ON THE EXISTENCE OF SOLUTIONS FOR NONLINEAR DIFFERENTIAL INCLUSIONS

BY

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Abstract. We consider a Cauchy problem for a nonlinear differential inclusion in separable and nonseparable Banach spaces under Filippov type assumptions and several existence results are obtained.

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1. Introduction

In this paper we study nonlinear differential inclusions of the form

$$(1) \quad x' \in Ax + F(t, x), \quad x(0) = x_0,$$

where A is a m -dissipative operator on a Banach space, $F : [0, T] \times X \rightarrow \mathcal{P}(X)$ is a set-valued map and $x_0 \in X$.

In [4, 12] it is shown that FILIPPOV's ideas ([7]) can be suitably adapted in order to prove the existence of C^0 -solutions of problem (1) provided X is separable. These approaches use a selection theorem due to KURATOWSKI and RYLL-NARDZEWSKI ([9]).

The aim of the present paper is twofold. On one hand, when X is separable, using Bressan-Colombo theorem concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values ([3]) we prove the existence of solutions continuously depending on a parameter for problem (1). In addition, as usual at a Filippov existence type

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theorem our result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. The result allows to obtain a continuous selection of the solution set of the problem considered. The proof follows the same pattern as in [5,11] where similar results are obtained for mild solutions when A is the infinitesimal generator of a strongly continuous semigroup on X .

On the other hand, DE BLASI and PIANIGIANI ([6]) established the existence of mild solutions for problem (1) with A the infinitesimal generator of a strongly continuous semigroup on an arbitrary, not necessarily separable Banach space X . Even if the ideas of Filippov are still present, the approach in ([6]) has a fundamental difference which consists in the construction of measurable selections of the multifunction. Our second result extends the above mentioned result to the case when A is an m -dissipative operator on X .

The paper is organized as follows. In section 2 we present the notations, definitions and preliminary results to be used in the sequel while in Section 3 we prove the main results.

2. Preliminaries

Let X be a real Banach space with the norm $\|\cdot\|$. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X . For any subset $A \subset X$ we denote by $cl(A)$ the closure of A .

Let $T > 0$, $I = [0, T]$ and denote by $\mathcal{L}(I)$ the σ -algebra of all Lebesgue measurable subsets of I . If $A \subset I$, then $\chi_A(\cdot) : I \rightarrow \{0, 1\}$ denotes the characteristic function of A .

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(\cdot) : I \rightarrow X$ endowed with the norm $|x(\cdot)|_C = \sup_{t \in I} \|x(t)\|$ and $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(\cdot) : I \rightarrow X$ endowed with the norm $|x(\cdot)|_1 = \int_0^T \|x(t)\| dt$.

For $x, y \in X$ we denote by $(x, y)_\perp = \lim_{h \uparrow 0} \frac{\|x+hy\|^2 - \|x\|^2}{2h}$ the left sided directional derivative of $\frac{1}{2}\|\cdot\|^2$ at x in the direction y .

Consider $A : X \rightarrow \mathcal{P}(X)$ an operator. With $D(A)$ we denote its domain. We recall that A is called *dissipative* if $(x_1 - x_2, y_1 - y_2)_\perp \leq 0$ for any $x_1, x_2 \in D(A)$ and $y_1 = Ax_1, y_2 = Ax_2$.

A is called *m -dissipative* if it is dissipative and $R(I - \lambda A) = X$ for any (equivalently, for some) $\lambda > 0$.

Let $A : D(A) \subset X \rightarrow \mathcal{P}(X)$ be m -dissipative and $f(\cdot) \in L^1(I, X)$ which

define the differential equation

$$(2) \quad x' \in Ax + f(t).$$

A mapping $x(\cdot) : I \rightarrow X$ is called a *strong solution* of (2) if $x(t) \in D(A)$ a.e. on $(0, T)$, $x(\cdot)$ is locally absolute continuous on $(0, T]$ and there exists $g \in L^1_{loc}((0, T], X)$ such that $g(t) \in Ax(t)$ a.e. on $(0, T)$ and $x'(t) = g(t) + f(t)$ a.e. on $(0, T)$.

It is well-known that if X is reflexive then for any $x_0 \in D(A)$ equation (2) has a unique strong solution on I which satisfies $x(0) = x_0$ (e.g. [2]). In general, equation (2) may not have strong solutions and a way to overcome this difficulty is the concept of C^0 -solutions (e.g.[10]).

Definition 1. A function $x(\cdot) \in C(I, X)$ is called a C^0 -solution of (2) if satisfies: for every $c \in (0, T)$ and $\varepsilon > 0$ there exists

- (i) $0 < t_1 < \dots < c \leq t_n < T$, $t_k - t_{k-1} \leq \varepsilon$, $t_0 = 0$ for $k = 1, 2, \dots, n$;
- (ii) $f_1, f_2, \dots, f_n \in X$ with

$$\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|f(t) - f_k\| dt \leq \varepsilon;$$

- (iii) $v_0, v_1, \dots, v_n \in X$ with

$$\frac{v_k - v_{k-1}}{t_k - t_{k-1}} \in Av_k + f_k$$

for $k = 1, 2, \dots, n$ and $\|x(t) - v_k\| \leq \varepsilon$ for $t \in [t_{k-1}, t_k)$, $k = 1, 2, \dots, n$.

According to [10], if A is m -dissipative, $f \in L^1(I, X)$ and $x_0 \in \overline{D(A)}$, there exists a unique C^0 -solution of (2) with $x(0) = x_0$.

Denote by $x(\cdot, x_0, f) : I \rightarrow \overline{D(A)}$ the unique C^0 -solution of (2) which satisfies $x(0, x_0, f) = x_0$.

See [13] for the following theorem.

Theorem 2. Let X be a real Banach space, let $A : D(A) \subseteq X \rightsquigarrow X$ be an m -dissipative operator, let $\xi, \eta \in \overline{D(A)}$ and $f, g \in L^1(I, X)$. Then for any $t \in I$

$$(3) \quad \|x(t, \xi, f) - x(t, \eta, g)\| \leq \|\xi - \eta\| + \int_0^t \|f(s) - g(s)\| ds.$$

See [14] for the following lemma.

Lemma 3. *Let $u : I \rightarrow X$ be measurable and let $G : I \rightarrow \mathcal{P}(X)$ be a measurable closed-valued multifunction. Then, for any measurable function $v : I \rightarrow (0, \infty)$, there exists a measurable selection $g : I \rightarrow X$ of $G(\cdot)$ (i.e., such that $g(t) \in G(t)$ a.e. on I) such that $\|u(t) - g(t)\| < d(u(t), G(t)) + v(t)$ a.e. on I , where the distance between a point $x \in X$ and a subset $A \subset X$ is defined as usual by $d(x, A) = \inf\{\|x - a\|; a \in A\}$.*

A subset $D \subset L^1(I; X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u \cdot \chi_A + v \cdot \chi_B \in D$, where $B = I \setminus A$. We denote by $\mathcal{D}(I, X)$ the family of all decomposable closed subsets of $L^1(I, X)$.

Next, (S, d) is a separable metric space; we recall that a multifunction $G(\cdot) : S \rightarrow \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S; G(s) \subset C\}$ is closed.

See [3] for the following two lemmas.

Lemma 4. *Let $F^*(\cdot, \cdot) : I \times S \rightarrow \mathcal{P}(X)$ be a closed valued, $\mathcal{L}(I) \otimes \mathcal{B}(S)$ measurable multifunction such that $F^*(t, \cdot)$ is l.s.c. for any $t \in I$. Then the multifunction $G(\cdot) : S \rightarrow \mathcal{D}(I, X)$ defined by*

$$G(s) = \{v \in L^1(I, X); \quad v(t) \in F^*(t, s) \text{ a.e. on } I\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $\rho(\cdot) : S \rightarrow L^1(I, X)$ such that $d(0, F^*(t, s)) \leq \rho(s)(t)$ a.e. on I , $\forall s \in S$.

Lemma 5. *Let $G(\cdot) : S \rightarrow \mathcal{D}(I, X)$ be a l.s.c. multifunction with closed decomposable values and let $\varphi(\cdot) : S \rightarrow L^1(I, X)$, $\psi(\cdot) : S \rightarrow L^1(I, \mathbb{R})$, be continuous such that the multifunction $H(\cdot) : S \rightarrow \mathcal{D}(I, X)$ defined by*

$$H(s) = cl\{v(s) \in G(s); \quad \|v(t) - \varphi(s)(t)\| < \psi(s)(t) \text{ a.e. on } I\}$$

has nonempty values.

Then, $H(\cdot)$ has a continuous selection, i.e. there exists a continuous mapping $h(\cdot) : S \rightarrow L^1(I, X)$ such that $h(s) \in H(s)$, $\forall s \in S$.

If $A, B \subset X$, the Pompeiu-Hausdorff distance between A and B is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, where $d^*(A, B) = \sup\{d(a, B), a \in A\}$.

In what follows, X is a Banach space and Y is a metric space. An open (respectively, closed) ball in Y with center y and radius r will be denoted by $B_Y(y, r)$ (respectively, $\overline{B_Y(y, r)}$).

A multifunction $F : Y \rightarrow \mathcal{P}(X)$ with closed bounded nonempty values is said to be d_H -continuous at $y_0 \in Y$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_H(F(y), F(y_0)) \leq \varepsilon$ for any $y \in B_Y(y_0, \delta)$. F is called d_H -continuous if it is so at each point $y_0 \in Y$.

Let $\mathcal{L}$ be the σ -algebra of the (Lebesgue) measurable sets of $\mathbb{R}$ and let $A \in \mathcal{L}$ with $\mu(A) < \infty$. A multifunction $F : Y \rightarrow \mathcal{P}(X)$ with closed bounded nonempty values is said to be *Lusin measurable* if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset A$, with $\mu(A \setminus K_\varepsilon) < \varepsilon$ such that $F|_{K_\varepsilon}$ is d_H -continuous.

It is clear that if $F, G : A \rightarrow \mathcal{P}(X)$ and $f : A \rightarrow X$ are Lusin measurable, then so F restricted to B (measurable, $B \subset A$), $F+G$ and $t \mapsto d(f(t), F(t))$. Moreover, the uniform limit of a sequence of Lusin measurable multifunctions also is Lusin measurable.

Hypothesis H. i) $F : I \times X \rightarrow \mathcal{P}(X)$ has nonempty closed bounded values and $F(\cdot, x)$ is Lusin measurable on I for any $x \in X$.

ii) There exists $L(\cdot) \in L^1(I, (0, \infty))$ such that $\forall t \in I$

$$d_H(F(t, x_1), F(t, x_2)) \leq L(t)\|x_1 - x_2\|, \quad \forall x_1, x_2 \in X.$$

iii) There exists $q(\cdot) \in L^1(I, (0, \infty))$ such that $\forall t \in I$ we have $F(t, 0) \subset q(t)B(0, 1)$.

See [6] for the following lemma.

Lemma 6. i) Let $F_i : I \rightarrow \mathcal{P}(X)$, $i = 1, 2$, be two Lusin measurable multifunctions and let $\varepsilon_i > 0$, $i = 1, 2$, be such that $H(t) := (F_1(t) + \varepsilon_1 B) \cap (F_2(t) + \varepsilon_2 B) \neq \emptyset$, $\forall t \in I$. Then the multifunction $H : I \rightarrow \mathcal{P}(X)$ has a Lusin measurable selection $h : I \rightarrow X$.

ii) Assume that Hypothesis H is satisfied. Then, for any $x(\cdot) : I \rightarrow X$ continuous, $u(\cdot) : I \rightarrow X$ measurable and $\varepsilon > 0$

a) the multifunction $t \rightarrow F(t, x(t))$ is Lusin measurable on I ,

b) the multifunction $G : I \rightarrow \mathcal{P}(X)$ defined by $G(t) := (F(t, x(t)) + \varepsilon B) \cap B_X(u(t), d(u(t), F(t, x(t))) + \varepsilon)$ has a Lusin measurable selection $g : I \rightarrow X$.

In what follows we are concerned with the following Cauchy problem

$$(4) \quad x' \in Ax + F(t, x), \quad x(0) = x_0,$$

where A is an m -dissipative operator on the separable Banach space X , $F : I \times X \rightarrow \mathcal{P}(X)$ and $x_0 \in X$.

Definition 7. A continuous mapping $x : I \rightarrow \overline{D(A)}$ is said to be a C^0 -solution of problem (4) if $x(0) = x_0$ and there exists $f(\cdot) \in L^1(I, X)$ with $f(t) \in F(t, x(t))$ a.e. on I and $x(\cdot)$ is a C^0 -solution on I of the equation (2) in the sense of Definition 1.

We shall call $(x(\cdot), f(\cdot))$ a *trajectory-selection pair* of (4) if $f(t) \in F(t, x(t))$ a.e. on I and $x(\cdot)$ is a C^0 -solution of (4).

We shall use the following notation for the solution sets of (4)

$$\mathcal{S}(x_0) = \{x(\cdot); \quad x(\cdot) \text{ is a } C^0\text{-solution of (4)}\}.$$

Definition 8. If X is not separable, a continuous mapping $x : I \rightarrow \overline{D(A)}$ is said to be a C^0 -solution of problem (4) if $x(0) = x_0$ and there exists a Lusin measurable function $f(\cdot) \in L^1(I, X)$ with $f(t) \in F(t, x(t))$ a.e. on I and $x(\cdot)$ is a C^0 -solution on I of the equation (2) in the sense of Definition 1.

3. Main results

In order to establish our continuous version of Filippov theorem for the problem (4) we need the following hypotheses.

Hypothesis 3.1. i) $F : I \times X \rightarrow \mathcal{P}(X)$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable;

ii) there exists $L(\cdot) \in L^1(I, \mathbb{R}_+)$ such that for almost all $t \in I$, $F(t, \cdot)$ is $L(t)$ -Lipschitz in the sense that $d_H(F(t, x), F(t, y)) \leq L(t)\|x - y\|, \forall x, y \in X$.

Let $A : D(A) \subset X \rightarrow \mathcal{P}(X)$ be an m -dissipative operator.

Hypothesis 3.2. i) S is a separable metric space and $a(\cdot) : S \rightarrow X$, $c(\cdot) : S \rightarrow (0, \infty)$ are continuous mappings;

ii) there exist the continuous mappings $g(\cdot) : S \rightarrow L^1(I, X)$, $y(\cdot) : S \rightarrow C(I, X)$ and $\rho(\cdot) : S \rightarrow L^1(I, \mathbb{R}_+)$ such that $y(s)(\cdot)$ is a C^0 -solution of $x' \in Ax + g(s)(t)$, $\forall s \in S$ and $d(g(s)(t), F(t, y(s)(t))) \leq \rho(s)(t)$ a.e. on I , $\forall s \in S$.

Next we use the following notations

$$m(t) = \int_0^t L(u)du,$$

$$\xi(s)(t) = e^{m(t)}(Tc(s) + \|a(s) - y(s)(t)\|) + \int_0^T \rho(s)(u)e^{m(t)-m(u)}du.$$

Theorem 9. *Assume that Hypotheses 3.1 and 3.2 are satisfied. Then there exist the continuous mappings $x(\cdot) : S \rightarrow C(I, X)$ and $f(\cdot) : S \rightarrow L^1(I, \mathbb{R}_+)$ such that for any $s \in S$, $(x(s)(\cdot), f(s)(\cdot))$ is a trajectory-selection pair of $x' \in Ax + F(t, x)$, $x(0) = a(s)$ and*

$$(5) \quad \|x(s)(t) - y(s)(t)\| \leq \xi(s)(t), \quad \forall t \in I, s \in S,$$

$$(6) \quad \|f(s)(t) - g(s)(t)\| \leq L(t)\xi(s)(t) + \rho(s)(t) + c(s) \quad \text{a.e. on } I, \forall s \in S.$$

Proof. We make the following notations:

$$\varepsilon_n(s) = c(s)\frac{n+1}{n+2}, \quad n = 0, 1, \dots, \quad d(s) = \|a(s) - y(s)(0)\|,$$

$$\rho_n(s)(t) = \int_0^t \rho(s)(u) \frac{(m(t) - m(u))^{n-1}}{(n-1)!} du + \frac{(m(t))^{n-1}}{(n-1)!} (T\varepsilon_n(s) + d(s)), \quad n \geq 1.$$

Set also $x_0(s)(t) = y(s)(t)$, $\forall s \in S, t \in I$.

We consider the multifunctions $G_0(\cdot)$, $H_0(\cdot)$ defined, respectively, by

$$G_0(s) = \{v \in L^1(I, X); \quad v(t) \in F(t, y(s)(t)) \text{ a.e. on } I\},$$

$$H_0(s) = cl\{v \in G_0(s); \quad \|v(t) - g(s)(t)\| < \rho(s)(t) + \varepsilon_0(s)\}.$$

Since $d(g(s)(t), F(t, y(s)(t))) \leq \rho(s)(t) < \rho(s)(t) + \varepsilon_0(s)$, according to Lemma 3, the set $H_0(s)$ is not empty.

Set $F_0^*(t, s) = F(t, y(s)(t))$. Note that $d(0, F_0^*(t, s)) \leq \|g(s)(t)\| + \rho(s, t) = \rho^*(s)(t)$ and $\rho^*(\cdot) : S \rightarrow L^1(I, X)$ is continuous.

Applying Lemma 4 and Lemma 5 we obtain the existence of a continuous solution $f_0(\cdot)$ of $H_0(\cdot)$, i.e.

$$f_0(s)(t) \in F(t, y(s)(t)) \quad \text{a.e. on } I, \forall s \in S$$

and $\|f_0(s)(t) - g(s)(t)\| \leq \rho_0(s)(t) = \rho(s)(t) + \varepsilon_0(s), \forall s \in S, t \in I$.

Let $x_1(s)(\cdot)$ be the unique C^0 -solution of the problem $x' = Ax + f_0(s)(t)$, $x(0) = a(s)$ and, by Theorem 2, we have

$$\begin{aligned} \|x_1(s)(t) - x_0(s)(t)\| &\leq \|a(s) - y(s)(0)\| + \int_0^t \|f_0(s)(u) - g(s)(u)\| du \\ &\leq d(s) + \int_0^t \rho_0(s)(u) du < \rho_1(s)(t). \end{aligned}$$

We shall construct, using the same idea as in [5,11], two sequences of approximations $f_n(\cdot) : S \rightarrow L^1(I, X)$, $x_n(\cdot) : S \rightarrow C(I, X)$ with the following properties:

- a) $f_n(\cdot) : S \rightarrow L^1(I, X)$, $x_n(\cdot) : S \rightarrow C(I, X)$ are continuous;
- b) $f_n(s)(t) \in F(t, x_n(s)(t))$ a.e. on I , $s \in S$;
- c) $\|f_n(s)(t) - f_{n+1}(s)(t)\| \leq L(t)\rho_n(s)(t)$ a.e. on I , $s \in S$;
- d) $x_{n+1}(s)(\cdot)$ is the unique C^0 -solution of the problem $x' = Ax + f_n(s)(t)$, $x(0) = a(s)$. Suppose we have already constructed $f_i(\cdot)$, $x_i(\cdot)$ satisfying a)-c) and let $x_{n+1}(\cdot)$ be as in d). From c) and Theorem 2 we have

$$\begin{aligned} \|x_{n+1}(s)(t) - x_n(s)(t)\| &\leq \int_0^t \|f_n(s)(u) - f_{n-1}(s)(u)\| du \\ (7) \qquad \qquad \qquad &\leq \int_0^t L(u)\rho_n(s)(u) du < \rho_{n+1}(s)(t). \end{aligned}$$

On the other hand,

$$\begin{aligned} d(f_n(s)(t), F(t, x_{n+1}(s)(t))) &\leq L(t)\|x_{n+1}(s)(t) - x_n(s)(t)\| \\ (8) \qquad \qquad \qquad &< L(t)\rho_{n+1}(s)(t). \end{aligned}$$

To prove that $H_{n+1}(s)$ is not empty, we note first that the real function $t \rightarrow r_n(s)(t) = c(s) \frac{TL(t)m(t)^n}{(n+2)(n+3)n!}$ is measurable and strictly positive for any $s \in S$. Using (8) we get $d(f_n(s)(t), F(t, x_{n+1}(s)(t))) \leq L(t)\|x_{n+1}(s)(t) - x_n(s)(t)\| - r_n(s)(t) \leq L(t)\rho_{n+1}(s)(t)$ and, therefore, according to Lemma 4, there exists $v(\cdot) \in L^1(I, X)$ such that $v(t) \in F(t, x_n(s)(t))$ a.e. on I and $\|v(t) - f_n(s)(t)\| < d(f_n(s)(t), F(t, x_n(s)(t))) + r_n(s)(t)$ and hence $H_{n+1}(s)$ is not empty.

Set $F_{n+1}^* = F(t, x_{n+1}(s)(t))$ and note that we may write $d(0, F_{n+1}^*(t, s)) \leq L(t)\|x_{n+1}(s)(t) - x_n(s)(t)\| \leq \|f_n(s)(t)\| + L(t)\rho_{n+1}(s)(t) = \rho_{n+1}^*(s)(t)$ a.e. on I and $\rho_{n+1}^*(\cdot) : S \rightarrow L^1(I, X)$ is continuous.

By Lemmas 4 and 5 there exists a continuous mapping $f_{n+1}(\cdot) : S \rightarrow L^1(I, X)$ such that

$$\begin{aligned} f_{n+1}(s)(t) &\in F(t, x_{n+1}(s)(t)) \quad \text{a.e. on } I, \quad \forall s \in S, \\ \|f_{n+1}(s)(t) - f_n(s)(t)\| &\leq L(t)\rho_{n+1}(s)(t) \quad \text{a.e. on } I, \quad \forall s \in S. \end{aligned}$$

From (7) we obtain

$$\begin{aligned} \|x_{n+1}(s)(\cdot) - x_n(s)(\cdot)\|_C &\leq \|f_{n+1}(s)(\cdot) - f_n(s)(\cdot)\|_1 \\ (9) \quad &\leq \frac{(m(T))^n}{n!} (\|\rho(s)(\cdot)\|_1 + Tc(s) + d(s)). \end{aligned}$$

Therefore, $f_n(s)(\cdot), x_n(s)(\cdot)$ are Cauchy sequences in the Banach space $L^1(I, X)$ and $C(I, X)$ respectively. Let $f(\cdot) : S \rightarrow L^1(I, X)$ and $x(\cdot) : S \rightarrow C(I, X)$ be their limits. The function $s \rightarrow \|\rho(s)(\cdot)\|_1 + T(s) + d(s)$ is continuous, hence locally bounded. Therefore (9) implies that for every $s' \in S$ the sequence $f_n(s')(\cdot)$ satisfies the Cauchy condition uniformly with respect to s' on some neighbourhood of s . Hence $s \rightarrow f(s)(\cdot)$ is continuous from S into $L^1(I, X)$.

From (9), as before, $x_n(s)(\cdot)$ is Cauchy in $C(I, X)$ locally uniformly with respect to s . So, $s \rightarrow x(s)(\cdot)$ is continuous from S into $C(I, X)$. On the other hand, since $x_n(s)(\cdot)$ converges uniformly to $x(s)(\cdot)$ and

$$d(f_n(s)(t), F(t, x(s)(t))) \leq L(t)\|f_n(s)(t) - x(s)(t)\|$$

a.e. on $I, \forall s \in S$. Passing to the limit along a subsequence of $f_n(\cdot)$ converging pointwise to $f(\cdot)$ we obtain

$$f(s)(t) \in F(t, x(s)(t)) \quad \text{a.e. on } I, \quad \forall s \in S.$$

Let $x^*(s)(\cdot)$ be the unique C^0 -solution of the Cauchy problem $x' \in Ax + f(s)(t), x(0) = a(s)$. Since $\|x_{n+1}(s)(t) - x^*(s)(t)\| \leq \int_0^t \|f_n(s)(u) - f(s)(u)\| du$, letting $n \rightarrow \infty$ we get $x^*(s)(\cdot) \equiv x(s)(\cdot)$.

By adding inequalities c) for all n and using the fact that $\sum_{i \geq 1} \rho_i(s)(t) \leq \xi(s)(t)$ we obtain

$$\begin{aligned} \|f_{n+1}(s)(t) - g(s)(t)\| &\leq \sum_{l=0}^n \|f_{l+1}(s)(t) - f_l(s)(u)\| + \|f_l(s)(u) - g(s)(t)\| \\ (10) \quad &\leq \sum_{l=0}^n L(t)\rho_{l+1}(s)(t) + \rho(s)(t) + \varepsilon_0(s) \leq L(t)\xi(s)(t) + \rho(s)(t) + c(s). \end{aligned}$$

Similarly, by adding (7) we get

$$(11) \quad \|x_{n+1}(s)(t) - y(s)(t)\| \leq \sum_{l=0}^n \rho_l(s)(t) \leq \xi(s)(t).$$

By passing to the limit in (10) and (11) we obtain (5) and (6), respectively. The proof is complete.

Theorem 9 allows to obtain the next corollary which is a general result concerning continuous solutions of the solution set of (4).

Hypothesis 3.3. Hypothesis 3.1 is satisfied and there exists $\rho_0(\cdot) \in L^1(I, \mathbb{R}_+)$ such that $d(0, F(t, 0)) \leq \rho_0(t)$ a.e. on I .

Theorem 10. Assume that Hypothesis 3.3 is satisfied. Then there exists a function $x(\cdot, \cdot) : I \times X \rightarrow X$ such that:

- a) $x(\cdot, \xi) \in \mathcal{S}(\xi)$, $\forall \xi \in X$;
- b) $\xi \rightarrow x(\cdot, \xi)$ is continuous from X into $C(I, X)$.

Proof. We take $S = X$, $a(\xi) = \xi$, $\forall \xi \in X$, $c(\cdot) : X \rightarrow (0, \infty)$ an arbitrary function, $g(\cdot) = 0$, $y(\cdot) = 0$, $\rho(\xi)(t) = \rho_0(t)$, $\forall x \in X, t \in I$, and we apply Theorem 9 in order to obtain the conclusion of the theorem.

Remark 11. A similar result as in the Theorem 10 is obtained in [11] in the particular case when X is a Hilbert space and $A = -B$ with B maximal monotone. Moreover, the result in Theorem 10 may be obtained as a particular case of the main result (Theorem 9) in [1]. The result in [1] is obtained for a more general problem, namely a Volterra integral solution. We underline the fact that our Theorem 9 cannot be obtained from any of the results in [1].

We consider now the case when X is not separable.

Theorem 12. We assume that Hypothesis H is satisfied. Then for any $x_0 \in X$ there exists $x(\cdot) : I \rightarrow X$ a C^0 -solution of problem (4).

Proof. Let us first note that, if $z(\cdot) : I \rightarrow X$ is continuous, then every Lusin measurable selection $u : I \rightarrow X$ of the multifunction $t \mapsto F(t, z(t)) + B(0, 1)$ is Bochner integrable on I . More precisely, for any $t \in I$ we have $\|u(t)\| \leq d_H(F(t, z(t)) + B(0, 1), 0) \leq d_H(F(t, z(t)), F(t, 0)) + d_H(F(t, 0), 0) + 1 \leq L(t)\|z(t)\| + q(t) + 1$. Let $0 < \varepsilon < 1$, $\varepsilon_n = \frac{\varepsilon}{2^{n+2}}$.

Consider an arbitrary Lusin measurable function $f_0(\cdot) : I \rightarrow X$, that is Bochner integrable and define $x_0(\cdot)$ the unique C^0 -solution of the problem $x' \in Ax + f_0(t), x(0) = x_0$. Since $x_0(\cdot)$ is continuous, by Lemma 6 ii) there exists a Lusin measurable function $f_1(\cdot) : I \rightarrow X$ satisfying $f_1(t) \in (F(t, x_0(t)) + \varepsilon_1 B(0, 1)) \cap B(f_0(t), d(f_0(t), F(t, x_0(t))) + \varepsilon_1)$, $t \in I$.

Obviously, $f_1(\cdot)$ is Bochner integrable on I . Define $x_1(\cdot)$ the unique C^0 -solution of the problem $x' \in Ax + f_1(t), x(0) = x_0$. We construct, by induction, a sequence $x_n(\cdot) : I \rightarrow X$, $n \geq 2$, with $x_n(\cdot)$ the unique C^0 -solution of the problem $x' \in Ax + f_n(t), x(0) = x_0$, where $f_n(\cdot) : I \rightarrow X$ is a Lusin measurable function satisfying

$$(12) \quad \begin{aligned} f_n(t) &\in (F(t, x_{n-1}(t)) + \varepsilon_n B) \cap B(f_{n-1}(t), \\ &d(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n), \quad t \in I. \end{aligned}$$

At the same time, as we saw at the beginning of the proof, $f_n(\cdot)$ is also Bochner integrable. From (12), for $n \geq 2$ and $t \in I$ we have

$$\begin{aligned} \|f_n(t) - f_{n-1}(t)\| &\leq d(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n \leq d(f_{n-1}(t), F(t, x_{n-2}(t))) \\ &+ d_H(F(t, x_{n-2}(t)), F(t, x_{n-1}(t))) + \varepsilon_n \leq \varepsilon_{n-1} + L(t)\|x_{n-1} - x_{n-2}\| + \varepsilon_n. \end{aligned}$$

Since $\varepsilon_{n-1} + \varepsilon_n < \varepsilon_{n-2}$, we deduce that

$$(13) \quad \|f_n(t) - f_{n-1}(t)\| \leq \varepsilon_{n-2} + L(t)\|x_{n-1}(t) - x_{n-2}(t)\|, n \geq 2.$$

Denote $m(t) = \int_0^t L(u)du$ and $\rho_0(t) = d(f_0(t), F(t, x_0(t)))$, $t \in I$. Next, we prove by recurrence that for $n \geq 2$ and $t \in I$ we have

$$(14) \quad \begin{aligned} \|x_n(t) - x_{n-1}(t)\| &\leq \sum_{k=0}^{n-2} \int_0^t \varepsilon_{n-2-k} \frac{(m(t) - m(u))^k}{k!} du \\ &+ \varepsilon_0 \int_0^t \frac{(m(t) - m(u))^{n-1}}{(n-1)!} du + \int_0^t \frac{(m(t) - m(u))^{n-1}}{(n-1)!} \rho_0(u) du. \end{aligned}$$

We start with $n = 2$. By (12), (13) and (3) for $t \in I$, we have

$$\begin{aligned} \|x_2(t) - x_1(t)\| &\leq \int_0^t \|f_2(s) - f_1(s)\| ds \leq \int_0^t [\varepsilon_0 + L(s)\|x_1(s) - x_0(s)\|] ds \\ &\leq \varepsilon_0 t + \int_0^t L(s) \int_0^s \|f_1(u) - f_0(u)\| du ds \leq \varepsilon_0 t + \int_0^t L(s) \int_0^s (\rho_0(u) + \varepsilon_1) du ds \\ &\leq \varepsilon_0 t + \int_0^t (\rho_0(u) + \varepsilon_1) \int_0^t L(s) ds du = \varepsilon_0 t + \int_0^t (m(t) - m(s))(\rho_0(s) + \varepsilon_0) ds, \end{aligned}$$

i.e. (14) is satisfied for $n = 2$. Using again (12), (13) and (3) we have

$$\begin{aligned}
\|x_{n+1}(t) - x_n(t)\| &\leq \int_0^t \|f_{n+1}(u) - f_n(u)\| du \\
&\leq \int_0^t [\varepsilon_{n-1} + L(s)\|x_n(s) - x_{n-1}(s)\|] ds \\
&\leq \varepsilon_{n-1}t + \int_0^t L(s) \left[\sum_{k=0}^{n-2} \int_0^s \varepsilon_{n-2-k} \frac{(m(s) - m(u))^k}{k!} du \right. \\
&\quad \left. + \int_0^s \frac{(m(s) - m(u))^{n-1}}{(n-1)!} (\rho_0(u) + \varepsilon_0) du \right] ds \\
&= \varepsilon_{n-1}t + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \int_0^t \left[\int_0^s \frac{(m(s) - m(u))^k}{k!} L(s) du \right] ds \\
&\quad + \int_0^t L(s) \left(\int_0^s \frac{(m(s) - m(u))^{n-1}}{(n-1)!} L(s) [\rho_0(u) + \varepsilon_0] du \right) ds \\
&= \varepsilon_{n-1}t + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \int_0^t \left[\int_u^t \frac{(m(s) - m(u))^k}{k!} L(s) ds \right] du \\
&\quad + \int_0^t \left[\int_u^t \frac{(m(s) - m(u))^{n-1}}{(n-1)!} L(s) ds \right] [\rho_0(u) + \varepsilon_0] du \\
&= \varepsilon_{n-1}t + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \int_0^t \frac{(m(s) - m(u))^{k+1}}{(k+1)!} du \\
&\quad + \int_0^t \frac{(m(s) - m(u))^k}{k!} [\rho_0(u) + \varepsilon_0] du \\
&= \sum_{k=0}^{n-1} \varepsilon_{n-1-k} \int_0^t \frac{(m(s) - m(u))^k}{k!} du \\
&\quad + \int_0^t \frac{(m(s) - m(u))^n}{n!} [\rho_0(u) + \varepsilon_0] du,
\end{aligned}$$

so that (14) holds true for $n + 1$.

It follows from (14) that for $n \geq 2$ and $t \in I$ we have

$$(15) \quad \|x_n(t) - x_{n-1}(t)\| \leq a_n,$$

where

$$a_n = \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \frac{m(t)^k}{k!} + \frac{m(t)^{n-1}}{(n-1)!} \left[\int_0^T \rho_0(u) du + \varepsilon_0 \right].$$

Obviously, the series with n -th term a_n is convergent.

So, from (15) we have that $x_n(\cdot)$ converges uniformly on I to a continuous function $x(\cdot) : I \rightarrow X$.

On the other hand, by (13) we have $\|f_n(t) - f_{n-1}(t)\| \leq \varepsilon_{n-2} + L(t)a_{n-1}$, $t \in I, n \geq 3$, which implies that the sequence $f_n(\cdot)$ converges to a Lusin measurable function $f(\cdot) : I \rightarrow X$. Since $x_n(\cdot)$ is bounded and $\|f_n(t)\| \leq L(t)\|x_{n-1}(t)\| + q(t) + 1$ we deduce that $f(\cdot)$ is also Bochner integrable.

Let $x^*(\cdot) : I \rightarrow X$ be the unique C^0 -solution of the Cauchy problem $x' \in Ax + f(t)$, $x(0) = x_0$. By (3) $\|x_n(t) - x^*(t)\| \leq \int_0^t \|f_n(s) - f(s)\| ds$ and using the Lebesgue dominated convergence theorem we obtain $x(\cdot) = x^*(\cdot)$. On the other hand, from (12) we get $f_n(t) \in F(t, x_n(t)) + \varepsilon_n B(0, 1)$, $t \in I, n \geq 1$ and letting $n \rightarrow \infty$ we have $f(t) \in F(t, x(t))$, $t \in I$ and the proof is complete.

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