \geq 1$, the relations $\rho_{\mathcal{A}}^n$ are symmetric, and the relation $\rho_{\mathcal{A}} = \bigcup_{n \geq 1} \rho_{\mathcal{A}}^n$ is reflexive and symmetric. Let $\rho_{\mathcal{A}}^*$ be the transitive closure of $\rho_{\mathcal{A}}$.

Theorem 2.1. *The relation $\rho_{\mathcal{A}}^*$ is a strongly regular relation.*

Proof. Notice that $\rho_{\mathcal{A}}^*$ is an equivalence relation. In order to prove that it is strongly regular, we have to show first that:

$$(2.1) \quad x\rho_{\mathcal{A}}y \Rightarrow xz \overset{=}{\rho_{\mathcal{A}}} yz, \quad zx \overset{=}{\rho_{\mathcal{A}}} zy.$$

for every $z \in H$. Suppose that $x\rho_{\mathcal{A}}y$, thus there exists $n \geq 1$ such that $x\rho_{\mathcal{A}}^n y$. Therefore there exists $(B, C) \in \mathfrak{R}_n^{\mathcal{A}}$ such that $x \in B$ and $y \in C$. We obtain the following three cases:

Case 1. Suppose that $(B, C) \in \mathfrak{S}_n^{\mathcal{A}}$ so we have $B = \prod_{i=1}^n x_i$, $C = \prod_{i=1}^n y_i$ and there exist $\sigma \in S_n$ such that $\prod_{i=1}^n y_i = \prod_{i=1}^n x_{\sigma(i)}$. Set $z = y_{n+1} = x_{n+1}$ and let τ be the permutation of S_{n+1} such that:

$$\tau(i) = \begin{cases} \sigma(i), & \text{if } i \in \{1, 2, \dots, n\}; \\ n+1, & \text{if } i = n+1. \end{cases}$$

So $Cz = \prod_{i=1}^{n+1} x_{\tau(i)}$ and hence $(Bz, Cz) \in \mathfrak{S}_{n+1}^{\mathcal{A}}$.

Case 2. Suppose that $(B, C) \in \wp_n^{\mathcal{A}}$ so we have $B = \prod_{i=1}^n x_i$, $C = \prod_{i=1}^n y_i$ and there exists $a \in \mathcal{A}$ such that $\{x_1, \dots, x_n\} \cap H_a = \{y_1, \dots, y_n\} \cap H_a \neq \emptyset$ and $\sigma_a \in S_n$ such that for all $1 \leq j \leq n$ if $y_j \in H_a$, then $y_j = x_{\sigma_a(j)}$. If $z \notin \mathcal{C}$, then it is easy to see that $(Bz, Cz) \in \wp_{n+1}^{\mathcal{A}}$. If $z \in \mathcal{C}$, then there exists $b \in \mathcal{A}$ such that $z \in H_b$. Set $z = y_{n+1} = x_{n+1}$ and let τ_a be the following permutation of S_{n+1} :

$$\tau_a(i) = \begin{cases} \sigma_a(i), & \text{if } i \in \{1, 2, \dots, n\}; \\ n+1, & \text{if } i = n+1. \end{cases}$$

So for $1 \leq j \leq n+1$ if $y_j \in H_a$, then $y_j = x_{\tau_a(j)}$. Therefore $(Bz, Cz) \in \wp_{n+1}^{\mathcal{A}}$ and hence $(Bz, Cz) \in \mathfrak{R}_{n+1}^{\mathcal{A}}$.

Case 3. Suppose that $(B, C) \in \mathcal{J}_n^{\mathcal{A}}$ so we have $B = \prod_{i=1}^n x_i$, $C = \prod_{i=1}^n y_i$, $\{x_1, \dots, x_n\} \cap \mathcal{C} = \{y_1, \dots, y_n\} \cap \mathcal{C} = \emptyset$. If $z \notin \mathcal{C}$, then $(Bz, Cz) \in \mathcal{J}_{n+1}^{\mathcal{A}}$ and if $z \in \mathcal{C}$, then $(Bz, Cz) \in \wp_{n+1}^{\mathcal{A}}$.

Therefore for all $v \in xz$ and for all $y \in yz$, we have $v \in xz \subseteq Bz$ and $u \in yz \subseteq Cz$, so $v\rho_{\mathcal{A}}^m u$ for some $m \in \mathbb{N}$, because $(Bz, Cz) \in \mathfrak{R}_m^{\mathcal{A}}$ for some $m \in \mathbb{N}$ and hence $v\rho_{\mathcal{A}} u$. Thus $xz \overset{=}{\rho_{\mathcal{A}}} yz$. In the same way we obtain that $x\rho_{\mathcal{A}}y \Rightarrow zx \overset{=}{\rho_{\mathcal{A}}} zy$. Moreover, if $x\rho_{\mathcal{A}}^* y$, then there exists $m \in \mathbb{N}$ and $(u_0 = x, u_1, \dots, u_m = y) \in H^m$ such that $x = u_0\rho_{\mathcal{A}}u_1\rho_{\mathcal{A}}\dots\rho_{\mathcal{A}}u_{m-1}\rho_{\mathcal{A}}u_m = y$, whence, by (2.1), we obtain that

$$xz = u_0z \overset{=}{\rho_{\mathcal{A}}} u_1z \overset{=}{\rho_{\mathcal{A}}} u_2z \overset{=}{\rho_{\mathcal{A}}} \dots \overset{=}{\rho_{\mathcal{A}}} u_{m-1} \overset{=}{\rho_{\mathcal{A}}} u_mz = yz.$$

Finally, for all $v \in xz = u_0z$ and for all $u \in u_mz = yz$, taking $z_1 \in u_1z, z_2 \in u_2z, \dots, z_{m-1} \in u_{m-1}z$, we have $v\rho_{\mathcal{A}}z_1\rho_{\mathcal{A}}z_2\rho_{\mathcal{A}}\dots\rho_{\mathcal{A}}z_{m-1}\rho_{\mathcal{A}}u$, and so $v\rho_{\mathcal{A}}^*u$. Therefore $x\rho_{\mathcal{A}}^*y \Rightarrow xz \overset{=}{\rho_{\mathcal{A}}^*} yz$.

Similarly we can prove that $x\rho_{\mathcal{A}}^*y \Rightarrow zx \overset{=}{\rho_{\mathcal{A}}^*} zy$, hence $\rho_{\mathcal{A}}^*$ is strongly regular. $\square$

Theorem 2.2. *The quotient $\frac{H}{\rho_{\mathcal{A}}^*}$ is an abelian group with the generators $\{\rho_{\mathcal{A}}^*(x_0), \rho_{\mathcal{A}}^*(a) \mid x_0 \in (H - \mathcal{C}), a \in \mathcal{A}\}$.*

Proof. By Theorem 2.1, $\rho_{\mathcal{A}}^*$ is a strongly regular equivalence, so the quotient $\frac{H}{\rho_{\mathcal{A}}^*}$ is a group under the following operation $\rho_{\mathcal{A}}^*(x) \otimes \rho_{\mathcal{A}}^*(y) = \rho_{\mathcal{A}}^*(z), \forall z \in xy$.

Since $\gamma^* \subseteq \rho_{\mathcal{A}}^*$, we conclude that $\frac{H}{\rho_{\mathcal{A}}^*}$ is an abelian group. For all $(x, y) \in (H - \mathcal{C})^2$ since $\{x, y\} \cap \mathcal{C} = \emptyset$, we have $(\{x\}, \{y\}) \in \mathfrak{R}_1^{\mathcal{A}}$ and hence $x\rho_{\mathcal{A}}^*y$ so $\rho_{\mathcal{A}}^*(x) = \rho_{\mathcal{A}}^*(y)$. Now suppose that $\rho_{\mathcal{A}}^*(h)$ is given. If $h \in (H - \mathcal{C})$, then $\rho_{\mathcal{A}}^*(h) = \rho_{\mathcal{A}}^*(x_0)$ and if $h \in \mathcal{C}$, then there exists $a \in \mathcal{A}$ such that $h \in H_a$. Therefore $\rho_{\mathcal{A}}^*(h) = [\rho_{\mathcal{A}}^*(a)]^k$ for some $k \in \mathbb{N}$. $\square$

Now suppose that $\mathcal{A} = \{a\}$, so $\mathcal{C} = H_a$. Put $\eta_a^n \stackrel{def}{=} \rho_{\mathcal{A}}^n$, $\eta_a = \bigcup_{n \geq 1} \eta_a^n$ and $\eta_a^* \stackrel{def}{=} \rho_{\mathcal{A}}^*$. So we have the following corollary:

Corollary 2.3. *The quotient $\frac{H}{\eta_a^*}$ is a cyclic group.*

Proof. By the proof of Theorem 2.2, we conclude that the equivalence classes determined by $\rho_{\mathcal{A}}^*$ of all elements of $(H - \mathcal{C})$ coincide and the equivalence class of every element of $\mathcal{C}$ is a power of $\eta_a^*(a)$.

If $H - \mathcal{C} \neq \emptyset$, and $x_0 \in H - \mathcal{C}$ then there exists $u \in H - \mathcal{C}$ such that $x_0 \in au$ since H is a hypergroup. We obtain $\eta_a^*(x_0) = \eta_a^*(a) \eta_a^*(u) = \eta_a^*(a) \eta_a^*(x_0)$ whence $\eta_a^*(a) = e_{\frac{H}{\eta_a^*}}$ the identity element of the group $\frac{H}{\eta_a^*}$.

Hence $\frac{H}{\eta_a^*} = \langle \eta_a^*(x_0) \rangle$. Moreover, since $a \in x_0v$ for some $v \in H - \mathcal{C}$, we obtain that $\eta_a^*(x_0) = (\eta_a^*(x_0))^{-1}$ which means that the order of $\eta_a^*(x_0)$ is two, so in this case $\frac{H}{\eta_a^*}$ is a cyclic group of order two.

Now if $H - \mathcal{C} = \emptyset$, which means that H is a cyclic hypergroup, then $\frac{H}{\eta_a^*} = \langle \eta_a^*(a) \rangle$.

Therefore $\frac{H}{\eta_a^*}$ is a cyclic group. $\square$

Example 2.4. Let S_3 be the permutation group of order 3, i.e., $S_3 = \{(1), (12), (13), (23), (123), (132)\}$ and $a = (123)$ be a cycle of order 3 in S_3 . Then $\frac{S_3}{\eta_a^*} \cong \mathbb{Z}_2$.

Proof. It is easy to see that $\mathfrak{R}_1^A = \{(x, y) \mid x, y \text{ are odd permutations or } x = y\}$ and $\mathfrak{R}_2^A = \{(z, t) \mid z, t \text{ are odd or } z, t \text{ are even permutations}\}$ and $\mathfrak{R}_n^A = \mathfrak{R}_2^A = \eta_a^* = \gamma^*$, so $\frac{S_3}{\eta_a^*} \cong \mathbb{Z}_2$. $\square$

Theorem 2.5. *The relation η_a^* is the least strongly regular equivalence such that the quotient $\frac{H}{\eta_a^*}$ is a cyclic group and the equivalence classes of all elements of $(H - C)$ are equal.*

Proof. Let R be a strongly regular equivalence such that $\frac{H}{R}$ is a cyclic group and the equivalence classes of R of all elements of $(H - C)$ are equal. Suppose that $\varphi : H \rightarrow \frac{H}{R}$ is the canonical projection. φ is a good homomorphism. We show that $\eta_a^* \subseteq R$.

Let $n \geq 1$ and $x \eta_a^n y$. So there exists $(B, C) \in \mathfrak{R}_n^A$ such that $x \in B$ and $y \in C$. We have three cases:

Case 1. Suppose that $(B, C) \in \mathfrak{S}_n^A$ so we have $B = \prod_{i=1}^n x_i$, $C = \prod_{i=1}^n y_i$, and there exist $\sigma \in S_n$ such that $\prod_{i=1}^n y_i = \prod_{i=1}^n x_{\sigma(i)}$. Since $\varphi(x) = \otimes_{i=1}^n \varphi(x_i)$ and $\varphi(y) = \otimes_{i=1}^n \varphi(y_i) = \otimes_{i=1}^n \varphi(x_{\sigma(i)})$, it follows that $\varphi(x) = \varphi(y)$, because $\frac{H}{R}$ is commutative. Therefore xRy .

Case 2. Suppose that $(B, C) \in \mathfrak{P}_n^A$ so we have $B = \prod_{i=1}^n x_i$, $C = \prod_{i=1}^n y_i$, $\{x_1, \dots, x_n\} \cap H_a = \{y_1, \dots, y_n\} \cap H_a \neq \emptyset$ and there exists $\sigma \in S_n$ such that for all $1 \leq j \leq n$ if $y_j \in H_a$, then $y_j = x_{\sigma(j)}$. Renumber of the elements of the sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ such that $\{y_1, \dots, y_m\} \subseteq C$, where $1 \leq m \leq n$ and $x_k, y_k \notin C$ for all $m+1 \leq k \leq n$. So we have $\varphi(x) = (\otimes_{j=1}^m \varphi(x_j)) \otimes (\otimes_{k=m+1}^n \varphi(x_k))$ and $\varphi(y) = (\otimes_{j=1}^m \varphi(x_{\sigma(j)})) \otimes (\otimes_{k=m+1}^n \varphi(y_k))$. For all $m+1 \leq t, l \leq n$, $\varphi(x_t) = \varphi(y_l)$ and since $\frac{H}{R}$ is a commutative group, we have $\varphi(x) = \varphi(y)$ and hence xRy .

Case 3. Suppose that $(B, C) \in \mathfrak{J}_n^A$ so we have $B = \prod_{i=1}^n x_i$, $C = \prod_{i=1}^n y_i$, $\{x_1, \dots, x_n\} \cap C = \{y_1, \dots, y_n\} \cap C = \emptyset$. So for all $1 \leq i, j \leq n$, $\varphi(x_i) = \varphi(y_j)$, thus $\varphi(x) = \varphi(y)$ and hence xRy .

In all cases we have xRy and hence $x \eta_a^n y \Rightarrow xRy$ whence $x \eta_a^* y \Rightarrow xRy$ by transitivity of R . Therefore $\eta_a^* \subseteq R$. $\square$

3. Transitivity condition of ρ_A

In this section we introduce the concept of ρ_A^* -part of a hypergroup H and we determine necessary and sufficient conditions such that the relation ρ_A to be transitive. For this purpose, we will follow arguments similar to those exploited by FRENİ in [8] in the study of fundamental relation γ in hypergroups.

Definition 3.1. Let M be a non-empty subset of H and $\prod_{i=1}^n x_i \cap M \neq \emptyset$. We say that M is a $\rho_{\mathcal{A}}^*$ -part of H if the following conditions hold:

- (P1) For all $\sigma \in S_n$, $\prod_{i=1}^n x_{\sigma(i)} \subseteq M$;
- (P2) If $\exists a \in \mathcal{A}$, $\{x_1, \dots, x_n\} \cap H_a \neq \emptyset$ implies that for all $\prod_{i=1}^n y_i$ such that $\{y_1, \dots, y_n\} \cap H_a = \{x_1, \dots, x_n\} \cap H_a$ there exists $\sigma_a \in S_n$ for which $\forall y_j \in \{y_1, \dots, y_n\} \cap H_a$, $y_j = x_{\sigma_a(j)}$, then $\prod_{i=1}^n y_i \subseteq M$;
- (P3) If $\{x_1, \dots, x_n\} \cap \mathcal{C} = \emptyset$ then for all $\prod_{i=1}^n y_i$ such that $\{y_1, \dots, y_n\} \cap \mathcal{C} = \emptyset$, we have $\prod_{i=1}^n y_i \subseteq M$.

Proposition 3.2. Let M be a non-empty subset of a hypergroup H . The following conditions are equivalent:

- (i) M is a $\rho_{\mathcal{A}}^*$ -part;
- (ii) $x \in M, x\rho_{\mathcal{A}}y \Rightarrow y \in M$;
- (iii) $x \in M, x\rho_{\mathcal{A}}^*y \Rightarrow y \in M$.

Proof. (i) $\Rightarrow$ (ii) Suppose that the pair $(x, y) \in H^2$ is such that $x \in M$ and $x\rho_{\mathcal{A}}y$. Then there exists $(B, C) \in \mathfrak{R}_n^{\mathcal{A}}$ such that $x \in B$ and $y \in C$. So we have three cases.

Case 1. Let $(B, C) \in \mathfrak{S}_n^{\mathcal{A}}$ so we have $B = \prod_{i=1}^n x_i$, $C = \prod_{i=1}^n y_i$ and there exists $\sigma \in S_n$ such that $\prod_{i=1}^n y_i = \prod_{i=1}^n x_{\sigma(i)}$. Since $x \in \prod_{i=1}^n x_i \cap M$, by (P1) we have $\prod_{i=1}^n x_{\sigma(i)} \subseteq M$ and hence $y \in M$.

Case 2. Let $(B, C) \in \wp_n^{\mathcal{A}}$. So $B = \prod_{i=1}^n x_i$, $C = \prod_{i=1}^n y_i$, there exists $a \in \mathcal{A}$, such that $\{x_1, \dots, x_n\} \cap H_a = \{y_1, \dots, y_n\} \cap H_a \neq \emptyset$ and there exists $\sigma_a \in S_n$ such that for all $1 \leq j \leq n$ if $y_j \in H_a$, then $y_j = x_{\sigma_a(j)}$. Since $x \in \prod_{i=1}^n x_i \cap M$, by (P2) we have $\prod_{i=1}^n y_i \subseteq M$ and hence $y \in M$.

Case 3. Suppose that $(B, C) \in \mathcal{I}_n^{\mathcal{A}}$ so we have $B = \prod_{i=1}^n x_i$, $C = \prod_{i=1}^n y_i$, $\{x_1, \dots, x_n\} \cap \mathcal{C} = \{y_1, \dots, y_n\} \cap \mathcal{C} = \emptyset$. Since $x \in \prod_{i=1}^n x_i \cap M$, by (P3) we have $\prod_{i=1}^n y_i \subseteq M$ and hence $y \in M$.

(ii) $\Rightarrow$ (iii) Suppose that $(x, y) \in H^2$ is such that $x \in M$ and $x\rho_{\mathcal{A}}^*y$. So there exist $m \in \mathbb{N}$ and $(w_0 = x, w_1, \dots, w_{m-1}, w_m = y) \in H^m$ such that $x = w_0\rho_{\mathcal{A}}w_1\rho_{\mathcal{A}}w_2\dots\rho_{\mathcal{A}}w_{m-1}\rho_{\mathcal{A}}w_m = y$. Since $x \in M$, applying (ii) m times, we obtain $y \in M$.

(iii) $\Rightarrow$ (i) Let $x \in \prod_{i=1}^n x_i \cap M$. We shall check the conditions (P1), (P2) and (P3).

(P1) Let $\sigma \in S_n$ be given, thus $(\prod_{i=1}^n x_i, \prod_{i=1}^n x_{\sigma(i)}) \in \mathfrak{S}_n^{\mathcal{A}}$. So for all $y \in \prod_{i=1}^n x_{\sigma(i)}$ we have $x\rho_{\mathcal{A}}^*y$ and hence $y \in M$. Therefore $\prod_{i=1}^n x_{\sigma(i)} \subseteq M$.

(P2) Let $\{x_1, \dots, x_n\} \cap H_a \neq \emptyset$ for some $a \in \mathcal{A}$. Now suppose that for all $\prod_{i=1}^n y_i$ if $\{y_1, \dots, y_n\} \cap H_a = \{x_1, \dots, x_n\} \cap H_a$ there exists $\sigma_a \in S_n$ such that

$\forall y_j \in \{y_1, \dots, y_n\} \cap H_a$, $y_j = x_{\sigma_a(j)}$. Thus $(\prod_{i=1}^n x_i, \prod_{i=1}^n y_i) \in \wp_n^A$ and hence for all $y \in \prod_{i=1}^n y_i$ we have $x \wp_{\mathcal{A}}^* y$. Therefore $y \in M$ and so $\prod_{i=1}^n y_i \subseteq M$.

(P3) Let $\{x_1, \dots, x_n\} \cap \mathcal{C} = \emptyset$ and $\prod_{i=1}^n y_i$ be such that $\{y_1, \dots, y_n\} \cap \mathcal{C} = \emptyset$. Thus $(\prod_{i=1}^n x_i, \prod_{i=1}^n y_i) \in \mathcal{J}_n^A$ and hence for all $y \in \prod_{i=1}^n y_i$ we have $x \wp_{\mathcal{A}}^* y$. Therefore $y \in M$ and so $\prod_{i=1}^n y_i \subseteq M$. $\square$

Notation 3.3. Let x be an arbitrary element of a hypergroup H .

For $n \geq 1$, set $P_n(x) = P_{\mathfrak{S}_n}(x) \cup P_{J_n}(x) \cup P_{\wp_n}(x)$, where

$$P_{\mathfrak{S}_n}(x) = \bigcup \left\{ \prod_{i=1}^n x_{\sigma(i)} \mid x \in \prod_{i=1}^n x_i \right\};$$

$$P_{\wp_n}(x) = \bigcup \left\{ \prod_{i=1}^n y_i \mid x \in \prod_{i=1}^n x_i, \{y_1, \dots, y_n\} \cap H_a = \{x_1, \dots, x_n\} \cap H_a, \right.$$

$$\left. \exists \sigma_a \in S_n, y_j \in \{y_1, \dots, y_n\} \cap H_a \Rightarrow y_j = x_{\sigma_a(j)} \right\};$$

$$P_{J_n}(x) = \bigcup \left\{ \prod_{i=1}^n y_i \mid x \in \prod_{i=1}^n x_i, \{y_1, \dots, y_n\} \cap \mathcal{C} = \{x_1, \dots, x_n\} \cap \mathcal{C} = \emptyset \right\}.$$

Denote $P(x) = \bigcup_{n \geq 1} P_n(x)$.

Notice that if $x \notin \mathcal{C}$, then $P_1(x) = H - \mathcal{C}$ and

$$P_2(x) = \bigcup_{z \in H} \{dz, zd \mid d \in (H - \mathcal{C}), \exists b \in (H - \mathcal{C}), x \in bz \cup zb\},$$

while if $x \in a^n \subseteq \mathcal{C}$ for some $n \in \mathbb{N}$, then $P_m(x) = a^n$ for all $m \geq 1$.

We have the following:

Proposition 3.4. For every $x \in H$, $P(x) = \{y \in H \mid x \wp_{\mathcal{A}} y\}$.

Proof. Suppose that $x \in H$ and $y \in P(x)$ are given. Then there exist B and C such that $x \in B$, $y \in C$ and:

$$(i) \ y \in P_{\mathfrak{S}_n}(x) \Rightarrow (B, C) \in \mathfrak{S}_n^A;$$

$$(ii) \ y \in P_{\wp_n}(x) \Rightarrow (B, C) \in \wp_n^A;$$

$$(iii) \ y \in P_{J_n}(x) \Rightarrow (B, C) \in \mathcal{J}_n^A.$$

Therefore $x \wp_{\mathcal{A}} y$ and so $P(x) \subseteq \{y \in H \mid x \wp_{\mathcal{A}} y\}$.

The proof of the reverse of the inclusion is similar to the above. $\square$

Lemma 3.5. Suppose that H is a hypergroup and M is a $\wp_{\mathcal{A}}^*$ -part of H . If $x \in M$, then $P(x) \subseteq M$.

Proof. It follows by Definition 3.1. $\square$

Theorem 3.6. *Let H be a hypergroup. The following conditions are equivalent:*

- (i) $\rho_{\mathcal{A}}$ is transitive;
- (ii) for every $x \in H$, $\rho_{\mathcal{A}}^*(x) = P(x)$;
- (iii) for every $x \in H$, $P(x)$ is a $\rho_{\mathcal{A}}^*$ -part of H .

Proof. (i) $\Rightarrow$ (ii) By Proposition 3.4, for all pair $(x, y) \in H^2$ we have:
 $y \in \rho_{\mathcal{A}}^*(x) \Leftrightarrow x\rho_{\mathcal{A}}^*y \Leftrightarrow x\rho_{\mathcal{A}}y \Leftrightarrow y \in P(x)$.

(ii) $\Rightarrow$ (iii) By Proposition 3.2, if M is a non-empty subset of H , then M is a $\rho_{\mathcal{A}}^*$ -part of H if and only if it is a union of equivalence classes modulo $\rho_{\mathcal{A}}^*$. Particularly, every equivalence class modulo $\rho_{\mathcal{A}}^*$ is a $\rho_{\mathcal{A}}^*$ -part of H .

(iii) $\Rightarrow$ (i) Suppose that $x\rho_{\mathcal{A}}y$ and $y\rho_{\mathcal{A}}z$ so $x \in P(y)$ and $y \in P(z)$. By Lemma 3.5, we have $P(y) \subseteq P(z)$ and hence $x \in P(z)$. By Proposition 3.4 it follows that $x\rho_{\mathcal{A}}z$ and the proof is complete. $\square$

4. A characterization of a new derived hypergroup of a hypergroup

Let H be a hypergroup and A be an invertible subhypergroup of H . For all $x \in H$, there exists $y \in H$ such that $x \in Ay$. Therefore the relation ${}_AE$ on H defined by: $x {}_AE y \Leftrightarrow x \in Ay$ is a regular equivalence and a congruence. Moreover the quotient $\frac{H}{{}_AE} = \{Ax \mid x \in H\}$ is a hypergroup with respect to the hyperoperation $Ax \odot Ay = \{Az \mid z \in xAy\}$. and the canonical projection $\pi : H \rightarrow \frac{H}{{}_AE}$ is an almost strong homomorphism (see [2]). This means that if $\pi(x) = \pi(a) \odot \pi(t)$, then there exist $a', t' \in H$ such that $\pi(a) = \pi(a'), \pi(t) = \pi(t')$ and $x \in a't'$.

In this section we suppose that $\mathcal{A} = \{a\}$ and $\eta_a^*(a) \neq e_{\frac{H}{\eta_a^*}}$.

Proposition 4.1. *For all $x \in (H - \mathcal{C})$, $\eta_a^*(x) = e_{\frac{H}{\eta_a^*}}$.*

Proof. Suppose that $\eta_a^*(x) \neq e_{\frac{H}{\eta_a^*}}$. By Theorem 2.5, $\frac{H}{\eta_a^*} = \langle \eta_a^*(a) \rangle$, so $\eta_a^*(x) = \eta_a^*(t)$ such that $t \in a^n$ for some $n \in \mathbb{N}$. Therefore $x \eta^* t$ and hence there exist the elements $x = x_1, \dots, x_{n-2}, x_{n-1}, x_n = t$ in H such that $x = x_1 \eta_a \dots x_{n-2} \eta_a x_{n-1} \eta_a x_n = t$. Thus $x_{n-1} \eta_a t$ and by Proposition 3.4 it follows that $x_{n-1} \in P(t)$. Since $t \in a^n$, $P(t) = a^n$ and hence $x_{n-1} \in a^n$. Similarly from $x_{n-2} \eta_a x_{n-1}$ we obtain $x_{n-2} \in a^n$. After $(n-1)$ steps, we conclude that $x = x_1 \in a^n \subseteq \mathcal{C}$ and it is a contradiction. $\square$

For all pair $(a, b) \in H^2$ and for all pair (A, B) of non-empty subsets of H , we set:

$$\begin{aligned} a/b &= \{x \in H \mid a \in xb\}, \quad a \setminus b = \{x \in H \mid b \in ax\}, \\ A/B &= \bigcup_{a \in A, b \in B} a/b, \quad A \setminus B = \bigcup_{a \in A, b \in B} a \setminus b. \end{aligned}$$

Denote

$$D_1 = \bigcup_{(x,y) \in H^2} xy/yx, \quad D_2 = \bigcup_{(x,y) \in H^2} xy \setminus yx, \quad D_3 = H - \mathcal{C}$$

and $D_c = D_1 \cup D_2 \cup D_3$.

Define the derived hypergroup $D_c(H)$ as the intersection of all subhypergroups, that are complete parts and contain D_c . Set $D = D_1 \cup D_2$. Since $D \subseteq D_c(H)$, it follows that $\frac{H}{D_c(H)E}$ is a commutative group (see [8]).

Proposition 4.2. *If H is a hypergroup, then $\eta_a^* \subseteq_{D_c(H)} E$.*

Proof. Let $x, y \in (H - \mathcal{C})$, so $x, y \in D_c(H)$. Since H is a hypergroup, there exists $d \in H$ such that $x \in dy$. Moreover $D_c(H)$ is a complete part, so it is closed by Proposition 1.3 and hence $d \in D_c(H)$. Therefore $x \mathrel{D_c(H)E} y$ and so x and y have the same equivalence class under the relation $_{D_c(H)}E$.

Now suppose that $x \in a^n \subseteq \mathcal{C}$. If $n = 1$, then $x \mathrel{D_c(H)E} a$. Let $n > 1$, we have $\pi(x) = \pi(a)^n$, where $\pi : H \rightarrow \frac{H}{D_c(H)E}$ is the canonical projection. Thus $\pi(x) = \pi(a) \odot \pi(t)$, where $t \in a^{n-1}$. Since π is almost strong, there exist $a', t' \in H$ such that $\pi(a) = \pi(a'), \pi(t) = \pi(t')$ and $x \in a't'$, so $a't' \subseteq D_c(H)aD_c(H)t$. Since $D_c(H)$ is the kernel of π , $D_c(H)$ is normal and hence $a't' \subseteq D_c(H)at$. From $at \subseteq a^n$, there exists $s \in a^n$ such that $x \in D_c(H)s$. Therefore $_{D_c(H)}E(x) =_{D_c(H)}E(s)$, for some $s \in a^n$.

Hence $\frac{H}{D_c(H)E}$ is a cyclic group. By Theorem 2.5 it follows that $\eta_a^* \subseteq_{D_c(H)} E$. $\square$

Theorem 4.3. *Let H be a hypergroup. If ϕ_c is the canonical projection $H \xrightarrow{\phi_c} \frac{H}{\eta_a^*}$, then $D_c(H) = \phi_c^{-1}(e_{\frac{H}{\eta_a^*}})$.*

Proof. Let $z \in D_c$, so $z \in D$ or $z \in D_3$. If $z \in D$, then $\eta_a^*(z) = e_{\frac{H}{\eta_a^*}}$ (see [8]). If $z \in D_3$, then $z \in (H - \mathcal{C})$. By Proposition 4.1 we have $\eta_a^*(z) =$

$e_{\frac{H}{\eta_a^*}}$. Therefore $D_c \subseteq \phi_c^{-1}(e_{\frac{H}{\eta_a^*}})$. Since η_a^* is strongly regular, $\phi_c^{-1}(e_{\frac{H}{\eta_a^*}})$ is a complete part subhypergroup of H , whence $D_c(H) \subseteq \phi_c^{-1}(e_{\frac{H}{\eta_a^*}})$.

Conversely, for all $x \in \phi_c^{-1}(e_{\frac{H}{\eta_a^*}})$ we have $\phi_c(x) = e_{\frac{H}{\eta_a^*}} = \phi_c(\varepsilon)$, where $\varepsilon \in D_c(H)$. So we obtain $\eta_a^*(x) = \eta_a^*(\varepsilon)$ and hence $x \eta_a^* \varepsilon$. From Proposition 4.2 it follows that $x \eta_a^*(H) \varepsilon$ and hence $x \in D_c(H)\varepsilon$. Since $D_c(H)$ is complete part, by Proposition 1.3 we have $D_c(H)x = D_c(H)$ whence $x \in D_c(H)$. Thus $\phi_c^{-1}(e_{\frac{H}{\eta_a^*}}) \subseteq D_c(H)$ and the proof is complete. $\square$

Proposition 4.4. *If M is a non-empty subset of a hypergroup H , then*

- (i) $\phi_c^{-1}(\phi_c(M)) = D_c(H)M = MD_c(H)$;
- (ii) M is an η_a^* -part if and only if $\phi_c^{-1}(\phi_c(M)) = M$.

Proof. (i) For every $x \in D_c(H)M$, there exists a pair $(d, m) \in D_c(H) \times M$ such that $x \in dm$ and so $\phi_c(x) = \phi_c(d) \otimes \phi_c(m) = e_{\frac{H}{\eta_a^*}} \otimes \phi_c(m)$ by Theorem 4.3. So $\phi_c(x) = \phi_c(m) \in \phi_c(M)$ and hence $x \in \phi_c^{-1}(\phi_c(M))$. Conversely, for all $x \in \phi_c^{-1}(\phi_c(M))$, an element $m \in M$ exists such that $\phi_c(x) = \phi_c(m)$. Moreover, there exists $d \in H$ such that $x \in dm$, so $\phi_c(x) = \phi_c(m) = \phi_c(d) \otimes \phi_c(m)$, whence $\phi_c(d) = e_{\frac{H}{\eta_a^*}}$, and $d \in \phi_c^{-1}(e_{\frac{H}{\eta_a^*}}) = D_c(H)$. Therefore $x \in D_c(H)M$.

(ii) Let M be an η_a^* -part of H and set $x \in \phi_c^{-1}(\phi_c(M))$. Then there exists $m \in M$ such that $\phi_c(x) = \phi_c(m)$. So $m \eta_a^* x$ and by Proposition 3.2 from $m \in M$ it follows that $x \in M$ and hence $\phi_c^{-1}(\phi_c(M)) \subseteq M$. It is obvious that $M \subseteq \phi_c^{-1}(\phi_c(M))$ and so the proof is complete. $\square$

Theorem 4.5. *If H is a hypergroup, then η_a is transitive.*

Proof. By Theorem 3.6, it is enough to show that for all $x \in H$, $P(x)$ is an η_a^* -part of H . We check that $\phi_c^{-1}(\phi_c(P(x))) = P(x)$ and then we apply Proposition 4.4.

Set $z \in \phi_c^{-1}(\phi_c(P(x)))$, so there exists $k \in P(x)$ such that $\phi_c(z) = \phi_c(k)$ and hence $\eta_a^*(z) = \eta_a^*(k)$. By Proposition 3.4 from $k \in P(x)$ it follows that $x \eta_a k$. Thus $\eta_a^*(k) = \eta_a^*(x)$ and so $\eta_a^*(z) = \eta_a^*(x)$. Therefore there exist $z = x_1, x_2, \dots, x_{n-1}, x_n = x$ such that:

$$(4.1) \quad z = x_1 \eta_a x_2 \dots \eta_a x_{n-1} \eta_a x_n = x.$$

We have two cases:

Case 1. Set $x \in (H - \mathcal{C})$, so $(H - \mathcal{C}) \subseteq P(x)$. From (4.1) we have $z \eta_a x_2$ whence by Proposition 3.4, $x_2 \in P(z)$. If we suppose that $z \in a^n \subseteq \mathcal{C}$, then $P(z) = a^n$ thus $x_2 \in a^n$. Using again (4.1) we would obtain $x \in a^n \subseteq \mathcal{C}$ and it is a contradiction. Therefore $z \in (H - \mathcal{C})$ and hence $z \in P(x)$.

Case 2. Set $x \in a^n \subseteq \mathcal{C}$, so $P(x) = a^n$. By Proposition 3.4 and (4.1), we have $z \in a^n$ and hence $P(z) = a^n$. Therefore $z \in a^n = P(x)$.

Hence $\phi_c^{-1}(\phi_c(P(x))) \subseteq P(x)$ and since the converse inclusion holds obviously, it follows that $P(x) = \phi_c^{-1}(\phi_c(P(x)))$ and thus the proof is complete. $\square$

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