A STUDY ON THE PARA-HOLOMORPHIC SECTIONAL CURVATURE OF PARA-KÄHLER COTANGENT BUNDLES

BY

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Abstract. We obtain the conditions under which the total space T^*M of the cotangent bundle, endowed with a natural diagonal para-Kähler structure (G, P) , has constant para-holomorphic sectional curvature. Moreover we prove that (T^*M, G, P) cannot have nonzero constant para-holomorphic sectional curvature.

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1. Introduction

Considering natural lifts of the Riemannian metric g from the base manifold M to the total space T^*M of the cotangent bundle, some new interesting geometric structures were studied e.g. in [7]–[11], [19]–[23], [25]. The lifts to T^*M cannot be constructed in the same manner as the lifts to the total space TM of the tangent bundle (see [26]), and this fact explains the fundamental differences between the geometry of the two dual bundles.

Some almost para-Hermitian structures (called also almost hyperbolic Hermitian structures) were constructed on the total spaces of the tangent and cotangent bundles, in papers such as [2]–[6], [11]–[15] and [23].

BEJAN classified the almost para-Hermitian manifolds, obtaining in [2] 36 classes, up to duality. GADEA and MUÑOZ–MASQUÉ gave a classification à la Gray-Hervella, obtaining in [12] 136 classes, up to duality. Maybe the best known class of (almost) para-Hermitian manifolds, characterized by the

closedness of the associated 2-form, is the class of the (almost) para-Kähler manifolds, studied for example in [1].

PRVANOVIC introduced in [24] the notion of para-holomorphic sectional curvature of a para-Kähler manifold, analogously to the holomorphic sectional curvature of a Kähler manifold. She also introduced the para-holomorphic projective curvature tensor, or H-projectiv curvature tensor, and she obtained the explicit expression of the curvature tensor field for spaces with constant para-holomorphic sectional curvature, studied e.g. in [13].

In the present paper, we get the conditions under which the natural diagonal para-Kähler cotangent bundle (T^*M, G, P) , characterized in [11], has constant para-holomorphic sectional curvature. Moreover, we prove that under the obtained conditions, the constant para-holomorphic sectional curvature vanishes.

Throughout this paper, the manifolds, tensor fields and other geometric objects are assumed to be differentiable of class C^∞ (i.e. smooth). The Einstein summation convention is used, the range of the indices h, i, j, k, l, m, r , being always $\{1, \dots, n\}$.

2. Preliminary results

The cotangent bundle of a smooth n -dimensional Riemannian manifold may be endowed with a structure of $2n$ -dimensional smooth manifold, induced by the structure on the base manifold. If (M, g) is a smooth Riemannian manifold of dimension n , we denote its cotangent bundle by $\pi : T^*M \rightarrow M$. Every local chart on M , $(U, \varphi) = (U, x^1, \dots, x^n)$ induces a local chart $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \dots, q^n, p_1, \dots, p_n)$, on T^*M , as follows. For a cotangent vector $p \in \pi^{-1}(U) \subset T^*M$, the first n local coordinates $q^1, \dots, q^n$ are the local coordinates of its base point $x = \pi(p)$ in the local chart (U, φ) (in fact we have $q^i = \pi^*x^i = x^i \circ \pi$, $i = 1, \dots, n$). The last n local coordinates $p_1, \dots, p_n$ of $p \in \pi^{-1}(U)$ are the vector space coordinates of p with respect to the basis $(dx_{\pi(p)}^1, \dots, dx_{\pi(p)}^n)$, defined by the local chart (U, φ) , i.e. $p = p_i dx_{\pi(p)}^i$.

The concept of M -tensor field on the cotangent bundle of a Riemannian manifold, recalled in [7], can be introduced the same manner as the M -tensor fields on the tangent bundle (see [18]).

We recall the splitting of the tangent bundle to T^*M into the vertical distribution $VT^*M = \text{Ker } \pi_*$ and the horizontal one, determined by the

Levi Civita connection $\dot{\nabla}$ of the metric g :

$$(2.1) \quad TT^*M = VT^*M \oplus HT^*M.$$

If $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \dots, q^n, p_1, \dots, p_n)$ is a local chart on T^*M , induced from the local chart $(U, \varphi) = (U, x^1, \dots, x^n)$, the local vector fields $\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}$ on $\pi^{-1}(U)$ define a local frame for VT^*M over $\pi^{-1}(U)$ and the local vector fields $\frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n}$ define a local frame for HT^*M over $\pi^{-1}(U)$, where $\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma_{ih}^0 \frac{\partial}{\partial p_h}$, $\Gamma_{ih}^0 = p_k \Gamma_{ih}^k$, and $\Gamma_{ih}^k(\pi(p))$ are the Christoffel symbols of g .

The set of vector fields $\{\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\}_{i,j=\overline{1,n}}$, denoted by $\{\partial^i, \delta_j\}_{i,j=\overline{1,n}}$, defines a local frame on T^*M , adapted to the direct sum decomposition (2.1).

We consider the energy density defined by g in the cotangent vector p :

$$t = \frac{1}{2} \|p\|^2 = \frac{1}{2} g_{\pi(p)}^{-1}(p, p) = \frac{1}{2} g^{ik}(x) p_i p_k, \quad p \in \pi^{-1}(U).$$

We have $t \in [0, \infty)$ for all $p \in T^*M$.

3. The constant para-holomorphic sectional curvature of the natural diagonal para-Kähler cotangent bundles

In this section we shall determine the conditions under which the para-Kähler manifold (T^*M, G, P) obtained in [11] has constant para-holomorphic sectional curvature.

Now we recall some results obtained in [11], concerning the natural diagonal almost product (locally product) and (almost) para-Hermitian structures on the (total space of the) cotangent bundle, and in particular the natural diagonal (almost) para-Kähler structures on T^*M .

An *almost product structure* J on a differentiable manifold M is a $(1, 1)$ tensor field on M such that $J^2 = I$. The pair (M, J) is called an *almost product manifold*. When the almost product structure J is integrable, it is called a *locally product structure*, and (M, J) is a *locally product manifold*.

The results from [16] and [17] concerning the natural lifts, allowed us to introduce in [11] a $(1, 1)$ tensor field P on T^*M , which is a natural diagonal lift of the metric g from the base manifold to the cotangent bundle. Using the adapted frame $\{\partial^i, \delta_j\}_{i,j=\overline{1,n}}$ to T^*M , we locally define P by the relations:

$$(3.1) \quad P\delta_i = P_{ij}^{(1)} \partial^j, \quad P\partial^i = P_{(2)}^{ij} \delta_j,$$

where the M -tensor fields involved as coefficients have the forms

$$(3.2) \quad P_{ij}^{(1)} = a_1(t)g_{ij} + b_1(t)p_i p_j, \quad P_{(2)}^{ij} = a_2(t)g^{ij} + b_2(t)g^{0i}g^{0j},$$

a_1, b_1, a_2 , and b_2 being smooth functions of the energy density t .

The invariant expression of the defined structure is

$$PX_p^H = a_1(t)(X^\flat)_p^V + b_1(t)p(X)p_p^V, \quad P\alpha_p^V = a_2(t)(\alpha^\sharp)_p^H + b_2(t)g_{\pi(p)}^{-1}(p, \alpha)(p^\sharp)_p^H,$$

in every point p of the induced local card $(\pi^{-1}(U), \Phi)$ on T^*M , $\forall X \in \mathcal{T}_0^1(M)$, $\forall \alpha \in \mathcal{T}_1^0(M)$. The vector $p^\sharp$ is tangent to M in $\pi(p)$, $p^V = p_i \partial^i$ is the Liouville vector field on T^*M , and $(p^\sharp)^H = g^{0i} \delta_i$ is the geodesic spray.

In the sequel we shall recall the conditions under which the above tensor field P defines an almost product (locally product) structure on the total space of the cotangent bundle (see [11]).

The tensor field P , given by (3.1), defines an almost product structure of natural diagonal lift type on T^*M , if and only if its coefficients satisfy

$$(3.3) \quad a_1 = 1/a_2, \quad a_1 + 2tb_1 = 1/(a_2 + 2tb_2).$$

The natural diagonal almost product structure P is a locally product structure on T^*M (i.e. P is integrable) if and only if the base manifold is of constant sectional curvature c , and the coefficients b_1, b_2 are given by:

$$(3.4) \quad b_1 = (a_1 a_1' + c)/(a_1 - 2ta_1'), \quad b_2 = (a_1 a_2' - a_2^2 c)/(a_1 + 2cta_2).$$

A Riemannian manifold (M, g) , endowed with an almost product structure J , satisfying the relation

$$(3.5) \quad g(JX, JY) = -g(X, Y), \quad \forall X, Y \in \mathcal{T}_0^1(M),$$

is called a an *almost para-Hermitian manifold* (or an *almost hyperbolic Hermitian manifold*).

We considered in [11] a natural diagonal lifted metric on the total space T^*M of the cotangent bundle, defined by:

$$(3.6) \quad \begin{cases} G_p(X^H, Y^H) = c_1(t)g_{\pi(p)}(X, Y) + d_1(t)p(X)p(Y), \\ G_p(\alpha^V, \omega^V) = c_2(t)g_{\pi(p)}^{-1}(\alpha, \omega) + d_2(t)g_{\pi(p)}^{-1}(p, \alpha)g_{\pi(p)}^{-1}(p, \omega), \\ G_p(X^H, \alpha^V) = G_p(\alpha^V, X^H) = 0, \end{cases}$$

$\forall X, Y \in \mathcal{T}_0^1(M)$, $\forall \alpha, \omega \in \mathcal{T}_1^0(M)$, $\forall p \in T^*M$, where the coefficients c_1, c_2, d_1, d_2 are smooth functions of the energy density.

The conditions for G to be nondegenerate are assured if

$$c_1 c_2 \neq 0, (c_1 + 2td_1)(c_2 + 2td_2) \neq 0.$$

The metric G is positive definite if and only if $c_1 + 2td_1 > 0, c_2 + 2td_2 > 0$.

With respect the adapted frame on T^*M , (3.6) can be written in the form

$$(3.7) \quad \begin{cases} G(\delta_i, \delta_j) = G_{ij}^{(1)} = c_1(t)g_{ij} + d_1(t)p_i p_j, \\ G(\partial^i, \partial^j) = G_{(2)}^{ij} = c_2(t)g^{ij} + d_2(t)g^{0i}g^{0j}, \\ G(\partial^i, \delta_j) = G(\delta_i, \partial^j) = 0, \end{cases}$$

where c_1, c_2, d_1, d_2 are smooth functions of the density energy on T^*M .

Next we proved that (T^*M, G, P) is a para-Hermitian manifold if and only if the integrability conditions for P and the following relations hold good:

$$(3.8) \quad \frac{c_1}{a_1} = -\frac{c_2}{a_2} = \lambda, \quad \frac{c_1 + 2td_1}{a_1 + 2tb_1} = -\frac{c_2 + 2td_2}{a_2 + 2tb_2} = \lambda + 2t\mu,$$

where the proportionality coefficients $\lambda > 0$ and $\lambda + 2t\mu > 0$ are some functions depending on the energy density t .

Moreover, the para-Hermitian manifold (T^*M, G, P) is para-Kähler (i.e. the associated 2-form $\Omega(X, Y) = G(X, PY)$, $\forall X, Y \in \mathcal{T}_0^1(T^*M)$, is closed) if and only if $\mu = \lambda'$.

Remark 1. The natural diagonal para-Kähler structures on T^*M depend on two essential coefficients, a_1 and λ , which must satisfy the conditions $a_1 > 0, a_1 + 2tb_1 > 0, \lambda > 0, \lambda + 2t\lambda' > 0$, where b_1 is given by (3.4).

Adapting [8, Theorem 3.1] to the context of the natural diagonal lifted metric on T^*M , we obtain the following:

Proposition 3.1. *The Levi-Civita connection ∇ of the natural diagonal lifted metric G on the cotangent bundle of the Riemannian manifold (M, g) has the following expression with respect to the local adapted frame $\{\delta_i, \partial^j\}_{i,j=1,\dots,n}$:*

$$\begin{cases} \nabla_{\partial^i} \partial^j = Q^{ij}_h \partial^h, \quad \nabla_{\delta_i} \partial^j = -\Gamma_{ih}^j \partial^h + P_i^{hj} \delta_h \\ \nabla_{\partial^i} \delta_j = P_j^{ih} \delta_h, \quad \nabla_{\delta_i} \delta_j = \Gamma_{ij}^h \delta_h + S_{ijh} \partial^h, \end{cases}$$

where Γ_{ij}^h are the Christoffel symbols of the Levi-Civita connection $\hat{\nabla}$ on the base manifold, and the M -tensor fields involved as coefficients are given by

$$(3.9) \quad \begin{cases} Q^{ij}{}_h = \frac{1}{2}(\partial^i G_{(2)}^{jk} + \partial^j G_{(2)}^{ik} - \partial^k G_{(2)}^{ij})H_{kh}^{(2)}, \\ P_j{}^{ih} = \frac{1}{2}(\partial^i G_{jk}^{(1)} - R_{ljk}^0 G_{(2)}^{li})H_{(1)}^{kh}, \\ S_{ijh} = \frac{1}{2}(R_{hij}^0 - \partial^k G_{ij}^{(1)} H_{kh}^{(2)}), \end{cases}$$

R_{kij}^h being the components of the curvature tensor field of $\hat{\nabla}$.

The curvature tensor field K of the para-Kähler manifold (T^*M, G, P) has the following components with respect to the adapted frame $\{\delta_i, \partial^j\}_{i,j=1,\dots,n}$:

$$\begin{aligned} K(\delta_i, \delta_j)\delta_k &= QQQQ_{ijk}{}^h \delta_h = (P_i{}^{lh} S_{jkl} - P_j{}^{lh} S_{ikl} - R_{lij}^0 P_k{}^{lh} + R_{kij}^h) \delta_h, \\ K(\delta_i, \delta_j)\partial^k &= QQP P_{ij}{}^k{}_h \partial^h = (P_j{}^{kl} S_{ilh} - P_i{}^{kl} S_{jlh} - R_{lij}^0 Q^k{}_h - R_{hij}^k) \partial^h, \\ K(\partial^i, \partial^j)\delta_k &= PPQQ_{ij}{}^k{}_h \delta_h = (\partial^i P_k{}^{jh} - \partial^j P_k{}^{ih} + P_k{}^{jl} P_l{}^{ih} - P_k{}^{il} P_l{}^{jh}) \delta_h, \\ K(\partial^i, \partial^j)\partial^k &= PPPP^{ijk}{}_h \partial^h = (\partial^i Q^j{}_h{}^k - \partial^j Q^i{}_h{}^k + Q^{jk}{}_l Q^il{}_h - Q^{ik}{}_l Q^{jl}{}_h) \partial^h, \\ K(\partial^i, \delta_j)\delta_k &= PQQP^i{}_{jkh} \partial^h = (\partial^i S_{jkh} + S_{jkl} Q^il{}_h - S_{jlh} P_k{}^{il}) \partial^h, \\ K(\partial^i, \delta_j)\partial^k &= PQPQ^i{}_j{}^{kh} \delta_h = (\partial^i P_j{}^{kh} + P_l{}^{ih} P_j{}^{kl} - Q^{ik}{}_l P_j{}^{lh}) \delta_h, \end{aligned}$$

In [11] we proved that the above components are of the form

$$(3.10) \quad \begin{aligned} &\alpha_1 g_{hj} g^{ik} + \alpha_2 \delta_h^i \delta_j^k + \alpha_3 \delta_h^k \delta_j^i + \alpha_4 g^{ik} p_h p_j \alpha_5 \delta_j^k p_h g^{0i} + \alpha_6 \delta_h^k p_j g^{0i} + \\ &+ \alpha_7 \delta_j^i p_h g^{0k} + \alpha_8 g_{hj} g^{0i} g^{0k} + \alpha_9 \delta_h^i p_j g^{0k} + \alpha_{10} p_h p_j g^{0i} g^{0k}, \end{aligned}$$

where the coefficients $\alpha_1, \dots, \alpha_{10}$ are smooth functions on T^*M , depending on a_1, λ , their first three orders derivatives, the energy density, and the constant sectional curvature c of the base manifold. In a few components of K , some of the coefficients $\alpha_1, \dots, \alpha_{10}$ vanish.

In [24] PRVANOVIC defined the curvature tensor field of a para-Kähler manifold of constant para-holomorphic sectional curvature, k . In the case of the para-Kähler manifold (T^*M, G, P) , we obtain the expression:

$$\begin{aligned} K_0(X, Y)Z &= \frac{k}{4}[G(Y, Z)X - G(X, Z)Y + G(Y, PZ)PX \\ &\quad - G(X, PZ)PY - 2G(X, PY)PZ], \quad \forall X, Y, Z \in \mathcal{T}_0^1(T^*M), \end{aligned}$$

and the components with respect to the adapted frame $\{\delta_i, \partial^j\}_{i,j=1,\dots,n}$ are

$$\begin{aligned}
K_0(\delta_i, \delta_j)\delta_k &= \frac{k}{4}(G_{jk}^{(1)}\delta_i^h - G_{ik}^{(1)}\delta_j^h)\delta_h, \\
K_0(\delta_i, \delta_j)\partial^k &= \frac{k}{4}(G_{jl}^{(1)}P_{ih}^{(1)} - G_{il}^{(1)}P_{jh}^{(1)})P_{(2)}^{kl}\partial^h, \\
K_0(\partial^i, \partial^j)\delta_k &= \frac{k}{4}(G_{(2)}^{jl}P_{(2)}^{ih} - G_{(2)}^{il}P_{(2)}^{jh})P_{kl}^{(1)}\delta_h, \\
K_0(\partial^i, \partial^j)\partial^k &= \frac{k}{4}(G_{(2)}^{jk}\delta_h^i - G_{(2)}^{ik}\delta_h^j)\partial^h, \\
K_0(\partial^i, \delta_j)\delta_k &= \frac{k}{4}[G_{jk}^{(1)}\delta_h^i - G_{(2)}^{il}(P_{kl}^{(1)}P_{jh}^{(1)} - 2P_{jl}^{(1)}P_{kh}^{(1)})]\partial^h, \\
K_0(\partial^i, \delta_j)\partial^k &= \frac{k}{4}(-G_{(2)}^{ik}\delta_j^h + G_{jl}^{(1)}P_{(2)}^{kl}P_{(2)}^{ih} - 2G_{(2)}^{il}P_{jl}^{(1)}P_{(2)}^{kh})\delta_h.
\end{aligned}$$

The para-Kähler manifold (T^*M, G, P) has constant para-holomorphic sectional curvature if and only if the equality $K = K_0$ is satisfied. So, we have to study the conditions of vanishing for all the components of the difference $K - K_0$. To this aim, we shall use the following result, provided in [7].

Lemma 3.2. *If $\alpha_1, \dots, \alpha_{10}$ are smooth functions on T^*M such that the relation (3.10) is satisfied, then $\alpha_1 = \dots = \alpha_{10} = 0$.*

Next we shall prove the characterization theorem for the natural diagonal para-Kähler cotangent bundles of constant para-holomorphic sectional curvature.

Theorem 3.3. *The natural diagonal para-Kähler manifold (T^*M, G, P) , characterized in [11], has constant para-holomorphic sectional curvature k , if and only if the constant sectional curvature c of the base manifold and the essential coefficients mentioned in Remark 1 satisfy the following conditions: $c = 0$ (i.e. the base manifold M is flat) and*

$$(3.11) \quad a_1'' = \frac{4a_1'^2(a_1 - a_1't)}{a_1(a_1 - 2a_1't)}, \quad c_1 = \mathcal{C}, \quad \lambda = \frac{\mathcal{C}}{a_1},$$

where $\mathcal{C}$ is a real constant. Moreover, the para-Kähler manifold (T^*M, G, P) of natural diagonal lift type cannot have nonzero constant para-holomorphic sectional curvature.

Proof. The components of $K - K_0$ with respect to the adapted frame $\{\delta_i, \partial^j\}_{i,j=1,\dots,n}$ are combinations of the form (3.10), hence in the study of their vanishing conditions, we can apply Lemma 3.2. Thus, analyzing the terms of the component $(K - K_0)(\partial^i, \delta_j)\partial^k$, we have that the term containing $g^{hk}\delta_j^i$ vanishes if and only if the derivative of the proportionality factor λ has the expression

$$(3.12) \quad \lambda' = \lambda \frac{a_1 k(a_1 - 2ta'_1)\lambda - a_1^2 a'_1 - 2c(a_1 - ta'_1)}{a_1(a_1^2 + 2ct)}.$$

Imposing the conditions of vanishing for the coefficients of $g^{hi}\delta_j^k$ and $g^{ik}\delta_j^h$, and taking into account the relation (3.12) we obtain the equations $-4a_1c + a_1^2k\lambda - 2ck\lambda t = 0$, $4a_1c - 3a_1^2k\lambda + 6ck\lambda t - 4a_1k^2\lambda^2t = 0$, from which we have that $k = \frac{4a_1c}{\lambda(a_1^2 - 2ct)}$, and then

$$(3.13) \quad -8a_1c(a_1^2 + 2ct)^2/(a_1^2 - 2ct)^2 = 0.$$

Since $a_1a_2 = 1$, a_1 cannot vanish. Hence (3.13) leads to the following cases:

Case I) $a_1^2 = -2ct$ may be discussed only when the base manifold is of negative constant sectional curvature. In this situation we obtain $a_1 = 2ta'_1$, which cannot hold, since $a_1 - 2ta'_1$ appears as a denominator in the integrability conditions (3.4).

Case II) $c = 0$ (i.e. the base manifold is flat) leads to $k = 0$ too. From (3.12) we obtain the differential equation $\lambda'/\lambda = -a'_1/a_1$, with the solution $\lambda = \frac{\mathcal{C}}{a_1}$, where $\mathcal{C}$ is a real constant.

Replacing the above value of λ into (3.8), we obtain that $\frac{c_1}{a_1} = \frac{\mathcal{C}}{a_1}$, so the coefficient c_1 involved in the definition of the metric G is a real constant.

By substituting the value of λ , and taking into account that $c = k = 0$, we obtain that all the components of $K - K_0$ vanish, except $(K - K_0)(\partial^i, \partial^j)\partial^k$, which becomes

$$2 \frac{a_1 a_1''(a_1 - 2a_1' t) + 4a_1'^2(a_1' t - a_1)}{a_1 - 2a_1' t} \left[\frac{1}{(a_1 - 2a_1' t)^2} (g^{jk}g^{0i} - g^{ik}g^{0j})p_h \right. \\ \left. + \frac{1}{a_1^2} (\delta_h^i g^{0j} - \delta_h^j g^{0i})g^{0k} \right] \partial^h,$$

and which vanishes if and only if a_1 satisfy the differential equation in (3.11). $\square$

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