DOUBLY WARPED PRODUCT SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD OF QUASI-CONSTANT CURVATURE

BY

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Abstract. We establish a sharp inequality for a doubly warped product submanifold in a Riemannian manifold of quasi-constant curvature.

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1. Introduction

The notion of a Riemannian manifold (M, g) of *quasi-constant curvature* was introduced by CHEN and YANO [6] and it defines a Riemannian manifold whose curvature tensor R satisfies the condition

$$\begin{aligned} R(X, Y, Z, W) = & a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ (1) \quad & + b[g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W) \\ & + g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)], \end{aligned}$$

where a and b are scalar functions and T is a 1-form given by $g(X, P) = T(X)$, with P a fixed unit vector field. It is easy to see that if R is of the form (1), then the manifold is conformally flat. If $b = 0$, then the manifold is called to be a *space of constant curvature*.

An n -dimensional ($n > 2$) non-flat Riemannian manifold M is said to be a *quasi-Einstein manifold* (see [1]) if its Ricci tensor S satisfies the condition $S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y)$, where α and β are scalar functions such

that $\beta \neq 0$ and A is a non-zero 1-form denoted by $g(X, U) = A(X)$, for any vector field X and where U is a fixed unit vector field. It is easy to see that any Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f_1, f_2 be positive, differentiable functions on M_1 and M_2 , respectively. The *doubly warped product* $M = {}_{f_2}M_1 \times_{f_1} M_2$ (see [10]) is the product manifold $M_1 \times M_2$ equipped with the metric $g = f_2^2 g_1 + f_1^2 g_2$. More precisely, if $\pi_1 : M_1 \times M_2 \rightarrow M_1$ and $\pi_2 : M_1 \times M_2 \rightarrow M_2$ are canonical projections, then the metric g is defined by $g = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2$. The functions f_1 and f_2 are called *warping functions*. If either $f_1 \equiv 1$ or $f_2 \equiv 1$, but not both, then we get a warped product. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we obtain a Riemannian product manifold. If neither f_1 nor f_2 is constant, then we have a non-trivial doubly warped product (see [10]).

For a doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$, let D_1 and D_2 denote the distributions obtained from the vectors tangent to leaves and fibres, respectively.

Assume that $x : {}_{f_2}M_1 \times_{f_1} M_2 \rightarrow N$ is an isometric immersion of a doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ into a Riemannian manifold N . We denote by σ the second fundamental form of x and by $H_i = \frac{1}{n_i} \text{trace} \sigma_i$ the partial mean curvatures, where $\text{trace} \sigma_i$ is the trace of σ restricted to M_i and $n_i = \dim M_i$ ($i = 1, 2$). The immersion x is called *mixed totally geodesic* if $\sigma(X, Z) = 0$, for any vector fields X and Z tangent to D_1 and D_2 , respectively.

In [5], CHEN proved the following result for a warped product submanifold of a Riemannian manifold of constant sectional curvature:

Theorem 1.1. *Let $x : M_1 \times_f M_2 \rightarrow N(c)$ be an isometric immersion of an n -dimensional warped product $M_1 \times_f M_2$ into an m -dimensional Riemannian manifold $N(c)$ of constant sectional curvature c . Then we have*

$$(2) \quad \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 c,$$

where $n_i = \dim M_i$, $n = n_1 + n_2$, Δ is the Laplacian operator of M_1 . The equality case of (2) holds identically if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_i , $i = 1, 2$ are the partial mean curvature vectors.

As a generalization of Chen's result, in [8], ÖZGÜR and MURATHAN considered warped product submanifolds of a Riemannian manifold of quasi-

constant curvature and obtained the following sharp inequality for a warped product isometrically immersed in a Riemannian manifold of quasi-constant curvature:

Theorem 1.2. *Let $x : M_1 \times_f M_2 \rightarrow N^m$ be an isometric immersion of an n -dimensional warped product $M_1 \times_f M_2$ into an m -dimensional Riemannian manifold N^m of quasi-constant curvature. Then we have:*

$$(3) \quad \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 a - \frac{b}{n_2} \sum_{1 \leq i \leq n_1} \sum_{n_1+1 \leq s \leq n} (T(e_i)^2 + T(e_s)^2) + \frac{b}{n_2} (n-1) \|P^\top\|^2,$$

where $n_i = \dim M_i$. The equality sign of (3) holds identically if and only if the immersion x is mixed totally geodesic with $\text{tr} \sigma_1 = \text{tr} \sigma_2$.

Recently, in [7], OLTEANU established the following general inequality for arbitrary isometric immersions of doubly warped product manifolds in arbitrary Riemannian manifolds:

Theorem 1.3. *Let x be an isometric immersion of an n -dimensional doubly warped product $M =_{f_2} M_1 \times_{f_1} M_2$ into an m -dimensional arbitrary Riemannian manifold $\widetilde{M}$. Then we have*

$$(4) \quad n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4} \|H\|^2 + n_1 n_2 \max \widetilde{K},$$

where $n_i = \dim M_i$, $n = n_1 + n_2$, Δ_i is the Laplacian operator of M_i , $i = 1, 2$ and $\max \widetilde{K}(p)$ denotes the maximum of the sectional curvature function of $\widetilde{M}$ restricted to 2-plane sections of the tangent space $T_p M$ of M at each point p in M . Moreover, the equality case of (4) holds if and only if the following two statements hold:

- (1) x is a mixed totally geodesic immersion satisfying $n_1 H_1 = n_2 H_2$, where H_i , $i = 1, 2$ are the partial mean curvature vectors of M_i .
- (2) at each point $p = (p_1, p_2) \in M$, the sectional curvature function $\widetilde{K}$ of $\widetilde{M}$ satisfies $\widetilde{K}(u, v) = \max \widetilde{K}(p)$ for each unit vector $u \in T_{p_1} M_1$ and each unit vector $v \in T_{p_2} M_2$.

Moreover, in [9], the present author and ÖZGÜR studied C -totally real doubly warped product submanifolds in (κ, μ) -contact space forms and non-Sasakian (κ, μ) -contact metric manifolds.

Motivated by the studies of the above mentioned authors, in the present paper, we establish a sharp inequality for a doubly warped product submanifold in a Riemannian manifold of quasi-constant curvature.

The paper is organized as follows: Section 2 is devoted to preliminaries. In section 3, we give a sharp inequality for a doubly warped product submanifold in a Riemannian manifold of quasi-constant curvature.

2. Preliminaries

Let M be an n -dimensional Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(u, v)$ the sectional curvature of M associated with a plane section $\pi \subset T_p M$, where $\{u, v\}$ is an orthonormal basis of π . For any n -dimensional subspace $L \subseteq T_p M$, $2 \leq n \leq m$, its scalar curvature $\tau(L)$ is denoted by $\tau(L) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$, where $\{e_1, \dots, e_n\}$ is any orthonormal basis of L (see [4]). When $L = T_p M$, then the scalar curvature $\tau(L)$ is just the scalar curvature $\tau(p)$ of M at p .

For an n -dimensional submanifold M in a Riemannian m -manifold N , we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and N , respectively. The Gauss and Weingarten formulas are given by $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$ and $\tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp Y$, respectively, for vector fields X, Y tangent to M and ξ normal to M , where σ denotes the second fundamental form, $\nabla^\perp$ the normal connection and A the shape operator of M (see [2]).

Denote by R and $\tilde{R}$ the Riemannian curvature tensors of M and N , respectively. Then the equation of Gauss is given by

$$(5) \quad \begin{aligned} R(X, Y, Z, W) &= \tilde{R}(X, Y, Z, W) + g(\sigma(Y, Z), \sigma(X, W)) \\ &\quad - g(\sigma(X, Z), \sigma(Y, W)), \end{aligned}$$

for all vector fields X, Y, Z, W tangent to M (see [2]).

For any orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space $T_p M$, the mean curvature vector is given by

$$(6) \quad H(p) = \frac{1}{n} \sum_{i=1}^n \sigma(e_i, e_i),$$

where $n = \dim M$.

We set $\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r)$, $i, j \in \{1, \dots, n\}$, $r \in \{n+1, \dots, m\}$, the coefficients of the second fundamental form σ with respect to $e_1, \dots, e_n$,

$e_{n+1}, \dots, e_m$, and

$$(7) \quad \|\sigma\|^2 = \sum_{i,j=1}^n g(\sigma(e_i, e_j), \sigma(e_i, e_j)).$$

Let M be an n -dimensional Riemannian manifold and $\{e_1, \dots, e_n\}$ be an orthonormal basis of M . For a differentiable function f on M , the Laplacian Δf of f is denoted by $\Delta f = \sum_{j=1}^n \{(\nabla_{e_j} e_j) f - e_j e_j f\}$.

We will need the following Chen's Lemma for later use:

Lemma 2.1 ([3]). *Let $n \geq 2$ and $a_1, a_2, \dots, a_n, \lambda$ be real numbers such that*

$$(8) \quad \left(\sum_{i=1}^n a_i \right)^2 = (n-1) \left(\sum_{i=1}^n a_i^2 + \lambda \right).$$

Then $2a_1 a_2 \geq \lambda$, with equality holding if and only if $a_1 + a_2 = a_3 = \dots = a_n$.

3. Doubly warped product submanifolds

In this section, we establish a sharp relationship between the warping functions f_1 and f_2 of a doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ isometrically immersed in a Riemannian manifold of quasi-constant sectional curvature and the squared mean curvature $\|H\|^2$.

Decomposing the vector field P on M uniquely into its tangent and normal components $P^\top$ and $P^\perp$, respectively, we have

$$(9) \quad P = P^\top + P^\perp.$$

Now, let us begin with the following theorem:

Theorem 3.1. *Let $x : {}_{f_2}M_1 \times_{f_1} M_2 \rightarrow N$ be an isometric immersion of an n -dimensional doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ into an m -dimensional Riemannian manifold N of quasi-constant curvature. Then we have:*

$$(10) \quad \begin{aligned} & n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4} \|H\|^2 + n_1 n_2 a \\ & - b \sum_{1 \leq i \leq n_1} \sum_{n_1+1 \leq s \leq n} (T(e_i)^2 + T(e_s)^2) + b(n-1) \|P^\top\|^2, \end{aligned}$$

where $n_i = \dim M_i$, $n = n_1 + n_2$ and Δ_i is the Laplacian of M_i , $i = 1, 2$. The equality case of (10) holds identically if and only if the immersion x is mixed totally geodesic with $\text{tr} \sigma_1 = \text{tr} \sigma_2$.

Proof. Let $M =_{f_2} M_1 \times_{f_1} M_2$ be a doubly warped product submanifold of a Riemannian manifold N of quasi-constant curvature. Since $_{f_2} M_1 \times_{f_1} M_2$ is a doubly warped product, we have

$$(11) \quad \nabla_X Y = \nabla_X^1 Y - \frac{f_2^2}{f_1^2} g_1(X, Y) \nabla^2(\ln f_2),$$

$$(12) \quad \nabla_X Z = Z(\ln f_2)X + X(\ln f_1)Z,$$

for any vector fields X, Y on M_1 and Z on M_2 , where ∇^1 and ∇^2 are Levi-Civita connections of the Riemannian metrics g_1 and g_2 , respectively. Here, $\nabla^2(\ln f_2)$ denotes the gradient of $(\ln f_2)$ with respect to the metric g_2 .

If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$(13) \quad K(X \wedge Z) = \frac{1}{f_1} \{(\nabla_X^1 X)f_1 - X^2 f_1\} + \frac{1}{f_2} \{(\nabla_Z^1 Z)f_2 - Z^2 f_2\}.$$

If we choose a local orthonormal frame $\{e_1, \dots, e_{n_1}, e_{n_1+1}, \dots, e_n\}$ such that $e_1, \dots, e_{n_1}$ are tangent to M_1 , $e_{n_1+1}, \dots, e_n$ are tangent to M_2 and e_{n_1+1} is parallel to the mean curvature vector H , we obtain

$$(14) \quad n_2 \frac{\Delta f_1}{f_1} + n_1 \frac{\Delta f_2}{f_2} = \sum_{1 \leq j \leq n_1 < s \leq n} K(e_j \wedge e_s),$$

for each $s \in \{n_1 + 1, \dots, n\}$.

From the equation of Gauss, for $X = W = e_i$ and $Y = Z = e_j$ such that $i \neq j$, we have $2\tau = n^2 \|H\|^2 - \|\sigma\|^2 + 2b(n-1)\|P^\top\|^2 + n(n-1)a$, where $\|\sigma\|^2$ is the squared norm of the second fundamental form σ of M in N and τ is the scalar curvature of $M =_{f_2} M_1 \times_{f_1} M_2$.

We set

$$(15) \quad \delta = 2\tau - \frac{n^2}{2} \|H\|^2 - 2b(n-1)\|P^\top\|^2 - n(n-1)a.$$

Then, we can write equation (15) as follows

$$(16) \quad n^2 \|H\|^2 = 2(\delta + \|\sigma\|^2).$$

For a chosen local orthonormal frame, the relation (16) takes the following form

$$\left(\sum_{i=1}^n \sigma_{ii}^{n+1} \right)^2 = 2 \left[\delta + \sum_{i=1}^n (\sigma_{ii}^{n+1})^2 + \sum_{i \neq j} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^m \sum_{i,j=1}^n (\sigma_{ij}^r)^2 \right].$$

If we put $a_1 = \sigma_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} \sigma_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^n \sigma_{tt}^{n+1}$, then the above equation turns into

$$\left(\sum_{i=1}^3 a_i \right)^2 = 2 \left[\delta + \sum_{i=1}^3 a_i^2 + \sum_{1 \leq i \neq j \leq n} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^m \sum_{i,j=1}^n (\sigma_{ij}^r)^2 \right. \\ \left. - \sum_{2 \leq j \neq k \leq n_1} \sigma_{jj}^{n+1} \sigma_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} \sigma_{ss}^{n+1} \sigma_{tt}^{n+1} \right].$$

Hence, a_1 , a_2 and a_3 satisfy the Chen's Lemma (for $n = 3$), which implies that $(\sum_{i=1}^3 a_i)^2 = 2(\lambda + \sum_{i=1}^3 a_i^2)$ with

$$\lambda = \delta + \sum_{1 \leq i \neq j \leq n} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^m \sum_{i,j=1}^n (\sigma_{ij}^r)^2 \\ - \sum_{2 \leq j \neq k \leq n_1} \sigma_{jj}^{n+1} \sigma_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} \sigma_{ss}^{n+1} \sigma_{tt}^{n+1}.$$

Then we get $2a_1a_2 \geq \lambda$, with equality holding if and only if $a_1 + a_2 = a_3$. Equivalently, we have

$$(17) \quad \sum_{1 \leq j < k \leq n_1} \sigma_{jj}^{n+1} \sigma_{kk}^{n+1} + \sum_{n_1+1 \leq s < t \leq n} \sigma_{ss}^{n+1} \sigma_{tt}^{n+1} \\ \geq \frac{\delta}{2} + \sum_{1 \leq \alpha < \beta \leq n} (\sigma_{\alpha\beta}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^m \sum_{\alpha,\beta=1}^n (\sigma_{\alpha\beta}^r)^2.$$

Equality holds if and only if

$$(18) \quad \sum_{i=1}^{n_1} \sigma_{ii}^{n+1} = \sum_{t=n_1+1}^n \sigma_{tt}^{n+1}.$$

By making use of the Gauss equation again, we have

$$n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} = \tau - \sum_{1 \leq j < k \leq n_1} K(e_j \wedge e_k) - \sum_{n_1+1 \leq s < t \leq n} K(e_s \wedge e_t)$$

$$\begin{aligned}
&= \tau - \frac{n_1(n_1 - 1)}{2}a - \sum_{r=n+1}^m \sum_{1 \leq j < k \leq n_1} [\sigma_{jj}^r \sigma_{kk}^r - (\sigma_{jk}^r)^2] \\
(19) \quad &- b \sum_{1 \leq j < k \leq n} (T(e_k)^2 + T(e_j)^2) - \frac{n_2(n_2 - 1)}{2}a \\
&- \sum_{r=n+1}^m \sum_{n_1+1 \leq s < t \leq n_1} [\sigma_{ss}^r \sigma_{tt}^r - (\sigma_{st}^r)^2] - b \sum_{n_1+1 \leq s < t \leq n} (T(e_s)^2 + T(e_t)^2).
\end{aligned}$$

In view of the equations (14), (17) and (19) we obtain

$$\begin{aligned}
&n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \leq \tau - \frac{n(n-1)}{2}a + n_1 n_2 a - \frac{\delta}{2} \\
&- \frac{1}{2} \sum_{r=n+2}^m \sum_{\alpha, \beta=1}^n (\sigma_{\alpha\beta}^r)^2 + \sum_{r=n+2}^m \sum_{1 \leq j < k \leq n_1} [(\sigma_{jk}^r)^2 - \sigma_{jj}^r \sigma_{kk}^r] \\
&+ \sum_{r=n+2}^m \sum_{n_1+1 \leq s < t \leq n_1} [(\sigma_{st}^r)^2 - \sigma_{ss}^r \sigma_{tt}^r] - b \sum_{1 \leq j < k \leq n} (T(e_k)^2 + T(e_j)^2) \\
&- b \sum_{n_1+1 \leq s < t \leq n} (T(e_s)^2 + T(e_t)^2) \\
(20) \quad &= \tau - \frac{n(n-1)}{2}a + n_1 n_2 a - \frac{\delta}{2} - \sum_{r=n+1}^m \sum_{j=1}^{n_1} \sum_{t=n_1+1}^n (\sigma_{jt}^r)^2 \\
&- \frac{1}{2} \sum_{r=n+2}^m \left(\sum_{j=1}^{n_1} \sigma_{jj}^r \right)^2 - \frac{1}{2} \sum_{r=n+2}^m \left(\sum_{t=n_1+1}^n \sigma_{tt}^r \right)^2 \\
&- b \sum_{1 \leq j < k \leq n} (T(e_k)^2 + T(e_j)^2) - b \sum_{n_1+1 \leq s < t \leq n} (T(e_s)^2 + T(e_t)^2) \\
&\leq \tau - \frac{n(n-1)}{2}a + n_1 n_2 a - \frac{\delta}{2} \\
&- b \sum_{1 \leq j < k \leq n} (T(e_k)^2 + T(e_j)^2) - b \sum_{n_1+1 \leq s < t \leq n} (T(e_s)^2 + T(e_t)^2) \\
&= \frac{n^2}{4} \|H\|^2 + n_1 n_2 a - b \sum_{1 \leq i \leq n_1} \sum_{n_1+1 \leq s \leq n} (T(e_i)^2 + T(e_s)^2) + b(n-1) \|P^\top\|^2,
\end{aligned}$$

which gives us (10).

By using of (18) and (20), it can be easily seen that the equality sign of (10) holds if and only if

$$(21) \quad \sigma_{jt}^r = 0, \quad n+1 \leq r \leq m$$

and

$$(22) \quad \sum_{i=1}^{n_1} \sigma_{ii}^r = \sum_{t=n_1+1}^n \sigma_{tt}^r = 0,$$

for $1 \leq j \leq n_1$, $n_1 + 1 \leq t \leq n$ and $n+2 \leq r \leq m$. The equation (21) means that the second fundamental form σ of $f_2 M_1 \times_{f_1} M_2$ in N is as follows $\sigma(D_1, D_2) = \{0\}$. Hence, the immersion x is mixed totally geodesic. From (18) and (22), we also get $\sum_{j=1}^{n_1} \sigma(e_j, e_j) = \sum_{s=n_1+1}^n \sigma(e_s, e_s)$, which implies that $tr\sigma_1 = tr\sigma_2$.

Conversely, assume that N is an m -dimensional Riemannian manifold of quasi-constant curvature and the immersion x is mixed totally geodesic with $tr\sigma_1 = tr\sigma_2$. Then, inequalities (17) and (20) reduce to equalities. Hence, we obtain the equality sign of (10) and finish the proof of the theorem. $\square$

As a consequence of Theorem 3.1 we can give the following corollary:

Corollary 3.2. *Let $x : f_2 M_1 \times_{f_1} M_2 \rightarrow N$ be an isometric immersion of an n -dimensional doubly warped product $f_2 M_1 \times_{f_1} M_2$ into an m -dimensional Riemannian manifold N of quasi-constant curvature. If the vector field P is tangent to M , then we have:*

$$(23) \quad n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4} \|H\|^2 + n_1 n_2 a + b(n-1) \\ - b \sum_{1 \leq i \leq n_1} \sum_{n_1+1 \leq s \leq n} (T(e_i)^2 + T(e_s)^2).$$

If the vector field P is normal to M , then we have:

$$(24) \quad n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4} \|H\|^2 + n_1 n_2 a.$$

The equality case of (23) and (24) holds identically if and only if the immersion x is mixed totally geodesic with $tr\sigma_1 = tr\sigma_2$.

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