ON STRONGLY SUM DIFFERENCE QUOTIENT LABELING OF ONE-POINT UNION OF GRAPHS, CHAIN AND CORONA GRAPHS

BY

A.D. AKWU

Abstract. In this paper we study strongly sum difference quotient labeling of some graphs that result from three different constructions. The first construction produces one-point union of graphs. The second construction produces chain graph, i.e., a concatenation of graphs. A chain graph will be strongly sum difference quotient graph if any graph in the chain, accepts strongly sum difference quotient labeling. The third construction is the corona product; strongly sum difference quotient labeling of corona graph is obtained.

Mathematics Subject Classification 2010: 05B15, 05C38, 20N05.

Key words: graph labeling, chain graphs, corona graphs, one-point union of graphs, complete graphs.

1. Introduction

Graph theory terminology and notation are taken from [3]. A labeling of a graph G is a one-to-one mapping from the vertex set of G into a set of integers. Labeled graphs serve as useful mathematical models for a broad range of applications such as coding theory, x -ray crystallographic analysis, circuit design, communication network et cetera. By a graph we mean a finite, undirected, connected graph without loops or multiple edges. The concept of strongly sum difference quotient graphs was introduced by ADIGA and SWAMY [1].

Definition 1.1. A graph with n vertices is said to be *strongly sum difference quotient* (SSDQ) graph if its vertices can be labeled with integers $1, 2, \dots, n$ such that the sum difference quotient function f_{sdq} is injective, i.e.,

the values $f_{sdq}(e)$ on the edges are all distinct. We define the sum difference quotient function $f_{sdq} : E(G) \rightarrow Q$ by $f_{sdq}(e) = |\frac{f(v)+f(w)}{f(v)-f(w)}|$, if e join vertex v and vertex w .

Definition 1.2 ([2]). Given two graphs G and H , the *corona* (crown) of G with H , denoted by $G \odot H$, is the graph with

$$V(G \odot H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i),$$

$$E(G \odot H) = E(G) \cup \bigcup_{i \in V(G)} E(H_i) \cup \{(i, u_i) : i \in V(G) \text{ and } u_i \in V(H_i)\}.$$

In other words, a corona graph is obtained from two graphs G and H , taking one copy of G , which is supposed to have order p , and p copies of H , and then joining by an edge the k^{th} vertex of G to every vertex in the k^{th} copy of H . An special kind of corona graph is given by $C_n \odot K_1$, i.e., a cycle with pendant points.

Definition 1.3 ([5]). A graph G in which a vertex is distinguished from other vertices is called a *rooted* graph and the vertex is called a *root* of G .

In Section 2, we give a strongly sum difference quotient labeling of one-point union of graphs while in Section 3 we also give SSDQ labeling of chain graph. In Section 4, we give SSDQ labeling of the corona graph $C_n \odot mK_1$.

2. SSDQ labelings of one-point union of graphs

Let G be a rooted graph, the graph $G^{(n)}$ obtained identifying the root of n -copies of G is called a *one – point union* of n copies of G .

Theorem 2.1. *Let $B_1, B_2, \dots, B_m$ be graphs that have SSDQ labelings. Then there exists a graph G that is a one-point union of $B_1, B_2, \dots, B_m$ that accepts a SSDQ labeling.*

Proof. Denote by f_s the strongly sum difference quotient labeling of graphs $B_s, 1 \leq s \leq m$. Without loss of generality, we may assume that f_s assigns the integers $1, 2, \dots, |B_s|$ to the vertices of graphs B_s . For every $1 \leq s \leq m - 1$, identify the vertex of B_s with label 1 with the vertex of fixed graph B_i with label 1, the graph so obtained is one-point union of graphs, denoted by G . The order of G is $1 - m + \sum_{s=1}^m |B_s|, 1 \leq s \leq m$

Now we shall obtain the *SSDQ* labeling of G . With the above assumption, B_s and B_i share the vertex labeled 1, which will be the common vertex between them. After labeling the common vertex, list the remaining vertices (without common vertex) of the graph G with the labels assigned to them by f_s in ascending order, that is $k_q, k \geq 2$. q is the number of times integer k appears in G . Then define a function ϕ on the vertices of G (without the common vertex) as follows: $\phi(k_q) = m - 1 + p$, $1 \leq p \leq q$, for $k=2$. Now for $k > 2$, $\phi((k)_q) = \phi(k_{p'}) + p$, $1 \leq p \leq q$, where $\phi(k_{p'}) = \phi((k-1)_q)$ for $p=q$. Using the labeling ϕ on the vertices of G , we have a *SSDQ* labelings of G . $\square$

Example. Consider the graphs $B_1 = C_3, B_2 = S_5, B_3 = P_4, B_4 = C_4$ with the following labels in Figure 1.

Figure 1: $B_1 = C_3, B_2 = S_5, B_3 = P_4, B_4 = C_4$

In Figure 2 we show a one-point union graph G obtained by using $B_1, B_2, \dots, B_m$ where B_2 is fixed. The vertex label of G is a strongly sum difference quotient (*SSDQ*) graph.

Figure 2: one – point union graph G

3. SSDQ labelings of chain graphs

Suppose now, that the graphs $B_1, B_2, \dots, B_m$ are blocks and that for every $i \in \{1, 2, \dots, m\}$, B_i and B_{i+1} have a vertex in common, in such a way that the block-cut point graph is a path. The graph G obtained from this concatenation will be called a *chain graph*.

Let G be a chain graph with m blocks ($m \geq 3$). Let u_i be the vertex that connect B_i to B_{i+1} , $1 \leq i \leq m-1$. So $\{u_1, u_2, \dots, u_{m-1}\}$ is the set of cut vertices of G . Let $d_i = d(u_i, u_{i+1})$, namely, the distance between u_i and u_{i+1} , $1 \leq i \leq m-2$. Therefore, each chain graph G has associated the string $d_1, d_2, \dots, d_{m-2}$.

Let G be any chain graph with blocks $B_1, B_2, \dots, B_m$ such that B_i is any complete bipartite graph. Then G has a string $d_1, d_2, \dots, d_{m-2}$ where $d_i \in \{1, 2\}$. In this case, it is possible to obtain *SSDQ* labeling of G .

Lemma 3.1. *The complete bipartite graph $K_{n,n}$ is SSDQ graph if and only if $n \leq 2$.*

Proof. Assume that $K_{n,n}$ is *SSDQ* graph. We now show that $K_{n,n}$ is not *SSDQ* for any $n \geq 3$. On contrary, let us assume that $K_{n,n}$ is *SSDQ* graph for some $n \geq 3$. Let $\{X, Y\}$ be a partition of the graph $K_{n,n}$, where $n \geq 3$. It is clear that $|X| = |Y| = n$, labels assigned to X and Y are distinct and the union of labels assigned to X and Y is $\{1, 2, \dots, 2n\}$.

Let x, y be labels assigned to the vertices of $K_{n,n}$, if $x = 1$ and $y = 2$, where $x \in X$ and $y \in Y$ then

$$(3.1) \quad f_{sdq}(x, y) = f_{sdq}(2y, y)$$

which implies that labels $y, 2y \in Y$. Also

$$(3.2) \quad f_{sdq}(x, y) = f_{sdq}(2x, 2y), \quad x, y \in n.$$

This implies that labels $y, 2y, 2x \in Y$. Lastly,

$$(3.3) \quad f_{sdq}(x_i, y_i) = f_{sdq}(x_j, y_j), \quad 1 \leq i, j \leq n,$$

whenever $y_i = 2x_i$ and $y_j = 2x_j$.

Combining equations (3.1), (3.2) and (3.3), we have the labels assigned to vertices in Y to be $\{2, 4, 3, 6, \dots, 2n\}$. This implies that $|Y| > n$, a contradiction. Therefore $n \leq 2$.

Conversely, let $n \leq 2$ and we need to prove that for $K_{n,n}$ is a *SSDQ* graph. The case $n = 1$ is trivial so we prove for the case $n = 2$.

If $n = 2$, then the number of edges in $K_{n,n}$ is 4. Assume that vertex labeled with label $x, 2x$ are not in the same partite set. Then $f_{sdq}(x_1, y_1) \neq f_{sdq}(x_1, y_2) \neq f_{sdq}(x_2, y_1) \neq f_{sdq}(x_2, y_2)$. Therefore $K_{n,n}$ is a *SSDQ* graph when $n \leq 2$. $\square$

Theorem 3.1. *Let $B_1, B_2, \dots, B_m$ be blocks of complete bipartite graph $K_{2,2}$ with *SSDQ* labeling, then any chain graph G , obtained by the concatenation of these blocks, has a *SSDQ* labeling.*

Proof. Let $d_1 = d_2 = \dots = d_{m-2} = 2$ the string associated to G , where the set of cut-vertices u_i $1 \leq i \leq m-1$ of G are not adjacent.

Let $B_i = K_{x_i, y_i}$ $1 \leq i \leq m$ with bipartite sets X_i and Y_i of order x_i and y_i respectively. In the construction of the desired labeling, the cut-vertices between B_i and B_{i+1} belongs to X_i and X_{i+1} . The vertices of the cut-vertex $u_1, u_2, \dots, u_{m-1}$ between block B_i and B_{i+1} $1 \leq i \leq m-1$ are labeled as $2i+1$. The vertices of X_1 are labeled with the integers $1, u_1$ and the vertices of Y_1 are labeled with the integers $u_1 - 1 = 2, q$ where $q = \sum_{i=1}^m x_i y_i - m + 1$. The vertices of X_2 are labeled with the integers $3 = u_1, 5 = u_2$ and the vertices of Y_2 are labeled with the integers $q - 1, u_1 + 1$.

Since $d_i = 2$, the vertices of X_{i+2} are labeled with the integers $u_i + 1, u_i + 2$ and the vertices of Y_{i+2} are labeled with the integers $u_{i+2} - 1, q - (i + 1)$ such that they are adjacent to the vertices of X_{i+2} . Suppose there exist an edge $x_i y_i$ in G such that the label of vertex x_i is a multiple of y_i or vice-versa (except edge $x_1 y_1$), then replace the label of the vertex y_i with the label of the vertex y_{i-1} and vice-versa.

Also, suppose there exist edges $x_i y_i$ and $x'_i y'_i$ in G such that the label of vertex x'_i is a multiple of the label of the vertex x_i and the label of vertex y'_i is a multiple of the label of the vertex y_i , then replace the label of the vertex y'_i with the label of the vertex y_{i-2} and vice-versa. Note that for every block B_i ($1 \leq i \leq m$), the numbering obtained is a *SSDQ* labeling. This complete the proof. $\square$

4. *SSDQ* labelings of corona graphs

An special kind of corona graph is given by $C_n \odot K_1$, i. e., a cycle with pendants points. In 1979, FRUCHT [4] shown that this corona graph is graceful. Following Frucht's idea, BARRIENTOS [2] obtained that any cycle

with m pendant points attached ($m \geq 1$) is graceful. We obtained the *SSDQ* labelings of the corona $C_n \odot mK_1$, ($m \geq 1$).

Theorem 4.1 ([1]). *For all $n \geq 3$, the cycle C_n is a *SSDQ* graph.*

Theorem 4.2. *The corona $C_n \odot mK_1$ is a *SSDQ* graph for every positive integers $n \geq 3$ and $m \geq 1$.*

Proof. Describe the cycle by the circuit $v_1, v_2, \dots, v_n, v_1$ and denote by $v_{i,j}$ the vertices of degree one adjacent to vertex v_i , ($1 \leq i \leq n$) and ($1 \leq j \leq m$). We distinguish two cases:

Case 1. n is even, the labeling f of $C_n \odot mK_1$ is define below:

$$\begin{aligned} f(v_{2k+1}) &= (m+1)k + 1, \quad 0 \leq k \leq \frac{n}{2} - 1, \\ f(v_{2k}) &= (m+1)(n-k) + 1, \quad 1 \leq k \leq \frac{n}{2}, \end{aligned}$$

and $f(v_{i,j}) = (m+1)n - f(v_i) + (2-j)$, $1 \leq i \leq n$, $1 \leq j \leq m$.

Case 2. n is odd, the labeling f of $C_n \odot mK_1$ is define below:

$$\begin{aligned} f(v_{2k+1}) &= (m+1)k + 1, \quad 0 \leq k \leq \frac{n-1}{2}, \\ f(v_{2k}) &= (m+1)(n-k) + 1, \quad 1 \leq k \leq \frac{n-1}{2}, \end{aligned}$$

and $f(v_{i,j}) = (m+1)n - f(v_i) + (2-j)$, $1 \leq i \leq n$, $1 \leq j \leq m$. Now we need to show that labeling f of the corona graph satisfies the required properties of *SSDQ* graph. The number of vertices in $C_n \odot mK_1$ is $n(m+1)$. From the labeling f above, the vertices of the corona are labeled with integers $1, 2, \dots, n(m+1)$.

Let $f_{sdq}(e_1) = \frac{|f(v_{2k+1}) + f(v_{2k})|}{|f(v_{2k+1}) - f(v_{2k})|} = \frac{|(m+1)n+2|}{|(m+1)(2k-n)|}$, $0 \leq k \leq \frac{n}{2}$, for n odd and $0 \leq k \leq \frac{n-1}{2}$, for n even.

Let $f_{sdq}(e_2) = \frac{|f(v_{2k+1}) + f(v_{i,j})|}{|f(v_{2k+1}) - f(v_{i,j})|} = \frac{|(m+1)(k+n) - f(v_i) + (2-j) + 1|}{|(m+1)(k-n) + f(v_i) - (2-j) + 1|}$, $0 \leq k \leq \frac{n}{2}$, for n odd and $0 \leq k \leq \frac{n-1}{2}$, for n even, $1 \leq i \leq n$, $1 \leq j \leq m$.

Note that $1 \leq f(v_i) \leq n(m+1)$ since $f(v_i)$ is the label assigned to the vertex v_i . Also $1 \leq j \leq m$ and $k \leq \frac{n}{2}$, then we have $f_{sdq}(e_2) \leq \frac{|(m+1)\frac{n}{2} - m + 3|}{|(m+1)\frac{n}{2} + m - 1|}$. Let

$$f_{sdq}(e_3) = \frac{|f(v_{2k}) + f(v_{i,j})|}{|f(v_{2k}) - f(v_{i,j})|} = \frac{|(m+1)(2n-k) - f(v_i) + (2-j) + 1|}{|f(v_i) - (m+1)k - (2-j) + 1|}.$$

Note that $1 \leq f(v_i) \leq n(m+1)$ since $f(v_i)$ is the label assigned to the vertex v_i . Also $1 \leq j \leq m$ and $k \leq \frac{n}{2}$. Then we have $f_{sdq}(e_3) \leq \frac{|(m+1)\frac{n}{2}-m+3|}{|(m+1)\frac{n}{2}+m+1|}$. Now to show that all edges of the corona graph are distinct for all values of k, i and j , i.e., $f_{sdq}(e_1) \neq f_{sdq}(e_2) \neq f_{sdq}(e_3)$.

Let $f_{sdq}(e_1) = f_{sdq}(e_2)$, $\frac{|f(v_{2k+1})+f(v_{2k})|}{|f(v_{2k+1})-f(v_{2k})|} = \frac{|f(v_{2k+1})+f(v_{i,j})|}{|f(v_{2k+1})-f(v_{i,j})|}$. By simplifying the above equation, we have

$$(4.1) \quad \frac{|(m+1)n+2|}{|(m+1)(2k-n)|} \leq \frac{|(m+1)\frac{n}{2}-m+3|}{|(m+1)\frac{n}{2}+m-1|}.$$

By simplifying inequality (4.1), we have

$$(4.2) \quad |(m+1)n+2| \leq 0.$$

Inequality (4.2) is not true since $|x| \geq 0$ for any integer x . Therefore $f_{sdq}(e_1) \neq f_{sdq}(e_2)$.

Also, let $f_{sdq}(e_2) = f_{sdq}(e_3)$ and assume that

$$(4.3) \quad \frac{|(m+1)k-m+3|}{|(m+1)k+m-1|} = \frac{|(m+1)(n-k)-m+3|}{|(m+1)(n-k)+m-1|}.$$

Inequality (4.3) is not true since $n-k$ is a positive integer greater than k . Therefore inequality (4.3) is not true. Hence $f_{sdq}(e_2) \neq f_{sdq}(e_3)$.

Now to show that $f_{sdq}(e_1) \neq f_{sdq}(e_3)$. Let

$$f_{sdq}(e_1) = f_{sdq}(e_3), \quad \frac{|f(v_{2k+1})+f(v_{2k})|}{|f(v_{2k+1})-f(v_{2k})|} = \frac{|f(v_{2k})+f(v_{i,j})|}{|f(v_{2k})-f(v_{i,j})|}.$$

By simplifying the above equation, we have

$$(4.4) \quad \frac{|(m+1)n+2|}{|(m+1)(2k-n)|} \leq \frac{|(m+1)\frac{n}{2}-m+3|}{|(m+1)\frac{n}{2}+m+1|}.$$

By simplifying equation (4.4), we have

$$(4.5) \quad |(m+1)n+2| \leq 0.$$

Inequality (4.5) is not true since $|x| \geq 0$ for any integer x . Therefore $f_{sdq}(e_1) \neq f_{sdq}(e_3)$. Hence $f_{sdq}(e_1) \neq f_{sdq}(e_2) \neq f_{sdq}(e_3)$ and the edges satisfy $f_{sdq} : E(G) \rightarrow Q$. Therefore the labeling f is a *SSDQ* labeling of $C_n \odot mK_1$ and the corona graph is a *SSDQ* graph. $\square$

Example. To avoid an excessive number of figures, we represent the labelings of $C_n \odot mK_1$ in an array of n columns and $m + 1$ rows. The first row containing the labels of vertices of the cycle C_n ; from the second row of any column, are the labels of the end vertices as we can see below.

$$\begin{array}{cccccc}
 & 1 & 13 & 5 & 9 & & 1 & 21 & 5 & 17 & 9 & 13 \\
 C_4 \odot 3K_1 = & 14 & 2 & 10 & 6 & , C_6 \odot 3K_1 = & 22 & 2 & 18 & 6 & 14 & 10 \\
 & 15 & 3 & 11 & 7 & & 23 & 3 & 19 & 7 & 15 & 11 \\
 & 16 & 4 & 12 & 8 & & 24 & 4 & 20 & 8 & 16 & 12 \\
 \\
 & 1 & 17 & 5 & 13 & 9 & & 1 & 25 & 5 & 21 & 9 & 17 & 13 \\
 C_5 \odot 3K_1 = & 18 & 2 & 14 & 6 & 10 & , C_7 \odot 3K_1 = & 26 & 2 & 22 & 6 & 18 & 10 & 14 \\
 & 19 & 3 & 15 & 7 & 11 & & 27 & 3 & 23 & 7 & 19 & 11 & 15 \\
 & 20 & 4 & 16 & 8 & 12 & & 28 & 4 & 24 & 8 & 20 & 12 & 16
 \end{array}$$

REFERENCES

1. ADIGA, C.; SWAMY, C.S. – *Shivakumar On strongly sum difference quotient graphs*, Adv. Stud. Contemp. Math. (Kyungshang), 19 (2009), 31–38.
2. BARRIENTOS, C. – *Graceful labelings of chain and corona graphs*, Bull. Inst. Combin. Appl., 34 (2002), 17–26.
3. CHARTRAND, G.; LESNIAK, L. – *Graphs and Digraphs*, Second edition. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1986.
4. FRUCHT, R. – *Graceful numbering of wheels and related graphs*, Second International Conference on Combinatorial Mathematics (New York, 1978), pp. 219–229, Ann. New York Acad. Sci., 319, New York Acad. Sci., New York, 1979.
5. SHEE, S.C.; HO, Y.S. – *The cordiality of one-point union of n copies of a graph*, Discrete Math., 117 (1993), 225–243.

Received: 10.III.2012

Revised: 15.VI.2012

Revised: 19.IX.2012

Revised: 6.XI.2012

Accepted: 22.XI.2012

Department of Mathematics,
Federal University of Agriculture,
Makurdi,
NIGERIA
abolaopeyemi@yahoo.co.uk