THE RATE OF INCREASE OF MEAN VALUES OF FUNCTIONS IN WEIGHTED HARDY-ORLICZ SPACES

BY

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Abstract. Let $0 \leq q < \infty$. For each f in the weighted Hardy-Orlicz space $H_{\Phi,q}$ we show that $\frac{d\|f_r\|_{\Phi,q}}{dr}$ grows at most like $o(1/1-r)$ as $r \rightarrow 1$.

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1. Introduction and preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ the unit circle, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$ and $dm(\theta) = \frac{1}{2\pi}d\theta$ be the normalized Lebesgue measure on $\partial\mathbb{D}$. For $0 < p < \infty$ and $0 \leq q < \infty$, define

$$\|f_r\|_{p,q} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p (1-r^2)^q d\theta \right)^{1/p}.$$

The weighted Hardy space $H_{p,q}(\mathbb{D}) = H_{p,q}$ is the family of all holomorphic functions f in $\mathbb{D}$ satisfying $\|f\|_{p,q} = \sup_{0 < r < 1} \|f_r\|_{p,q} < \infty$. It is trivial that $H_{p,0}$ is just the classical Hardy space H^p and $H_{p,q}$ is a Banach space for $p \geq 1$. We write $\|f\|_p$ for $\|f\|_{p,0}$. For more about H^p and $H_{p,q}$ see [3], [10] and [11]. Let $ST^2(\mathbb{R})$ denote the class of strongly convex functions $\Phi : [-\infty, \infty) \rightarrow [0, \infty)$ (that is, Φ is non-negative, convex and non-decreasing with $\frac{\Phi(t)}{t} \rightarrow \infty$ as $t \rightarrow \infty$) which satisfy:

- (i) $\Phi(t) = 0$ for all $t < 0$ with $\Phi(0) = \Phi'(0) = 0$;
- (ii) $\Phi''(t)$ exists for all $t > 0$;

(iii) $\Phi(2t) \leq C\Phi(t)$ for some positive constant C and for all $t > 0$.

For example, the function $\Phi : [-\infty, \infty) \rightarrow [0, \infty)$ defined as

$$\Phi(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

is in $ST^2(\mathbb{R})$. For $\Phi \in ST^2(\mathbb{R})$, we define the Hardy-Orlicz space H^Φ by

$$H^\Phi = \left\{ f \in H(\mathbb{D}) : \sup_{0 < r < 1} \int_{\partial\mathbb{D}} \Phi(\log^+ |f(re^{i\theta})|) dm < \infty \right\},$$

where

$$\log^+(x) = \begin{cases} \log x, & \text{if } x \geq 1 \\ 0, & \text{if } x < 1. \end{cases}$$

Although the integral expression above does not define a norm in H^Φ , the distance $d(f, g) = \int_{\partial\mathbb{D}} \Phi(\log^+ |f(re^{i\theta}) - g(re^{i\theta})|) dm$ defines a translation-invariant metric on H^Φ , and turns H^Φ into a complete metric space. Obviously, the inequalities $\log^+ x \leq \log(1 + x) \leq 1 + \log^+ x$, $x \geq 0$ and $2\log^+ x \leq \log(1 + x^2) \leq 1 + 2\log^+ x$, $x \geq 0$ and the fact Φ is non-decreasing convex function imply that $\Phi(\log^+ x) \leq \Phi(\log(1 + x)) \leq \Phi(1 + \log^+ x) \leq \frac{1}{2}\Phi(2) + \frac{1}{2}\Phi(2\log^+ x) \leq \frac{1}{2}\Phi(2) + \frac{1}{2}C\Phi(\log^+ x)$ and $\Phi(\log^+ x) \leq \Phi(2\log^+ x) \leq \Phi(\log(1 + x^2)) \leq \Phi(1 + 2\log^+ x) \leq \frac{1}{2}\Phi(2) + \frac{1}{2}\Phi(4\log^+ x) \leq \frac{1}{2}\Phi(2) + \frac{1}{2}C\Phi(\log^+ x)$. Hence $f \in H^\Phi$ if and only if $\sup_{0 < r < 1} \int_{\partial\mathbb{D}} \Phi(\log(1 + |f(re^{i\theta})|)) dm < \infty$ or if and only if $\sup_{0 < r < 1} \int_{\partial\mathbb{D}} \Phi(\log(1 + |f(re^{i\theta})|^2)) dm < \infty$. For more about Hardy-Orlicz spaces see [6], [7] and [9]. For $0 \leq q < \infty$ and $\Phi \in ST^2(\mathbb{R})$, defined weighted Hardy-Orlicz space $H_{\Phi,q}$ as

$$H_{\Phi,q} = \left\{ f \in H(\mathbb{D}) : \sup_{0 \leq r < 1} \int_{\partial\mathbb{D}} \Phi(\log(1 + |f(re^{i\theta})|^2))(1 - |r|^2)^q dm < \infty \right\}.$$

Abusing notation, we will write

$$\|f\|_{\Phi,q} = \sup_{0 < r < 1} \int_{\partial\mathbb{D}} \Phi(\log(1 + |f(re^{i\theta})|^2))(1 - |r|^2)^q dm < \infty.$$

Note that $H_{\Phi,0}$ reduces to the Hardy-Orlicz space H_Φ .

It is well known that $\|f_r\|_p$ is a non-decreasing function of r . Furthermore, HARDY [4] showed that $\log \|f_r\|_p$ is a convex function of $\log r$. Other

properties of mean values of analytic functions can be found in [1], [2] and [8]. Recently, MARHREGHI [5] showed that $\frac{d\|f_r\|_p}{dr}$ grows at most like $o(1/(1-r))$ as $r \rightarrow 1$. This result was extended to weighted Hardy spaces by XIONG and LIU [10]. In this paper, we generalized this result to weighted Hardy-Orlicz spaces.

Throughout this paper constants are denoted by C , they are positive and not necessarily the same at each occurrence. The $O(1)$ and $o(1)$ are usual Landau symbols.

2. Some lemmas

In this section, we prove some lemmas that are helpful in the proof of main result of this paper. Suppose that $f \neq 0$ is in $H(\mathbb{D})$. Let $W(z) = \Phi(\log(1+|f(z)|^2))(1-|z|^2)^q$ for $0 \leq q < \infty$, and ∇ be the gradient operator.

Lemma 2.1. *Let $0 < r \leq 1$ and $D_r = \{z : |z| < r\}$. For $z_0 \in D_r$ and small $\varepsilon > 0$, let $D(z_0, \varepsilon) = \{z : |z - z_0| \leq \varepsilon\} \subset D_r$ and*

$$I_\varepsilon = \int_{\partial D(z_0, \varepsilon)} \left(\log(r/|z|) \frac{\partial W}{\partial n} - W \frac{\partial \log(r/|z|)}{\partial n} \right) dl.$$

If $z_0 \neq 0$ is a zero of f , then $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = 0$. If $z_0 = 0$, then $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = 2\pi\Phi(\log(1+|f(0)|^2))$.

Proof. Since $W(z) = \Phi(\log(1+|f(z)|^2))(1-|z|^2)^q$, direct computation gives

$$\nabla W(z) = (1-|z|^2)^q \nabla \Phi(\log(1+|f(z)|^2)) + \Phi(\log(1+|f(z)|^2)) \nabla(1-|z|^2)^q$$

and

$$|\nabla \Phi(\log(1+|f(z)|^2))| \leq 2|\Phi'(\log(1+|f(z)|^2))||f(z)||f'(z)| \left| \frac{1}{(1-|f(z)|^2)} \right|.$$

So

$$\begin{aligned} \left| \frac{\partial W(z)}{\partial n} \right| &\leq |\nabla W(z)| \leq (1-|z|^2)^q |\nabla \Phi(\log(1+|f(z)|^2))| \\ &\quad + \Phi(\log(1+|f(z)|^2)) |\nabla(1-|z|^2)^q| \\ &\leq 2(1-|z|^2)^q |\Phi'(\log(1+|f(z)|^2))||f(z)||f'(z)| \left| \frac{1}{(1-|f(z)|^2)} \right| \\ &\quad + 2|\Phi(\log(1+|f(z)|^2))|q|z|(1-|z|^2)^{q-1} \end{aligned}$$

and $|\frac{\partial \log(r/|z|)}{\partial n}| \leq |\nabla \log(r/|z|)| = \frac{1}{|z|}$. Write

$$\begin{aligned} I_\epsilon &= \int_{\partial D(z_0, \epsilon)} \left(\log(r/|z|) \frac{\partial W}{\partial n} - W \frac{\partial \log(r/|z|)}{\partial n} \right) dl \\ &= \int_0^{2\pi} \log(r/|z_0 + \epsilon e^{i\theta}|) \frac{\partial W(z_0 + \epsilon e^{i\theta})}{\partial n} \epsilon d\theta - \int_{\partial D(z_0, \epsilon)} W(z) \frac{\partial \log(r/|z|)}{\partial n} dl \\ &= I_1 - I_2. \end{aligned}$$

For convenience, let $C > 0$ be a constant independent of ϵ . If $z_0 \neq 0$ is a zero of order $k \geq 1$, then

$$I_1 = \int_0^{2\pi} \log(r/|z_0 + \epsilon e^{i\theta}|) \frac{\partial W(z_0 + \epsilon e^{i\theta})}{\partial n} d\theta \leq O(1)\epsilon$$

and $I_2 \leq O(1)\epsilon$. Taking limit $\epsilon \rightarrow 0$ we get $\lim_{\epsilon \rightarrow 0} I_2 = 0$. Hence $\lim_{\epsilon \rightarrow 0} I_\epsilon = 0$.

If $z_0 = 0$ and $f(z) \neq 0$, then $I_1 = \int_0^{2\pi} \log(r/|\epsilon e^{i\theta}|) \frac{\partial W(\epsilon e^{i\theta})}{\partial n} \epsilon d\theta \leq O(1)\epsilon$. Taking limit $\epsilon \rightarrow 0$ we get $\lim_{\epsilon \rightarrow 0} I_1 = 0$.

Similarly, $I_2 = \int_0^{2\pi} W(\epsilon e^{i\theta}) d\theta = (1 - \epsilon^2)^q \int_0^{2\pi} \Phi(\log(1 + |f(\epsilon e^{i\theta})|^2)) d\theta$. Taking limit $\epsilon \rightarrow 0$ we get $\lim_{\epsilon \rightarrow 0} I_2 = 2\pi\Phi(\log(1 + |f(0)|^2))$. Hence $\lim_{\epsilon \rightarrow 0} I_\epsilon = 2\pi\Phi(\log(1 + |f(0)|^2))$. $\square$

Lemma 2.2. *Let $f \in H_{\Phi, q}(\mathbb{D})$, $0 \leq q < \infty$ and $f \neq 0$. Then*

$$\begin{aligned} \lim_{r \rightarrow 1} \int_0^{2\pi} |\Phi(\log(1 + |f(re^{i\theta})|^2))| (1 - |z|^2)^q d\theta \\ = 2\pi\Phi(\log(1 + |f(0)|^2)) + \int_{|z| < 1} \log(1/|z|) G(z) dx dy, \end{aligned}$$

where $G(z) = \nabla^2 |\Phi(\log(1 + |f(z)|^2))| (1 - |z|^2)^q + 2\nabla |\Phi(\log(1 + |f(z)|^2))| \nabla (1 - |z|^2)^q + |\Phi(\log(1 + |f(z)|^2))| \nabla^2 (1 - |z|^2)^q$.

Proof. Since $f \neq 0$ is analytic in $\mathbb{D}$, for any $0 < R < 1$, we can choose $r, R < r < 1$ such that f has finite many zeros in $D_r = \{|z| < r\}$ and no zeros on the circle ∂D_r . Let $\{z_1, z_2, \dots, z_n\}$ be the set consisting of 0 and all zeros of f in D_r . Take $\epsilon > 0$ so small that $\{z : |z - z_k| < \epsilon\} \subset D_r$ for $1 \leq k \leq n$ and all these disks are disjoint.

Denote $\Omega = D_r \setminus \bigcup_{k=1}^n \{z : |z - z_k| < \epsilon\}$. The function $W(z) = \Phi(\log(1 + |f(z)|^2)) (1 - |z|^2)^q$ is twice differentiable in a neighborhood of $\bar{\Omega}$. Then by Green's theorem we have

$$(2.1) \quad \int_{\Omega} \log(r/|z|) \nabla^2 W(z) dx dy = \int_{\partial \Omega} \left(\log(r/|z|) \frac{\partial W}{\partial n} - W \frac{\partial \log(r/|z|)}{\partial n} \right) dl.$$

Direct computation gives $\nabla^2 W(z) = \nabla^2 |\Phi(\log(1 + |f(z)|^2))|(1 - |z|^2)^q + 2\nabla |\Phi(\log(1 + |f(z)|^2))| \nabla (1 - |z|^2)^q + \Phi(\log(1 + |f(z)|^2)) \nabla^2 (1 - |z|^2)^q = G(z)$. The direction of every small circle in $\partial\Omega$ is clockwise. For $\varepsilon \rightarrow 0$, by Lemma 2.1 formula (2.1) becomes $\int_{\partial D_r} (\log(r/|z|) \frac{\partial W}{\partial n} - W \frac{\partial \log(r/|z|)}{\partial n}) dl = 2\pi\Phi(\log(1 + |f(0)|^2)) + \int_{D_r} \log(r/|z|) G(z) dx dy$. The left-side of above formula is equal to $\int_0^{2\pi} W(\varepsilon e^{i\theta}) d\theta - 2\pi\Phi(\log(1 + |f(0)|^2))$. Proceeding to limit as $r \rightarrow 1$, through a sequence (r_n) such that f has no zeros on ∂D_{r_n} , accomplish the proof. $\square$

3. The rate of increase of $\|f_r\|_{\Phi,q}$

Theorem 3.1. *Let $\Phi \in ST^2(\mathbb{R})$ and $0 \leq q < \infty$. Let $f \in H_{\Phi,q}(\mathbb{D})$ and $f \neq 0$. Then $\frac{d\|f_r\|_{\Phi,q}}{dr} = o(1/1 - r)$ as $r \rightarrow 1$.*

Proof. As in the proof of Lemma 2.2, we choose suitable r and ε . Replacing $\log(r/|z|)$ by $\log(1/|z|)$ and using Green's theorem, we have

$$(3.1) \quad \int_{\Omega} \log(r/|z|) \nabla^2 W(z) dx dy = \int_{\partial\Omega} \left(\log(1/|z|) \frac{\partial W}{\partial n} - W \frac{\partial \log(r/|z|)}{\partial n} \right) dl.$$

For $\varepsilon \rightarrow 0$, using Lemma 2.1 for $r = 1$, formula (3.1) turns into

$$\begin{aligned} & 2\pi\Phi(\log(1 + |f(0)|^2)) + \int_{D_r} \log(1/|z|) \nabla^2 G(z) dx dy \\ &= \int_{\partial D_r} \left(\log(1/|z|) \frac{\partial W}{\partial n} - W \frac{\partial \log(1/|z|)}{\partial n} \right) dl \\ &= \int_0^{2\pi} \left(\log(1/r) \frac{\partial W(re^{i\theta})}{\partial r} - \frac{W(re^{i\theta})}{r} \right) r d\theta \\ &= \int_0^{2\pi} W(re^{i\theta}) d\theta - r \log r \int_0^{2\pi} \frac{\partial W(re^{i\theta})}{\partial r} d\theta \\ &= \int_0^{2\pi} |\Phi(\log(1 + |f(re^{i\theta})|^2))|(1 - r^2)^q d\theta - 2\pi r \log r \frac{d\|f_r\|_{\Phi,q}}{dr}. \end{aligned}$$

Now let $r \rightarrow 1$, through a sequence (r_n) such that f has no zeros on ∂D_{r_n} , by Lemma 2.2, we have $\lim_{r \rightarrow 1} \log r \frac{d\|f_r\|_{\Phi,q}}{dr} = 0$. Hence, $\lim_{r \rightarrow 1} \frac{d\|f_r\|_{\Phi,q}}{dr} = o(1/\log r) = o(1/1 - r)$. $\square$

Corollary 3.2. *Let $\Phi \in ST^2(\mathbb{R})$ and $0 \leq q < \infty$. Let $f \in H_\Phi(\mathbb{D})$ and $f \neq 0$. Then $\frac{d\|f_r\|_\Phi}{dr} = o(1/1-r)$ as $r \rightarrow 1$.*

Corollary 3.3. *If $f \in H_{\Phi,q}(\mathbb{D})$ with $0 \leq q < \infty$, then we have*

$$\begin{aligned}
 (3.2) \quad & 2 \lim_{r \rightarrow 1} \int_0^{2\pi} \Phi(\log(1 + |f(re^{i\theta})|^2))(1 - r^2)^q d\theta \\
 &= \int_{\mathbb{D}} (1 - |z|^2)^q \nabla^2(|\Phi(\log(1 + |f(z)|^2))|(1 - |z|^2)^q) dx dy \\
 &\quad + 4 \int_{\mathbb{D}} |\Phi(\log(1 + |f(z)|^2))|(1 - |z|^2)^q dx dy.
 \end{aligned}$$

Proof. Denote

$$J_\varepsilon = \int_{\partial D(z_0, \varepsilon)} \left((1 - |z|^2) \frac{\partial W}{\partial n} - W \frac{\partial(1 - |z|^2)}{\partial n} \right) dl,$$

where z_0 is a zero of f . Similar arguments as in the proof of Lemma 2.1, we have $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = 0$. Choosing suitable r and ε as in the proof of Theorem 3.1, by Green's theorem, we have

$$\begin{aligned}
 & \int_{\Omega} (1 - |z|^2)^q \nabla^2 W(z) - \nabla^2(1 - |z|^2) W(z) dx dy \\
 &= \int_{\partial D\Omega} \left((1 - |z|^2) \frac{\partial W}{\partial n} - W \frac{\partial(1 - |z|^2)}{\partial n} \right) dl.
 \end{aligned}$$

As $\varepsilon \rightarrow 0$ above formula turns into

$$\begin{aligned}
 (3.3) \quad & \int_{D_r} (1 - |z|^2)^q \nabla^2 W(z) + 4W(z) dx dy \\
 &= \int_{\partial D_r} \left((1 - |z|^2) \frac{\partial W}{\partial n} - W \frac{\partial(1 - |z|^2)}{\partial n} \right) dl \\
 &= \int_0^{2\pi} \left((1 - r^2) \frac{\partial W(re^{i\theta})}{\partial n} + 2rW(re^{i\theta}) \right) r d\theta.
 \end{aligned}$$

By Theorem 3.1, we have

$$\lim_{r \rightarrow 1} \int_0^{2\pi} r(1 - r^2) \frac{\partial W(re^{i\theta})}{\partial n} d\theta = 2 \lim_{r \rightarrow 1} \left[(1 - r^2) \frac{\partial}{\partial r} \int_0^{2\pi} W(re^{i\theta}) d\theta \right] = 0.$$

The proof is completed by letting $r \rightarrow 1$, through a sequence (r_n) such that f has no zeros on ∂D_{r_n} in formula (3.3).

Note that

$$\begin{aligned} \nabla^2(|\Phi(\log(1 + |f(z)|^2))|) &= 4[\Phi''(\log(1 + |f(z)|^2))|f(z)|^2 \\ &\quad + \Phi'(\log(1 + |f(z)|^2))]\frac{|f'(z)|^2}{1 + |f(z)|^2}. \end{aligned}$$

Taking $q = 0$ in formula (3.2) we obtain

$$\begin{aligned} &\int_0^{2\pi} (|\Phi(\log(1 + |f(re^{i\theta})|^2))|)d\theta \\ &= 2 \int_{\mathbb{D}} (1 - |z|^2) \left[\Phi''(\log(1 + |f(z)|^2))|f(z)|^2 + \Phi'(\log(1 + |f(z)|^2)) \right] \\ &\quad \frac{|f'(z)|^2}{1 + |f(z)|^2} dxdy + 2 \int_{\mathbb{D}} |\Phi(\log(1 + |f(z)|^2))| dxdy \end{aligned}$$

which is reminiscent of Hardy-Stein identity for Hardy-Orlicz space. $\square$

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