A MULTI-VALUED CHOQUET INTEGRAL WITH RESPECT TO A MULTISUBMEASURE

BY

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Abstract. JANG, KIM and KWON introduced a multi-valued Choquet integral for multifunctions with respect to real fuzzy measures and ZHANG, GUO and LIU established for this kind of integral some convergence theorems. The aim of this paper is to present another type of set-valued Choquet integral, called by us the Aumann-Choquet integral, for non-negative measurable functions with respect to multisubmeasures taking values in the class of all non-empty, compact and convex sets of $\mathbb{R}_+$ on which we use the order relation considered by GUO and ZHANG. For this kind of integral, we study some important properties and we prove that if we add some supplementary properties to the multisubmeasure then they are also preserved by the set-valued function defined as Aumann-Choquet integral.

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1. Introduction

In recent years, there have appeared numerous works concerning Choquet integral which has drawn much attention due to the multiple applications in mathematics, economics, theory of control, decision making, risk analysis and many other fields.

Motivated by some problems in potential theory, in 1953/1954, CHOQUET ([2]) had been introduced an integral for non-negative measurable functions with respect to non-additive measures which has the monotonicity and continuity properties on certain sequences of sets. Later, the theory of Choquet integral was extended for the fuzzy measure, that is monotonic

set functions. For this kind of Choquet integral was proved several important properties such as monotonicity and comonotonical additivity. Other properties of this integral was obtained under supplementary conditions imposed to the fuzzy measures ([14,15,21]).

Using an Aumann type procedure ([1]), JANG ET AL. [10] introduced the concept of set-valued Choquet integral for a multifunction taking values in the class of all closed, nonempty sets of $\mathbb{R}_+$ with respect to a nonnegative fuzzy measure, for which the selectors are real Choquet integrals. For this integral, in [9], [10], [11] were obtained some properties and in [8] were proved some convergence theorems for sequences of multifunctions taking values in the class of all closed, nonempty sets of $\mathbb{R}_+$ with respect to autocontinuous fuzzy measures, under Hausdorff convergence.

In [22], using the Kuratowski convergence, ZHANG, GUO and LIU studied some properties of the set-valued Choquet integral of multifunctions with respect to real fuzzy measures and proved some convergence theorems for this kind of integral.

In [4], HONGXIA and JUN proved that if are added supplementary properties to the fuzzy measures then they are also preserved by the multifunctions defined as the set-valued Choquet integral. In [6], [7] JANG presented some applications of Choquet integral for closed bounded interval-valued multifunctions.

Using the Aumann procedure, PRECUPANU and SATCO introduced and studied in [19], a set-valued integral of real functions with respect to finite additive multimeasures for which the selectors are Gould integrals with respect to finite additive measures taking their values in a Banach space (also see [17], [18]).

Based on analogous construction as in [19], we present in this paper a new type of set-valued Choquet integral for non-negative measurable functions with respect to multisubmeasures taking values in the class of all nonempty, compact and convex sets of $\mathbb{R}_+$.

The organisation of the paper is as follows: in section 2 we present notations and some basic concepts and in section 3 we recall some definitions and basic properties of classical Choquet integral of non-negative functions with respect to fuzzy measures ([14], [15], [16]). In section 4 we define another set-valued Choquet integral for non-negative bounded measurable functions with respect to multisubmeasures taking values in the class of all nonempty, compact and convex sets of $\mathbb{R}_+$, on which we use an order relation considered by GUO and ZHANG in [3]. We also present some remarkable

properties of this kind of integral such as homogeneity, monotonicity, linearity under some special conditions. In section 5, we proved that under supplementary conditions for the multisubmeasure such as null-additivity, converse null-additivity, pseudometric generating property or Darboux property then these are preserved by the corresponding Aumann Choquet integral.

2. Terminology and notations

Let S be a nonempty set, $\mathcal{A}$ an algebra of subsets of S and $X = [0, +\infty) = \mathbb{R}_+$.

We denote by: $\mathcal{P}_0(X)$, the family of all nonempty subsets of X , $\mathcal{P}_k(X)$, the family of all nonempty compact subsets of X , $\mathcal{P}_f(X)$, the family of all nonempty, closed subsets of X , $\mathcal{P}_b(X)$, the family of all nonempty, bounded subsets of X , $\mathcal{P}_{bf}(X)$, the family of all nonempty, bounded and closed subsets of X , $\mathcal{P}_{kc}(X)$, the family of nonempty, compact convex subsets of X .

For every $A, B \in \mathcal{P}_0(X)$ we denote by $e(A, B) = \sup_{x \in A} \inf_{y \in B} |x - y|$ and by $h : \mathcal{P}_0(Y) \times \mathcal{P}_0(Y) \rightarrow \mathbb{R}_+$ the Hausdorff pseudometric defined by $h(A, B) = \max\{e(A, B), e(B, A)\}$.

We observe that h becomes a metric on $\mathcal{P}_{bf}(X)$ ([5], ch.I.1, [13]).

We also denote by $|A| = h(A, \{0\})$, for every $A \in \mathcal{P}_0(X)$. By the definition of the Hausdorff metric, we have immediately the following:

Lemma 2.1. *For every $A, B \in \mathcal{P}_{kc}(\mathbb{R}_+)$, with $A = [a, b], B = [c, d]$ we have $h(A, B) = \max\{|a - c|, |b - d|\}$.*

In [3], GUO and ZHANG consider the following order relation on $\mathcal{P}_{kc}(X)$:

Definition 2.2. Let $A, B \in \mathcal{P}_0(X)$.

1) $A \lesssim B$ if:

- (i) for each $x_0 \in A$, there exists $y_0 \in B$ such that $x_0 \leq y_0$;
- (ii) for each $y_0 \in B$ there exists $x_0 \in A$ such that $x_0 \leq y_0$.

2) $A \lesssim_{\varphi} B$ if:

- (i') for each $x_0 \in A$, there exists $y_0 \in B$ such that $x_0 < y_0$;
- (ii') for each $y_0 \in B$ there exists $x_0 \in A$ such that $x_0 < y_0$.

It is easy to prove the following:

Proposition 2.3. *If $A, B \in \mathcal{P}_{kc}(\mathbb{R}_+)$ where $A = [a, b]$ and $B = [c, d]$, $a, b, c, d \in \mathbb{R}_+$, we have $A \lesssim B$ if and only if $a \leq c$ and $b \leq d$.*

Remark 2.4. 1) If $A, B \in \mathcal{P}_{kc}(\mathbb{R}_+)$, where $A = [a, b]$ and $B = [c, d]$, $a, b, c, d \in \mathbb{R}_+$, then $A \subseteq B$ means $c \leq a \leq b \leq d$.

2) It is easy to see that if A is a singleton, $A \in \mathcal{P}_{kc}(\mathbb{R}_+)$, the relation $\lesssim$ becomes the usual order relation on $\mathbb{R}_+$.

Definition 2.5 ([5]). For a net $(A_i)_{i \in I} \subset \mathcal{P}_0(\mathbb{R}_+)$, where I is a filtering set, we define $\limsup_{i \in I} A_i = \{x \in \mathbb{R}_+ : x = \lim_k x_{i_k}, x_{i_k} \in A_{i_k}\}$ and $\liminf_{i \in I} A_i = \{x \in \mathbb{R}_+ : x = \lim_i x_i, x_i \in A_i\}$.

We say that A is the *Kuratowski limit* (briefly *K-limit*) of (A_i) if $\limsup_{i \in I} A_i = \liminf_{i \in I} A_i = A$ and we denote it by $A_i \xrightarrow{K} A$ (or, simply $A_i \rightarrow A$).

This kind of convergence is called *Kuratowski convergence* (briefly *K-convergence*).

Lemma 2.6 ([23]). *If $A_i = [a_i, b_i] \subset \mathcal{P}_{kc}(\mathbb{R}_+)$, $i \in I$, then $A_i \xrightarrow{K} A = [a, b]$ if and only if $a_i \rightarrow a$, $b_i \rightarrow b$.*

Definition 2.7 ([22]). Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}_f(\mathbb{R}_+)$ be a sequence and $A \in \mathcal{P}_f(\mathbb{R}_+)$. We say that:

1) (A_n) is *increasing K-convergent* to A if $A_1 \lesssim A_2 \lesssim \dots$ and $A_n \xrightarrow{K} A$ and we denote it by $A_n \uparrow A$;

2) (A_n) is *decreasing K-convergent* to A if $A_1 \gtrsim A_2 \gtrsim \dots$ and $A_n \xrightarrow{K} A$ and we denote it by $A_n \downarrow A$.

Remark 2.8. From [5], ch.1, we have that if $(A_i)_{i \in I} \subset \mathcal{P}_{kc}(\mathbb{R})$ then *K-convergence* of $(A_i)_{i \in I}$ is equivalent to its convergence in Hausdorff metric. It is easy to prove that the same is true for $\mathcal{P}_{kc}(\mathbb{R}_+)$.

Definition 2.9 ([7]). If $[a, b], [c, d] \in \mathcal{P}_{kc}(\mathbb{R}_+)$ and $k \in \mathbb{R}_+$, then we define:

$$[a, b] + [c, d] = [a + c, b + d],$$

$$k[a, b] = [ka, kb],$$

$$[a, b][c, d] = [ac, bd],$$

$$[a, b] \wedge [c, d] = [a \wedge c, b \wedge d] \text{ and } [a, b] \vee [c, d] = [a \vee c, b \vee d],$$

where $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Remark 2.10. For all $A, B \in \mathcal{P}_{kc}(\mathbb{R}_+)$ we have:

- i) $A \lesssim A \vee B$ and $B \lesssim A \vee B$;
- ii) $A \wedge B \lesssim A$ and $A \wedge B \lesssim B$.

Definition 2.11 ([14]). Let be $f, g : S \rightarrow \mathbb{R}$ two functions. We say that f and g are *comonotonic* and we denote it by $f \sim g$, if $f(s) < f(s') \Rightarrow g(s) \leq g(s')$, for every $s, s' \in S$.

Definition 2.12. The set function $m : \mathcal{A} \rightarrow \mathbb{R}_+$ with $m(\emptyset) = 0$ is said to be:

- 1) a *fuzzy measure* if $m(A) \leq m(B)$, for every $A, B \in \mathcal{A}$ with $A \subseteq B$;
- 2) a *submeasure in Drewnowski sense* if:
 - (i) $m(A \cup B) \leq m(A) + m(B)$, for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$;
 - (ii) $m(A) \leq m(B)$, for every $A, B \in \mathcal{A}$ with $A \subseteq B$.
- 3) a *strict submeasure* if:
 - (i') $m(A \cup B) < m(A) + m(B)$, for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$;
 - (ii') $m(A) < m(B)$, for every $A, B \in \mathcal{A}$ with $A \subseteq B$.
- 4) an *additive measure* (shortly a *measure*) if $m(A \cup B) = m(A) + m(B)$, for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$.

The order in the Definition 2.12. is the usual order on $\mathbb{R}_+$.

Definition 2.13. A set-valued function $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ with $\mu(\emptyset) = \{0\}$ is said to be:

- 1) a *fuzzy multimeasure* if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{A}$ with $A \subseteq B$;
- 2) a *monotone set-function* if $\mu(A) \lesssim \mu(B)$, for every $A, B \in \mathcal{A}$ with $A \subseteq B$;
- 3) a *multisubmeasure* if the following conditions are satisfied:
 - (i) $\mu(A \cup B) \lesssim \mu(A) + \mu(B)$, for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$;
 - (ii) $\mu(A) \lesssim \mu(B)$, for every $A, B \in \mathcal{A}$ with $A \subseteq B$;

4) a *strict multisubmeasure* if:

- (i') $\mu(A \cup B) \lesssim \mu(A) + \mu(B)$, for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$;
- (ii') $\mu(A) \lesssim \mu(B)$, for every $A, B \in \mathcal{A}$ with $A \subseteq B$.

5) an *additive multimeasure* (shortly a *multimeasure*) if $\mu(A \cup B) = \mu(A) + \mu(B)$, for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$, where we denoted by $E + F = \overline{E + F}$, $\forall E, F \in \mathcal{A}$ in which $\overline{E}$ means the closure of E .

Unless stated otherwise, in what follows we consider that $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(X)$ is a multisubmeasure.

Definition 2.14 ([4]). 1) The set multifunction $\mu : \mathcal{A} \rightarrow \mathcal{P}_0(X)$ is called:

- i) *null-additive* if for every $A, B \in \mathcal{A}$ with $\mu(A) = \{0\}$ we have $\mu(A \cup B) = \mu(B)$;
- ii) *weakly null-additive* if for every $A, B \in \mathcal{A}$ with $\mu(A) = \mu(B) = \{0\}$ we have $\mu(A \cup B) = \{0\}$;
- iii) *converse null-additive* if for every $A, B \in \mathcal{A}$ with $A \subset B$ such that $\mu(A) = \mu(B)$ we have $\mu(B - A) = \{0\}$;
- iv) *order continuous* (briefly *o-continuous*) if for every $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}, A_n \downarrow A$ with $\mu(A) = \{0\}$ we have $\lim_{n \rightarrow \infty} \mu(A_n) = \{0\}$;
- v) *lower semi-continuous* if for $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}, A_n \uparrow A$ we have $h(\mu(A_n), \mu(A)) \rightarrow 0$.

Definition 2.15. The set multifunction $\mu : \mathcal{A} \rightarrow \mathcal{P}_0(X)$ is said to have:

- i) *the pseudometric generating property* (briefly *p.g.p.*) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $A, B \in \mathcal{A}$ with $\mu(A) \subset [0, \delta]$, $\mu(B) \subset [0, \delta]$ we have $\mu(A \cup B) \subset [0, \varepsilon]$;
- ii) *the Darboux property* (briefly *d.p.*) if for every $A \in \mathcal{A}$ with $\mu(A) \not\supseteq \{0\}$, there exists $p \in (0, 1)$ and a set $B \in \mathcal{A}$ such that $B \subset A$ and $\mu(B) = p\mu(A)$.
- iii) *the property (S)* if for any $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ such that $\lim_{n \rightarrow \infty} \mu(A_n) = \{0\}$ there exists a subsequence $(A_{n_k}) \subset (A_n)$ such that $\mu(\liminf_{k \rightarrow \infty} A_{n_k}) = \{0\}$.

Definition 2.16 ([4]). Let be $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $\lambda : \mathcal{A} \rightarrow \mathbb{R}_+$ a fuzzy measure. We say that:

- 1) μ is *absolutely continuous of type I* with respect to λ , denoted by $\mu \ll_I \lambda$, if for every $A \in \mathcal{A}$ with $\lambda(A) = 0$ we have $\mu(A) = \{0\}$;
- 2) μ is *absolutely continuous of type II* with respect to λ , denoted by $\mu \ll_{II} \lambda$, if for every $B \subset A$, $A, B \in \mathcal{A}$ for which $\lambda(A - B) = 0$ we have $\mu(A) = \mu(B)$;
- 3) μ is *absolutely continuous of type III* with respect to λ , denoted by $\mu \ll_{III} \lambda$, if for every $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ with $\lim_{n \rightarrow \infty} \lambda(A_n) = 0$ we have $\lim_{n \rightarrow \infty} \mu(A_n) = \{0\}$;
- 4) μ is *absolutely continuous of type IV* with respect to λ , denoted by $\mu \ll_{IV} \lambda$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $A \in \mathcal{A}$ with $\lambda(A) \leq \delta$ we have $\mu(A) \subseteq [0, \varepsilon]$.

Remark 2.17. If $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ is a null-additive multisubmeasure and λ is a fuzzy measure we have that $\mu \ll_I \lambda \Leftrightarrow \mu \ll_{II} \lambda$.

Indeed, let be $\mu \ll_I \lambda$ and $B \subset A$, $A \in \mathcal{A}$, with $\lambda(A - B) = 0$. Then from $\mu \ll_I \lambda$ we obtain that $\mu(A - B) = \{0\}$. Because μ is a multisubmeasure null-additive we have that $\mu(A) = \mu((A - B) \cup B) = \mu(B)$ which assures that $\mu \ll_{II} \lambda$.

Conversely, let be $\mu \ll_{II} \lambda$ and $A \in \mathcal{A}$ with $\lambda(A) = 0$. If we particularly take $B = \emptyset$, $B \subset A$ then $\lambda(A - B) = \lambda(A) = 0$. Since $\mu \ll_{II} \lambda$ we have that $\mu(A) = \mu(B) = \{0\}$, so $\mu \ll_I \lambda$.

Definition 2.18. We say that a property P holds μ -almost everywhere (μ -a.e.) on S if the property P is true on $S - N$, with $\mu(N) = \{0\}$.

Definition 2.19. Let be a set multifunction $\mu : \mathcal{A} \rightarrow \mathcal{P}_0(\mathbb{R}_+)$. The set function $m : \mathcal{A} \rightarrow X$ is a *selector* of μ if for every $A \in \mathcal{A}$ we have $m(A) \in \mu(A)$.

We denote by S_μ the set of all selectors of the set multifunction μ .

Remark 2.20. If $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ is a multisubmeasure, we can associate to μ two nonnegative set functions:

- (1) $m_1(A) = \inf \mu(A)$, $\forall A \in \mathcal{A}$ and $m_2(A) = \sup \mu(A)$, for every $A \in \mathcal{A}$.

According to [23], Proposition 2.1, m_1, m_2 are submeasures in Drewnowski sense, such that $m_1(A) \in \mu(A)$ and $m_2(A) \in \mu(A)$, for every $A \in \mathcal{A}$. Thus, every multisubmeasure has selectors that are fuzzy measures. Moreover, for $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ we have that $\mu(A) = [m_1(A), m_2(A)]$, for every $A \in \mathcal{A}$.

In what follows we shall also use:

Lemma 2.21. *If $A, B, C, D \in \mathcal{P}_{kc}(\mathbb{R}_+)$ with $A \subseteq B, C \subseteq D$ and $B \lesssim D$ then $A \lesssim C$.*

Proof. Since $A, B, C, D \in \mathcal{P}_{kc}(\mathbb{R}_+)$ then we can write $A = [a_1, a_2], B = [b_1, b_2], C = [c_1, c_2], D = [d_1, d_2]$, where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}_+$. From $A \subseteq B$ and $C \subseteq D$ we have $b_1 \leq a_1 \leq a_2 \leq b_2$ and, respectively, $d_1 \leq c_1 \leq c_2 \leq d_2$.

Since $B \lesssim D$ we obtain $b_1 \leq d_1$ and $b_2 \leq d_2$.

We have the following cases:

- a) $b_1 \leq a_1 \leq d_1 \leq c_1 \leq a_2 \leq b_2 \leq c_2 \leq d_2$;
- b) $b_1 \leq d_1 \leq a_1 \leq a_2 \leq b_2 \leq c_1 \leq c_2 \leq d_2$ and respectively,
- c) $b_1 \leq a_1 \leq a_2 \leq d_1 \leq c_1 \leq b_2 \leq c_2 \leq d_2$.

It is easy to see that in all cases, we obtain that $a_1 \leq c_1$ and $a_2 \leq c_2$, so $A \lesssim C$. $\square$

Definition 2.22 ([12]). Let be $\mu, \nu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ two multisubmeasures. We define, for every $A \in \mathcal{A}$, $(\mu \vee \nu)(A) = \mu(A) \vee \nu(A)$ and $(\mu \wedge \nu)(A) = \mu(A) \wedge \nu(A)$.

3. The Choquet integral

In this section we recall the definition and some basic properties of the real-valued Choquet integral.

Let be $f : S \rightarrow \mathbb{R}_+$ a measurable function and $m : \mathcal{A} \rightarrow \mathbb{R}_+$ a fuzzy measure.

Definition 3.1 ([14, 15, 16]). 1) *The Choquet integral* of the function f with respect to the fuzzy measure m on $A \in \mathcal{A}$ is defined by

$$(2) \quad (C) \int_A f dm = \int_0^\infty m(\{s : f(s) \geq t\} \cap A) dt,$$

where the integral on the right-hand is the Lebesgue integral.

2) The function f is called *Choquet integrable on A* if $(C) \int_A f dm < \infty$.

If $A = S$ then we write $(C) \int_S f dm$ or $(C) \int f dm$. We denote by $\mathcal{L}_C(m, A)$ the set of all Choquet integrable functions on $A \in \mathcal{A}$ and by $\mathcal{L}_C(m, S)$ (or *shortly* $\mathcal{L}_C(m)$) the set of all Choquet integrable functions on S .

We mention the following properties for Choquet integral defined by (2):

Theorem 3.2 ([14, 15, 16]). *Let be $m : \mathcal{A} \rightarrow \mathbb{R}_+$ a fuzzy measure and $f, g : S \rightarrow \mathbb{R}_+$ Choquet integrable functions on S . The following statements hold:*

- i) *if $f = 0$ then $(C) \int f dm = 0$;*
- ii) *for every $A \in \mathcal{A}$, $(C) \int \chi_A dm = m(A)$, where χ_A is the characteristic function of the set A ;*
- iii) *for any $\alpha \geq 0$ we have $(C) \int \alpha f dm = \alpha (C) \int f dm$;*
- iv) *if $\alpha > 0$, αm is a fuzzy measure and $(C) \int f d(\alpha m) = \alpha (C) \int f dm$;*
- v) *if $m(S) < \infty$ then for every $\alpha \in \mathbb{R}_+$ we have $(C) \int (f + \alpha) dm = (C) \int f dm + \alpha m(S)$;*
- vi) *for every $A, B \in \mathcal{A}$ with $A \subset B$, f is Choquet integrable on A and B and we have $(C) \int_A f dm \leq (C) \int_B f dm$.*

Theorem 3.3 ([14, 15, 16]). *Let be $m : \mathcal{A} \rightarrow \mathbb{R}_+$ a fuzzy measure and $f, g : S \rightarrow \mathbb{R}_+$ Choquet integrable functions on S .*

If $f \leq g$ on S , then $(C) \int f dm \leq (C) \int g dm$.

Proposition 3.4 ([14]). *Let be $m : \mathcal{A} \rightarrow \mathbb{R}_+$ a fuzzy measure and $f, g : S \rightarrow \mathbb{R}_+$ Choquet integrable functions on S . Then the following statements hold:*

- i) *$(C) \int (f \vee g) dm \geq (C) \int f dm \vee (C) \int g dm$ and $(C) \int (f \wedge g) dm \leq (C) \int f dm \wedge (C) \int g dm$, where $(f \vee g)(s) = \max\{f(s), g(s)\}$ and $(f \wedge g)(s) = \min\{f(s), g(s)\}$, for all $s \in S$;*
- ii) *for every $A, B \in \mathcal{A}$ we have $(C) \int_{A \cap B} f dm \leq (C) \int_A f dm \wedge (C) \int_B f dm$ and $(C) \int_{A \cup B} f dm \geq (C) \int_A f dm \vee (C) \int_B f dm$.*

Proposition 3.5 ([12, 14]). *Let be $m, n : \mathcal{A} \rightarrow \mathbb{R}_+$ two fuzzy measures such that $m(S) = n(S)$ and $f : S \rightarrow \mathbb{R}_+$ a measurable function, Choquet integrable on S with respect to m and n . Then:*

$$i) (C) \int f d(m + n) = (C) \int f dm + (C) \int f dn;$$

$$ii) \text{ if } m \leq n \text{ then } (C) \int f dm \leq (C) \int f dn.$$

One of the most important property of Choquet integral is the comonotonic additivity that is the Choquet integral is additive for functions which are comonotone.

Theorem 3.6 ([16]). *Let be $m : \mathcal{A} \rightarrow \mathbb{R}_+$ a fuzzy measure and $f, g : S \rightarrow \mathbb{R}_+$ measurable functions Choquet integrable on S such that f, g are comonotonic on S . Then $(C) \int (f + g) dm = (C) \int f dm + (C) \int g dm$.*

4. A multi-valued Choquet integral with respect to a multi-submeasure

In [10], JANG ET AL. introduced a multi-valued Choquet integral for a multifunction taking values in the class of all closed, nonempty sets of $\mathbb{R}_+$ with respect to a nonnegative fuzzy measure, for which the selectors are real Choquet integrals and in [6], [7] JANG presented some properties of the set-valued Choquet integral for closed bounded interval-valued multifunctions.

In this section, we introduce and study another multivalued Choquet integral, this time for non-negative functions with respect to multisubmeasures taking values in $\mathcal{P}_{kc}(\mathbb{R}_+)$.

In what follows we consider that $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ is a multisubmeasure.

From Remark 2.20 we have that $\mu(A) = [m_1(A), m_2(A)]$, for every $A \in \mathcal{A}$, and $S_\mu \neq \emptyset$ because contains the submeasures m_1 and m_2 . Let be $f : S \rightarrow \mathbb{R}_+$ a measurable function. We denote by $S_\mu^C(f)$ or S_μ^C the set of all fuzzy measures $m : \mathcal{A} \rightarrow \mathbb{R}_+$ which are selectors of the multisubmeasure μ with respect to which the function f is Choquet integrable. Using a similar procedure as in [19] for Aumann-Gould integral we give:

Definition 4.1. Let be $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure, $f : S \rightarrow \mathbb{R}_+$ a measurable function and $A \in \mathcal{A}$.

1) *The Aumann-Choquet integral (shortly (AC)-integral) of f on A with respect to the multisubmeasure μ is the set*

$$(3) \quad (AC) \int_A f d\mu = \{(C) \int_A f dm, m \in S_\mu^C(f)\},$$

where $(C) \int_A f d\mu$ is the Choquet integral defined by (2).

2) The function f is said to be *Aumann-Choquet integrable on A* if $(AC) \int_A f d\mu \neq \emptyset$.

If $A = S$ we denote the integral by $(AC) \int_S f d\mu$ or simply $(AC) \int f d\mu$.

We also denote by $\mathcal{L}_{AC}(\mu, A)$ the set of all Aumann-Choquet integrable functions with respect to μ on A and by $\mathcal{L}_{AC}(\mu, S)$ or $\mathcal{L}_{AC}(\mu)$ the set of all Aumann-Choquet integrable functions with respect to μ on S .

Proposition 4.2. *Let be $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f : S \rightarrow \mathbb{R}_+$ a measurable function such that $f \in \mathcal{L}_C(m_2)$, where m_2 is defined by (1). Then the function f is AC-integrable on every $A \in \mathcal{A}$.*

Proof. From Remark 2.20 we have that $\mu(A) = [m_1(A), m_2(A)]$, for every $A \in \mathcal{A}$, and since $f \in \mathcal{L}_C(m_2)$, then $f \in \mathcal{L}_C(m_2, A)$. Hence, from Definition 3.1,2), we have that $(C) \int_A f d\mu_2 < \infty$. Then, for every $m \in S_\mu(f)$ we have that $m(A) \leq m_2(A)$, for every $A \in \mathcal{A}$. From Theorem 3.5, ii) we obtain $(C) \int_A f d\mu < (C) \int_A f d\mu_2 < \infty$, which assures that $f \in \mathcal{L}_{AC}(\mu, A)$. $\square$

Proposition 4.3. *Let be $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f : S \rightarrow \mathbb{R}_+$ an AC-integrable function on S . Then $S_\mu^C(f)$ is a convex set.*

Proof. Let be $S_\mu^C(f) = \{m \in S_\mu; (C) \int f d\mu < \infty\} \neq \emptyset$ and $m, n \in S_\mu^C(f)$. Let be $\alpha \in (0, 1)$. Since $m \in S_\mu^C(f)$ we have $(C) \int f d\mu < \infty$, and from $n \in S_\mu^C(f)$ we have $(C) \int f d\mu < \infty$. If we denote by $\nu = \alpha m + (1 - \alpha)n$ then, evidently, $\nu \in S_\mu$. According to Theorems 3.2, iv) and 3.5 we obtain $(AC) \int f d\nu = (C) \int f d(\alpha m + (1 - \alpha)n) = \alpha(C) \int f d\mu + (1 - \alpha)(C) \int f d\mu < \infty$ and, consequently $\nu = \alpha m + (1 - \alpha)n \in S_\mu^C(f)$ and hence $S_\mu^C(f)$ is a convex set. $\square$

In the following, we point out some immediate properties of the Aumann-Choquet integral. $\square$

Proposition 4.4. *Let be $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f : S \rightarrow \mathbb{R}_+$ a measurable function. Then:*

- i) if $f=0$ on S then f is AC integrable on S and $(AC) \int f d\mu = \{0\}$;
- ii) if $\mu(A) = \{0\}$ then $(AC) \int_A f d\mu = \{0\}$.

Proof. i) Since $f = 0$ on S then for every $m \in S_\mu^C(f)$ and all $t \geq 0$ we have from Theorem 3.2, i) that $(C) \int f dm = \int_0^\infty m(\{s : f(s) \geq t\}) dt = 0$ hence $(AC) \int f d\mu = \{0\}$.

For ii) let us observe that since $\mu(A) = \{0\}$ then from Remark 2.20 and $\mu(A) \in P_{kc}(\mathbb{R}_+)$ we have that $\mu(A) = [m_1(A), m_2(A)] = \{0\}$ hence $m_1(A) = m_2(A) = 0$. Then, for every $m \in S_\mu^C$, we have $m(A) = 0$. Hence, $(C) \int_A f dm = \int_0^\infty m(\{s : f(s) \geq t\} \cap A) dt = 0$ and we have that $(AC) \int_A f d\mu = \{0\}$. $\square$

Proposition 4.5. Let be $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f, g : S \rightarrow \mathbb{R}_+$ measurable functions, (AC) -integrable with respect to μ on S , such that $f = g(\mu - a.e.)$. Then $(AC) \int f d\mu = (AC) \int g d\mu$.

Proof. Since $f, g \in \mathcal{L}_{AC}(\mu)$ then $(AC) \int f d\mu = \{(C) \int f dm, m \in S_\mu^C(f)\} \neq \emptyset$ and $(AC) \int g d\mu = \{(C) \int g dm, m \in S_\mu^C(g)\} \neq \emptyset$. Let be any $x_0 \in (AC) \int f d\mu$. Then there exists $m \in S_\mu^C(f)$ such that $x_0 = (C) \int f dm$. Since $f = g(\mu - a.e.)$ then there exists $N \in \mathcal{A}$ with $\mu(N) = \{0\}$ such that $f(s) = g(s)$, for every $s \in cN$ and at the same time we have $f = g(\mu - a.e.)$.

Thus $x_0 = (C) \int_{cN} f dm = (C) \int_{cN} g dm$ and consequently $(C) \int g dm \in (AC) \int g d\mu$. Hence $(AC) \int f d\mu \subseteq (AC) \int g d\mu$.

Analogously, we can prove that $(AC) \int g d\mu \subseteq (AC) \int f d\mu$. $\square$

Proposition 4.6. Let be $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f : S \rightarrow \mathbb{R}_+$ an AC -integrable function on S . Then f is Aumann-Choquet integrable on every $E \in \mathcal{A}$.

Proof. Since $f \in \mathcal{L}_{AC}(\mu)$ we have that

$(AC) \int f d\mu = \{(C) \int f dm, m \in S_\mu^C(f)\} \neq \emptyset$. Let be any $y_0 \in (AC) \int f d\mu$. Then there exists $m \in S_\mu^C(f)$ such that $y_0 = (C) \int f dm < \infty$. Let be any $E \in \mathcal{A}$. We have, for all $t \geq 0$, that $\{s : f(s) \geq t\} \cap E \subset \{s : f(s) \geq t\}$ and since $m : \mathcal{A} \rightarrow \mathbb{R}_+$ is a fuzzy measure selector of multisubmeasure μ we obtain that $m(\{s : f(s) \geq t\} \cap E) \leq m(\{s : f(s) \geq t\})$ and hence we have that $(C) \int_E f dm \leq (C) \int f dm < \infty$. Therefore $m \in S_\mu^C(f, E)$ and consequently, $(AC) \int_E f d\mu \neq \emptyset$, that is $f \in \mathcal{L}_{AC}(\mu, E)$. $\square$

Theorem 4.7. Let be $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure, $f : S \rightarrow \mathbb{R}_+$ an AC -integrable function with respect to μ on S and $\alpha > 0$. The following statements hold:

- i) αf is AC -integrable with respect to μ on S and $(AC) \int (\alpha f) d\mu = \alpha (AC) \int f d\mu$;

ii) f is AC-integrable with respect to $\alpha\mu$ on S and $(AC) \int f d(\alpha\mu) = a(AC) \int f d\mu$.

Proof. i) From AC-integrability of f on S we have that $(AC) \int f d\mu = \{(C) \int f dm, m \in S_\mu^C(f)\} \neq \emptyset$. Let be $y_0 \in (AC) \int f d\mu$. Then there exists $m \in S_\mu^C(f)$ such that $y_0 = (C) \int f dm < \infty$.

From Theorem 3.2, iii) we have that $(C) \int (\alpha f) dm = \alpha(C) \int f dm = \alpha y_0$. Then $m \in S_\mu^C(\alpha f)$ and we obtain $(AC) \int (\alpha f) d\mu = \{(C) \int (\alpha f) dm, m \in S_\mu^C(\alpha f)\} = \{\alpha(C) \int f dm, m \in S_\mu^C(f)\} = \{\alpha y, y \in (AC) \int f d\mu\} = a(AC) \int f d\mu$. Hence $(AC) \int (\alpha f) d\mu = a(AC) \int f d\mu$.

In order to prove ii), let us observe that we have $(AC) \int f d(\alpha\mu) = \{(C) \int f dm, m \in S_{\alpha\mu}^C(f)\}$.

Since f is AC-integrable, for any $z_0 \in (AC) \int f d\mu$ there exists $n \in S_\mu^C(f)$ such that $z_0 = (C) \int f dn < \infty$. From Theorem 3.2, iv) we obtain that $(C) \int f d(\alpha n) = \alpha(C) \int f dn < \infty$. Hence $\alpha n = m' \in S_{\alpha\mu}^C(f)$ and $(C) \int f dm' < \infty$. Then $f \in \mathcal{L}_{AC}(\alpha\mu)$ and $(AC) \int f d(\alpha\mu) = \{(C) \int f dm', m' \in S_{\alpha\mu}^C(f)\} = \{(C) \int f d(\alpha n), n \in S_\mu^C(f)\} = \{\alpha(C) \int f dn, n \in S_\mu^C(f)\} = a(AC) \int f d\mu$. $\square$

Theorem 4.8. Let be $\mu, \nu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ two multisubmeasures and $f : S \rightarrow \mathbb{R}_+$ a function such that $f \in \mathcal{L}_{AC}(\mu) \cap \mathcal{L}_{AC}(\nu)$. Then, for every $\alpha, \beta > 0$, f is AC-integrable with respect to $\alpha\mu + \beta\nu$ on S and $(AC) \int f d(\alpha\mu + \beta\nu) = a(AC) \int f d\mu + \beta(AC) \int f d\nu$.

Proof. Let be any $\alpha, \beta > 0$. From AC-integrability of f with respect to μ , respectively to ν , we have $(AC) \int f d\mu \neq \emptyset$ and $(AC) \int f d\nu \neq \emptyset$. From Theorem 4.7, ii) we have that $(AC) \int f d(\alpha\mu) = a(AC) \int f d\mu$, respectively $(AC) \int f d(\beta\nu) = \beta(AC) \int f d\nu$.

Let be $x_1 \in (AC) \int f d(\alpha\mu)$ and $x_2 \in (AC) \int f d(\beta\nu)$, arbitrarily chosen. Then there exists $m' \in S_{\alpha\mu}^C(f)$ and $n' \in S_{\beta\nu}^C(f)$ such that $x_1 = (C) \int f dm'$ and $x_2 = (C) \int f dn'$. Since $m' \in S_{\alpha\mu}^C(f)$ and $n' \in S_{\beta\nu}^C(f)$ then we obtain that $m' = \alpha m$, where $m \in S_\mu^C(f)$ and, respectively, $n' = \beta n$, where $n \in S_\nu^C(f)$.

Using Theorem 3.2, iv) and Theorem 3.5, i) we have that $m' + n' \in S_{\alpha\mu + \beta\nu}^C(f)$ and $(C) \int f d(m' + n') = (C) \int f d(\alpha m + \beta n) = (C) \int f d(\alpha m) + (C) \int f d(\beta n) = x_1 + x_2$, hence $f \in \mathcal{L}_{AC}(\alpha\mu + \beta\nu)$.

Moreover, using Theorems 3.2 and 3.5 we obtain that $(C) \int f d(m' + n') = (C) \int f d(\alpha m + \beta n) = \alpha(C) \int f dm + \beta(C) \int f dn \in a(AC) \int f d\mu +$

$\beta(AC) \int f dv$, hence $(AC) \int f d(\alpha\mu + \beta\nu) \subset a(AC) \int f d\mu + \beta(AC) \int f dv$. Conversely, let be $z \in a(AC) \int f d\mu + \beta(AC) \int f dv$. Then we can write $z = z_1 + z_2$, where $z_1 \in a(AC) \int f d\mu$ and $z_2 \in \beta(AC) \int f dv$. Hence there exists $m'_1 \in S_\mu^C(f)$ and $n'_1 \in S_\nu^C(f)$ such that $z_1 = a(C) \int f dm'_1$, and $z_2 = \beta(C) \int f dn'_1$. From properties of real Choquet integral (Theorems 3.2, iv) and 3.5, i)) we obtain that

$$\begin{aligned} z &= z_1 + z_2 = (C) \int f d(\alpha m'_1) + (C) \int f d(\beta n'_1) \\ &= \int_0^\infty (\alpha m'_1)(\{s : f(s) \geq t\}) dt + \int_0^\infty (\beta n'_1)(\{s : f(s) \geq t\}) dt \\ &= \int_0^\infty (\alpha m'_1 + \beta n'_1)(\{s : f(s) \geq t\}) dt = (C) \int f d(\alpha m'_1 + \beta n'_1) \end{aligned}$$

and hence $z \in (AC) \int f d(\alpha\mu + \beta\nu)$. Consequently we have that $(AC) \int f d(\alpha\mu + \beta\nu) = a(AC) \int f d\mu + \beta(AC) \int f dv$. $\square$

Theorem 4.9. *Let be $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f, g : S \rightarrow \mathbb{R}_+$ two AC-integrable functions with respect to μ on S . If f and g are comonotonic functions, then $f + g$ is AC integrable and $(AC) \int (f + g) d\mu \subseteq (AC) \int f d\mu + (AC) \int g d\mu$.*

Proof. Since f and g are AC-integrable functions we have that $(AC) \int f d\mu = \{(C) \int f dm, m \in S_\mu^C(f)\} \neq \emptyset$ and $(AC) \int g d\mu = \{(C) \int g dn, n \in S_\mu^C(g)\} \neq \emptyset$. Firstly, we observe that $S_\mu^C(f + g) = S_\mu^C(f) \cap S_\mu^C(g)$.

Indeed, $S_\mu^C(f + g) = \{m \in S_\mu^C : (C) \int (f + g) dm < \infty\}$. But, since f and g are comonotonic functions then $(C) \int (f + g) dm = (C) \int f dm + (C) \int g dm < \infty$.

Hence $(C) \int f dm < \infty$ and $(C) \int g dm < \infty$ and then $m \in S_\mu^C(f) \cap S_\mu^C(g)$. Thus $S_\mu^C(f + g) \subseteq S_\mu^C(f) \cap S_\mu^C(g)$.

Analogously, we can prove the conversely inclusion.

Now, let be any $y_0 \in (AC) \int (f + g) d\mu$. Then there exists $m \in S_\mu^C(f + g)$ such that we have $y_0 = (C) \int (f + g) dm$. Since f and g are comonotonic functions then using Theorem 3.6 we obtain that $y_0 = (C) \int (f + g) dm = (C) \int f dm + (C) \int g dm = y_1 + y_2$, where $y_1 = (C) \int f dm \in (AC) \int f d\mu$ and $y_2 = (C) \int g dm \in (AC) \int g d\mu$.

Hence, we have that $(AC) \int (f + g) d\mu \subseteq (AC) \int f d\mu + (AC) \int g d\mu$. $\square$

Proposition 4.10. *Let be $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f : S \rightarrow \mathbb{R}_+$ an AC-integrable function with respect to μ on S . Then $(AC) \int_S f d\mu$ is a convex set.*

Proof. Since f is AC integrable with respect to μ on S we have $(AC) \int_S f d\mu \neq \emptyset$. Let be $y_1, y_2 \in (AC) \int_S f d\mu$, arbitrarily chosen, such that $y_1 < y_2$. Then there exists $m, n \in S_\mu^C(f)$ such that $y_1 = (C) \int f dm < \infty$ and $y_2 = (C) \int f dn < \infty$. From Proposition 4.3, for every $\alpha \in (0, 1)$ we have that $m' = \alpha m + (1 - \alpha)n \in S_\mu^C(f)$ and from Theorems 3.2, iv) and 3.3 we obtain

$$\begin{aligned} y &= (C) \int f dm' = (C) \int f d(\alpha m + (1 - \alpha)n) \\ &= (C) \int f d(\alpha m) + (C) \int f d((1 - \alpha)n) \\ &= \alpha(C) \int f dm + (1 - \alpha)(C) \int f dn = \alpha y_1 + (1 - \alpha)y_2 < \infty. \end{aligned}$$

Then $y = \alpha y_1 + (1 - \alpha)y_2 \in (y_1, y_2) \subset (AC) \int_S f d\mu$ and hence $(AC) \int_S f d\mu$ is a convex set. $\square$

Theorem 4.11. Let be $\mu: \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f: S \rightarrow \mathbb{R}_+$ a measurable function. If $f \in \mathcal{L}_C(m_2)$, where $m_2: \mathcal{A} \rightarrow \mathbb{R}_+$ is defined by (1), then for every $A \in \mathcal{A}$, $f \in \mathcal{L}_{AC}(\mu, A)$ and

$$(4) \quad (AC) \int_A f d\mu \subseteq [(C) \int_A f dm_1, (C) \int_A f dm_2].$$

Proof. Let be $A \in \mathcal{A}$ fixed. From Remark 2.20, since $\mu(A) = [m_1(A), m_2(A)]$ then for every $m: \mathcal{A} \rightarrow \mathbb{R}_+$ with $m \in S_\mu$ we have

$$(5) \quad m_1(A) \leq m(A) \leq m_2(A).$$

Since $f \in \mathcal{L}_C(m_2)$, then $f \in \mathcal{L}_C(m_2, A)$, and hence $(C) \int_A f dm_2 < \infty$. From (5) we have that $(C) \int_A f dm < \infty$ and hence $(AC) \int_A f d\mu \neq \emptyset$. Obviously, for every $m \in S_\mu^C(f)$ we have $\int_0^\infty m_1(\{s: f(s) \geq t\} \cap A) dt \leq \int_0^\infty m(\{s: f(s) \geq t\} \cap A) dt \leq \int_0^\infty m_2(\{s: f(s) \geq t\} \cap A) dt$ and from $\int_0^\infty m(\{s: f(s) \geq t\} \cap A) dt = (C) \int_A f dm \in (AC) \int_A f d\mu$ we obtain that $(AC) \int_A f d\mu \subseteq [(C) \int_A f dm_1, (C) \int_A f dm_2]$. $\square$

If $m: \mathcal{A} \rightarrow \mathbb{R}_+$ is a strict submeasure then we proved in [21], ex. 2.15 that $\mu: \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ defined by $\mu(A) = [0, m(A)]$ is a strict multisubmeasure, called the strict multisubmeasure induced by m .

Now, we can easily obtain the following:

Corollary 4.12. *Let be $m:A \rightarrow \mathbb{R}_+$ a strict submeasure, $\mu:A \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ the strict multisubmeasure induced by the submeasure m and $f:S \rightarrow \mathbb{R}_+$ a measurable function.*

If $f \in L_C(m)$, then $f \in L_{AC}(\mu)$ and $(AC) \int_A f d\mu \subseteq [0, (C) \int_A f dm]$, for every $A \in \mathcal{A}$.

Proposition 4.13. *Let be $\mu:A \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f, g:S \rightarrow \mathbb{R}_+$ measurable functions such that $f, g \in \mathcal{L}_C(m_2)$, where m_2 is defined by (1). If $f \leq g$ on S then, for every $A \in \mathcal{A}$, we have that $(AC) \int_A f d\mu \lesssim (AC) \int_A g d\mu$.*

Proof. Since $f \in \mathcal{L}_C(m_2)$ from Theorem 4.11 we have that, for every $A \in \mathcal{A}$, $f \in \mathcal{L}_{AC}(\mu, A)$ and

$$(6) \quad (AC) \int_A f d\mu \subseteq [(C) \int_A f dm_1, (C) \int_A f dm_2].$$

Analogously we obtain from $g \in \mathcal{L}_C(m_2)$ that $g \in \mathcal{L}_{AC}(\mu, A)$ and

$$(7) \quad (AC) \int_A g d\mu \subseteq [(C) \int_A g dm_1, (C) \int_A g dm_2].$$

Since $f \leq g$ on S we obtain from Theorem 3.3 that for every $A \in \mathcal{A}$ $(C) \int_A f dm_1 \leq (C) \int_A g dm_1$ and $(C) \int_A f dm_2 \leq (C) \int_A g dm_2$. Then from Proposition 2.3 we have that $[(C) \int_A f dm_1, (C) \int_A f dm_2] \lesssim [(C) \int_A g dm_1, (C) \int_A g dm_2]$.

Using Lemma 2.21 we obtain that $(AC) \int_A f d\mu \lesssim (AC) \int_A g d\mu$, for every $A \in \mathcal{A}$. $\square$

Theorem 4.14. *Let be $\mu:A \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f:S \rightarrow \mathbb{R}_+$ a measurable functions such that $f \in \mathcal{L}_C(m_2)$, where m_2 is defined by (1). Then, for every $A, B \in \mathcal{A}$ with $A \subseteq B$ we have $(AC) \int_A f d\mu \lesssim (AC) \int_B f d\mu$.*

Proof. Since $f \in \mathcal{L}_C(m_2)$ then for $A, B \in \mathcal{A}$ with $A \subseteq B$ we have that $f \in \mathcal{L}_C(m_2, A)$, respectively $f \in \mathcal{L}_C(m_2, B)$. From $(C) \int_A f dm_1 \leq (C) \int_A f dm_2 < \infty$, respectively from $(C) \int_B f dm_1 \leq (C) \int_B f dm_2 < \infty$, we obtain that $f \in \mathcal{L}_C(m_1, A)$ and $f \in \mathcal{L}_C(m_1, B)$. Then, since $f \in \mathcal{L}_C(m_1, A) \cap \mathcal{L}_C(m_2, A)$ and $f \in \mathcal{L}_C(m_1, B) \cap \mathcal{L}_C(m_2, B)$ we have, from Theorem 4.11, that $f \in \mathcal{L}_{AC}(\mu, A)$ and $f \in \mathcal{L}_{AC}(\mu, B)$ and

$$(8) \quad (AC) \int_A f d\mu \subseteq [(C) \int_A f dm_1, (C) \int_A f dm_2],$$

respectively

$$(9) \quad (AC) \int_B f d\mu \subseteq [(C) \int_B f dm_1, (C) \int_B f dm_2].$$

Since $A \subseteq B$, from Theorem 3.2, vi) we obtain that $(C) \int_A f dm_1 \leq (C) \int_B f dm_1$ and $(C) \int_A f dm_2 \leq (C) \int_B f dm_2$.

From Proposition 2.3 we easily see that

$$(10) \quad [(C) \int_A f dm_1, (C) \int_A f dm_2] \lesssim [(C) \int_B f dm_1, (C) \int_B f dm_2].$$

Using (8), (9), (10) and Lemma 2.21 we obtain that for every $A, B \in \mathcal{A}$ with $A \subseteq B$ we have $(AC) \int_A f d\mu \lesssim (AC) \int_B f d\mu$. $\square$

Proposition 4.15. *Let be $\mu, \nu: \mathcal{A} \rightarrow P_{kc}(\mathbb{R}_+)$ multisubmeasures such that $\mu(A) = [m_1(A), m_2(A)]$ and $\nu(A) = [n_1(A), n_2(A)]$, for every $A \in \mathcal{A}$, where m_1, m_2, n_1, n_2 are defined by (1). If $f: S \rightarrow \mathbb{R}_+$ is a measurable function such that $f \in L_C(m_2)$ and $f \in L_C(n_2)$, then the following statements hold:*

- i) *if $f \in \mathcal{L}_C(m_2 \vee n_2)$ then $(AC) \int_A f d\mu \cup (AC) \int_A f d\nu \lesssim (AC) \int_A f d(\mu \vee \nu)$, for every $A \in \mathcal{A}$;*
- ii) *if $f \in \mathcal{L}_C(m_2 \wedge n_2)$ then $(AC) \int_A f d(\mu \wedge \nu) \lesssim (AC) \int_A f d\mu \cap (AC) \int_A f d\nu$, for every $A \in \mathcal{A}$.*

Proof. Firstly, let us observe that since for every $A \in \mathcal{A}$, $\mu(A) = [m_1(A), m_2(A)]$, $\nu(A) = [n_1(A), n_2(A)]$ we have from Definition 2.22 that $(\mu \vee \nu)(A) = \mu(A) \vee \nu(A) = [m_1(A) \vee n_1(A), m_2(A) \vee n_2(A)]$, respectively $(\mu \wedge \nu)(A) = \mu(A) \wedge \nu(A) = [m_1(A) \wedge n_1(A), m_2(A) \wedge n_2(A)]$.

Let be $A \in \mathcal{A}$ arbitrarily chosen. To prove i) let us observe that since $f \in \mathcal{L}_C(m_2)$ from Theorem 4.11 we have $f \in \mathcal{L}_{AC}(\mu, A)$ and $(AC) \int_A f d\mu \subseteq [(C) \int_A f dm_1, (C) \int_A f dm_2]$. Analogously from $f \in \mathcal{L}_C(n_2)$ we obtain that $f \in \mathcal{L}_{AC}(\nu, A)$ and $(AC) \int_A f d\nu \subseteq [(C) \int_A f dn_1, (C) \int_A f dn_2]$, for every $A \in \mathcal{A}$. Then $(AC) \int_A f d\mu \cup (AC) \int_A f d\nu \subseteq [(C) \int_A f dm_1 \wedge (C) \int_A f dn_1, (C) \int_A f dm_2 \vee (C) \int_A f dn_2]$.

Let be any $y_0 \in (AC) \int_A f d(\mu \vee \nu)$. Then there exists $m \in S_{\mu \vee \nu}^C(f)$ such that $y_0 = (C) \int_A f dm$.

Since $m \in S_{\mu \vee \nu}^C(f)$ we have $m_1 \vee n_1 \leq m \leq m_2 \vee n_2$.

Because $m_1 \leq m_1 \vee n_1$, $m_1 \in S_\mu^C(f)$ we obtain

$$(11) \quad x_0 = (C) \int_A f dm_1 \in (AC) \int_A f d\mu \cup (AC) \int_A f d\nu$$

and from Theorem 3.5, iii) we have $x_0 = (C) \int_A f dm_1 \leq (C) \int_A f dm = y_0$.

Hence, the condition i) from Definition 2.2 is true. Analogously we can prove ii) from Definition 2.2 and so we have that $(AC) \int_A f d\mu \cup (AC) \int_A f d\nu \lesssim (AC) \int_A f d(\mu \vee \nu)$, for every $A \in \mathcal{A}$, and hence i) is true.

Analogously, we can prove ii). $\square$

5. Some properties of the multifunction defined as an Aumann-Choquet integral

In this section we shall prove that for $\mu: \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f: S \rightarrow \mathbb{R}_+$ an AC-integrable function if we impose supplementary conditions to μ , the same properties are also preserved by the multifunction I defined by

$$(12) \quad I(E) = (AC) \int_E f d\mu, E \in \mathcal{A}.$$

From Theorem 4.15 we can easily obtain:

Proposition 5.1. *The multifunction I defined by (12) is a monotone set multifunction.*

Proposition 5.2. *Let be $\mu: \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f: S \rightarrow \mathbb{R}_+$ a function such that $f \in \mathcal{L}_{AC}(\mu)$. If μ is weakly null-additive, then the multifunction defined by (12) is weakly null-additive.*

Proof. Since μ is a weakly null-additive multisubmeasure then for every $A, B \in \mathcal{A}$ with $\mu(A) = \mu(B) = \{0\}$ we have $\mu(A \cup B) = \{0\}$.

Let be $A, B \in \mathcal{A}$ with $I(A) = I(B) = \{0\}$. Firstly, let us observe that every fuzzy measure selector m of μ is weakly null-additive.

Indeed, since $\mu(A) = [m_1(A), m_2(A)]$, where m_1, m_2 are defined by (1) then, for every $A, B \in \mathcal{A}$ with $\mu(A) = \mu(B) = \{0\}$ we have $[m_1(A), m_2(A)] = \{0\}$ and $[m_1(B), m_2(B)] = \{0\}$.

Hence $m_1(A)=0=m_2(A)$ and $m_1(B)=m_2(B)=0$ and for every $m \in S_\mu$ we have $m(A) = m(B) = 0$. From $\mu(A \cup B) = [m_1(A \cup B), m_2(A \cup B)] = \{0\}$ we easily obtain that $m(A \cup B) = 0$. Consequently, m is null-additive.

Since $f \in \mathcal{L}_{AC}(\mu)$ we have from Proposition 4.6 that $I(A \cup B) = (AC) \int_{A \cup B} f d\mu = \{(C) \int_{A \cup B} f dm, m \in S_\mu^C(f)\} \neq \emptyset$.

Let be any $y \in I(A \cup B)$. Then there exists $m \in S_\mu^C(f)$, weakly null-additive, such that $y = (C) \int_{A \cup B} f dm$.

Hence, from [24], th.7 we obtain that $y = (C) \int_{A \cup B} f dm = 0$ and so $I(A \cup B) = \{0\}$, which assures that $I(\cdot)$ is weakly null-additive. $\square$

Proposition 5.3. *Let be $\mu: \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f: S \rightarrow \mathbb{R}_+$ a function such that $f \in \mathcal{L}_{AC}(\mu)$. If μ is a null-additive multisubmeasure then the multifunction defined by (12) is null-additive.*

Proof. We shall prove that if $A, B \in \mathcal{A}$ and $I(A) = \{0\}$ we obtain $I(A \cup B) = I(B)$. As in the proof of Proposition 5.2, we can show that if μ is null-additive then every $m \in S_\mu$ is a null-additive fuzzy measure. From $S_\mu^C \subseteq S_\mu$ we have that every $m \in S_\mu^C$ is null-additive.

Now, let be $x_0 \in I(A \cup B)$ arbitrarily chosen. Then there exists $m' \in S_\mu^C$, null-additive, such that $x_0 = (C) \int_{A \cup B} f dm'$.

Moreover, since m' is null-additive we obtain $x_0 = (C) \int_{A \cup B} f dm' = \int_0^\infty m'(\{s : f(s) \geq t\} \cap (A \cup B)) dt = \int_0^\infty m'(\{s : f(s) \geq t\} \cap B) dt = (C) \int_B f dm' \in I(B)$, which assures that $I(A \cup B) \subseteq I(B)$.

Analogously, we can prove the converse inclusion. Indeed, if $y_0 \in I(B)$ is arbitrarily chosen then there exists $m'' \in S_\mu^C$, null-additive such that $y_0 = (C) \int_B f dm''$. Since m'' is null-additive then we have that $y_0 = (C) \int_B f dm'' = \int_0^\infty m''(\{s : f(s) \geq t\} \cap B) dt = \int_0^\infty m''(\{s : f(s) \geq t\} \cap (A \cup B)) dt = (C) \int_{A \cup B} f dm''$, hence $y_0 \in I(A \cup B)$ and the proof is completed. $\square$

In what follows we shall prove that if we impose supplementary conditions for the multisubmeasure μ , such as converse null-additivity, pseudo generating property or Darboux property, then they are also preserved by the corresponding Aumann-Choquet integral.

Indeed we have:

Proposition 5.4. *Let be $\mu: \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure such that every $m \in S_\mu$ is converse null-additive and $f: S \rightarrow \mathbb{R}_+$ an AC-integrable function with respect to μ on S . Then the multifunction defined by (12) is converse null-additive.*

Proof. To prove that I is converse null-additive let be any $A, B \in \mathcal{A}$ with $A \subset B$ and $I(A) = I(B)$.

Firstly, we prove that μ is converse null-additive.

Indeed, since every $m \in S_\mu$ is a converse null-additive fuzzy measure, from $S_\mu^C(f) \subseteq S_\mu$ we obtain that every $m \in S_\mu^C(f)$ is converse null-additive. Since from (1) we have $\mu(E) = [m_1(E), m_2(E)]$, for every $E \in \mathcal{A}$, where m_1, m_2 are converse null-additive, if $\mu(A) = \mu(B)$ then $[m_1(A), m_2(A)] = [m_1(B), m_2(B)]$ and thus $m_i(A) = m_i(B), i = 1, 2$. Because $m_i, i = 1, 2$, are converse null-additive we have $m_1(B - A) = 0 = m_2(B - A)$, and consequently $\mu(B - A) = \{0\}$ which assures that μ is a converse null-additive multisubmeasure.

From $f \in \mathcal{L}_{AC}(\mu)$ and Proposition 4.6 we have that $f \in \mathcal{L}_{AC}(\mu, B - A)$ and hence $I(B - A) = (AC) \int_{B-A} f d\mu = \{(C) \int_{B-A} f dm, m \in S_\mu^C(f)\} \neq \emptyset$. From [24], th.8, since m is a converse null-additive fuzzy measure then the set function defined by $\nu(A) = (C) \int_A f dm, A \in \mathcal{A}$, is converse null-additive. Hence $(C) \int_{B-A} f dm = 0$, for every $m \in S_\mu^C(f)$, and we have that $I(B - A) = (AC) \int_{B-A} f d\mu = \{0\}$ and consequently I is converse null-additive. $\square$

Proposition 5.5. *Let be $\mu: \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f: S \rightarrow \mathbb{R}_+$ an AC-integrable function with respect to μ on S . If μ has the pseudometric generating property then the same is true for the multifunction defined by (12).*

Proof. Firstly, let us observe that since μ has p.g.p then every fuzzy measure $m \in S_\mu$ has p.g.p. Indeed, since μ has p.g.p, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $A, B \in \mathcal{A}$ with $\mu(A) \subseteq [0, \delta]$ and $\mu(B) \subseteq [0, \delta]$ we have $\mu(A \cup B) \subseteq [0, \varepsilon]$. From (1) we have that $\mu(A) = [m_1(A), m_2(A)] \subseteq [0, \delta]$ and $\mu(B) = [m_1(B), m_2(B)] \subseteq [0, \delta]$ and for every $m \in S_\mu$ we have $m(A) \leq m_2(A) \leq \delta$ and $m(B) \leq m_2(B) \leq \delta$. From $\mu(A \cup B) = [m_1(A \cup B), m_2(A \cup B)] \subseteq [0, \varepsilon]$ we obtain that $m(A \cup B) \leq m_2(A \cup B) \leq \varepsilon$, hence $m \in S_\mu$ has p.g.p.

Since $S_\mu^C(f) \subseteq S_\mu$, then every $m \in S_\mu^C(f)$ has p.g.p. Now, for every $\varepsilon > 0$ we consider $\delta > 0$ from the definition of p.g.p. of μ and let be $A, B \in \mathcal{A}$ with $I(A) \subseteq [0, \delta]$ and $I(B) \subseteq [0, \delta]$. Let be any $y \in I(A \cup B)$, where $I(A \cup B) = (AC) \int_{A \cup B} f d\mu = \{(C) \int_{A \cup B} f dm, m \in S_\mu^C(f)\}$. Then there exists $m \in S_\mu^C(f)$ with p.g.p. such that $y = (C) \int_{A \cup B} f dm$ and from [20], th.5 the set function defined as Choquet integral has p.g.p. and hence we obtain that $y \leq \varepsilon$.

Hence $I(A \cup B) = (AC) \int_{A \cup B} f d\mu \subseteq [0, \varepsilon]$ and I has p.g.p. $\square$

Theorem 5.6. *Let be $\mu: \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f: S \rightarrow \mathbb{R}_+$ a function AC-integrable with respect to μ on S such that $f(s) \neq 0$,*

for every $s \in S$. If every $m \in S_\mu$ has the Darboux property, then the multifunction defined by (12) has the Darboux property.

Proof. Since $f \in \mathcal{L}_{AC}(\mu)$ we have from Proposition 4.6, that $I(A) = (AC) \int_A f d\mu \neq \emptyset$, for every $A \in \mathcal{A}$. Let be any $m \in S_\mu^C(f)$ with Darboux property.

Since $m(A) \neq 0$ and $f(s) \neq 0$, for every $s \in S$ we have $I(A) = (AC) \int_A f d\mu \not\subseteq \{0\}$. We shall prove that for any $p \in (0, 1)$ there exists $B \in \mathcal{A}, B \subset A$ such that $I(B) = pI(A)$.

Let be any $y_0 \in I(A)$. Then there exists $m' \in S_\mu^C(f)$ with Darboux property, such that $y_0 = (C) \int_A f dm' = \int_0^\infty m'(\{s : f(s) \geq t\} \cap A) dt$.

Since m' has Darboux property then for any $p \in (0, 1)$ there exists $B \subset A$ such that $m'(B) = pm'(A)$.

Thus, we have that $y_0 = \int_0^\infty m'(\{s : f(s) \geq t\} \cap A) dt = \int_0^\infty \frac{1}{p} m'(\{s : f(s) \geq t\} \cap B) dt$ and so $y_0 = \frac{1}{p} y'_0$, where $y'_0 \in I(B)$. Consequently, we have $pI(A) \subseteq I(B)$. It is easy to see that $I(B) \subseteq pI(A)$, hence the multifunction defined by (12) has the Darboux property. $\square$

In what follows, we prove some absolutely continuity properties for the multifunction I with respect to μ .

Proposition 5.7. *If $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ is a multisubmeasure and $f : S \rightarrow \mathbb{R}_+$ is an AC-integrable function with respect to μ on S then $I \ll_I \mu$, where $I(A) = (AC) \int_A f d\mu$, for every $A \in \mathcal{A}$.*

Proof. Let be any $A \in \mathcal{A}$ with $\mu(A) = \{0\}$. Then we have $\mu(A) = [m_1(A), m_2(A)] = \{0\}$ and, consequently, $m_1(A) = 0$ and $m_2(A) = 0$.

Then, for every $m \in S_\mu^C$, we have $m(A) = 0$ and we obtain that $(AC) \int_A f d\mu = \{(C) \int_A f dm, m \in S_\mu^C(f)\} = \{0\}$, hence $I \ll_I \mu$. $\square$

Remark 5.8. From Remark 2.17 and Proposition 5.3 we obtain that if μ is a null-additive multisubmeasure then $I \ll_I \mu \Leftrightarrow I \ll_{II} \mu$ where I is given by (12).

Hence we have:

Proposition 5.9. *If $\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ is a null-additive multisubmeasure and $f : S \rightarrow \mathbb{R}_+$ is an AC-integrable function with respect to μ on S then $I \ll_{II} \mu$, where $I(A) = (AC) \int_A f d\mu$, for every $A \in \mathcal{A}$.*

Denoting by γ the fuzzy measure defined by $\gamma(A) = |\mu(A)|$, for every $A \in \mathcal{A}$ we have:

Proposition 5.10. *If $\mu: \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ is a multisubmeasure and $f: S \rightarrow \mathbb{R}_+$ is a measurable function such that $f \in \mathcal{L}_C(\gamma)$, then $I \ll_{IV} \gamma$, where I is the multifunction given by $I(A) = (AC) \int_A f d\mu$, for every $A \in \mathcal{A}$.*

Proof. According to the Definition 2.16 we shall prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $A \in \mathcal{A}$ with $\gamma(A) < \delta$ we have $I(A) \subseteq [0, \varepsilon]$.

Let be $A \in \mathcal{A}$, arbitrarily chosen. From hypothesis we obtain that $f \in \mathcal{L}_C(m_2, A)$ and from Theorem 4.11, f is (AC)-integrable with respect to μ on A . From $f \in \mathcal{L}_C(m_2)$ we have $(C) \int f dm_2 = \int_0^\infty m_2(\{s : f(s) \geq t\}) dt < \infty$ and from [24], th.13, [4], th.4.4, for every $\varepsilon > 0$ there exists $0 < a < b$ such that

$$(13) \quad \int_0^a m_2(\{s : f(s) \geq t\}) dt + \int_b^\infty m_2(\{s : f(s) \geq t\}) dt < \frac{\varepsilon}{2}.$$

Let us consider $\delta = \frac{\varepsilon}{2(b-a)}$ and $\gamma(A) < \delta$. Then $\gamma(A) = |\mu(A)| = m_2(A) < \delta$. To prove that $I(A) \subset [0, \varepsilon]$ let be any $y \in I(A) = (AC) \int_A f d\mu = \{(C) \int_A f dm, m \in S_\mu^C(f)\}$. Then there exists $m \in S_\mu^C(f)$ such that $y = (C) \int_A f dm$ and $m(A) \leq m_2(A)$. We obtain

$$(14) \quad \begin{aligned} y &= (C) \int_A f dm = \int_0^\infty m(\{s : f(s) \geq t\} \cap A) dt \\ &\leq \int_0^\infty m_2(\{s : f(s) \geq t\} \cap A) dt. \end{aligned}$$

From $\int_0^\infty m_2(\{s : f(s) \geq t\} \cap A) dt = \int_0^a m_2(\{s : f(s) \geq t\} \cap A) dt + \int_a^b m_2(\{s : f(s) \geq t\} \cap A) dt + \int_b^\infty m_2(\{s : f(s) \geq t\} \cap A) dt$ and using (14) we have that $y = (C) \int_A f dm < \frac{\varepsilon}{2} + \int_a^b m_2(\{s : f(s) \geq t\} \cap A) dt < \frac{\varepsilon}{2} + \delta(b-a) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon$.

Hence, for every $y \in I(A)$ we have $y < \varepsilon$, that is $I(A) \subset [0, \varepsilon]$ and $I \ll_{IV} \gamma$. $\square$

Proposition 5.11. *Let be $\mu: \mathcal{A} \rightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure and $f: S \rightarrow \mathbb{R}_+$ a measurable function such that $f \in \mathcal{L}_C(\gamma)$. If the multisubmeasure μ is o-continuous then, $I \ll_{III} \gamma$ where I is defined by (12).*

Proof. We shall prove that for any $(A_n)_n \subset \mathcal{A}$ with $\gamma(A_n) \rightarrow 0$ we have $|I(A_n)| \rightarrow 0$. Since for every $A \in \mathcal{A}$ we have that $\mu(A) = [m_1(A), m_2(A)]$ then $\mu(A_n) = [m_1(A_n), m_2(A_n)]$. Because $\gamma(A_n) = |\mu(A_n)| = m_2(A_n) \rightarrow 0$

and since $m_1(A_n) \leq m_2(A_n)$, $\forall n \in \mathbb{N}$ we have that $m_1(A_n) \rightarrow 0$. From [24], th.3,4 we have that $\nu_1(A_n) \rightarrow 0$ and $\nu_2(A_n) \rightarrow 0$, where we denote by $\nu_i(A) = (C) \int_A f dm_i$, $i = 1, 2$.

Since $f \in \mathcal{L}_C(\gamma)$ we easily obtain that $f \in \mathcal{L}_C(m_1)$ and from Theorem 4.11 we have that $f \in \mathcal{L}_{AC}(\mu, A_n)$ and

$$(15) \quad (AC) \int_{A_n} f d\mu \subseteq [(C) \int_{A_n} f dm_1, (C) \int_{A_n} f dm_2], \text{ for every } n \in \mathbb{N}.$$

From (15) we have $h(I(A_n), \{0\}) \leq h([(C) \int_{A_n} f dm_1, (C) \int_{A_n} f dm_2], [0, 0]) = \max\{(C) \int_{A_n} f dm_1, (C) \int_{A_n} f dm_2\} = (C) \int_{A_n} f dm_2 \rightarrow 0$.

Hence $h(I(A_n), \{0\}) \rightarrow 0$ and we have $I \ll_{III} \gamma$. $\square$

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REFERENCES

1. AUMANN, R.J. – *Integrals of set-valued functions*, J. Math. Anal. Appl., 12 (1965), 1–12.
2. CHOQUET, G. – *Theory of capacities*, Ann. Inst. Fourier, Grenoble 5 (1953–1954), 131–295 (1955).
3. GUO, C.; ZHANG, D. – *On set-valued fuzzy measures*, Inform. Sci., 160 (2004), 13–25.
4. HONGXIA, S.; JUN, L.; YASUDA, M. – *Monotone set-valued functions defined by set-valued Choquet integrals*, Tenth International Conference on Fuzzy Set Theory and Applications, 2010.
5. HU, S.; PAPAGEORGIOU, N.S. – *Handbook of Multivalued Analysis*, Vol. I, Theory, Mathematics and its Applications, 419, Kluwer Academic Publishers, Dordrecht, 1997.
6. JANG, L.C. – *Interval-valued Choquet integrals and their applications*, J. Appl. Math. and Computing, 16 (2004), 429–443.
7. JANG, L.-C. – *A note on the monotone interval-valued set function defined by the interval-valued Choquet integral*, Commun. Korean Math. Soc., 22 (2007), 227–234.
8. JANG, L.C.; KIM, T.; JEON, J. – *On set-valued Choquet integrals and convergence theorems*, Adv. Stud. Contemp. Math. (Kyungshang), 6 (2003), 63–76.
9. JANG, L.C.; KIM, T.K.; JEON, J.D. – *On set-valued Choquet integrals and convergence theorems. II*, Bull. Korean Math. Soc., 40 (2003), 139–147.

10. JANG, L.C.; KWON, J.S. – *On the representation of Choquet integrals of set-valued functions, and null sets*, Fuzzy Sets and Systems, 112 (2000), 233–239.
11. JANG, L.C.; KIL, B.M.; KIM, Y.K.; KWON, J.S. – *Some properties of Choquet integrals of set-valued functions*, Fuzzy Sets and Systems, 91 (1997), 95–98.
12. KIM, H.M.; JANG, L.-C.; KIM, T. – *On closed set-valued Choquet integrals with respect to non-monotonic fuzzy measures*, Proc. Jangjeon Math. Soc., 7 (2004), 109–116.
13. KLEIN, E.; THOMPSON, A.C. – *Theory of Correspondences. Including Applications to Mathematical Economics*, Canadian Mathematical Society Series of Monographs and Advanced Texts, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1984.
14. MUROFUSHI, T.; SUGENO, M. – *A theory of fuzzy measures: representations, the Choquet integral, and null sets*, J. Math. Anal. Appl., 159 (1991), 532–549.
15. MUROFUSHI, T.; SUGENO, M. – *An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure*, Fuzzy Sets and Systems, 29 (1989), 201–227.
16. PAP, E. – *Null-Additive Set Functions*, Mathematics and its Applications, 337, Kluwer Academic Publishers Group, Dordrecht; Ister Science, Bratislava, 1995.
17. PRECUPANU, A.-M.; CROITORU, A. – *A Gould type integral with respect to a multimeasure. I*, An. Ştiinţ. Univ. “Al.I. Cuza” Iaşi. Mat. (N.S.), 48 (2002), 165–200 (2003).
18. PRECUPANU, A.-M.; CROITORU, A. – *A Gould type integral with respect to a multimeasure. II*, An. Ştiinţ. Univ. “Al.I. Cuza” Iaşi. Mat. (N.S.), 49 (2003), 183–207.
19. PRECUPANU, A.-M.; SATCO, B. – *The Aumann-Gould integral*, Mediterr. J. Math., 5 (2008), 429–441.
20. OUYANG, Y.; LI, J. – *Some properties of monotone set functions defined by Choquet integral*, J. Southeast Univ. (English Ed.), 19 (2003), 423–426.
21. SOFIAN-BOCA, F.-N. – *Another Gould type integral with respect to a multisubmeasure*, An. Ştiinţ. Univ. “Al.I. Cuza” Iaşi. Mat. (N.S.), 57 (2011), 13–30.
22. ZHANG, D.; GUO, C.; LIU, D. – *Set-valued Choquet integrals revisited*, Fuzzy Sets and Systems, 147 (2004), 475–485.
23. ZHANG, D.L.; WANG, Z.X. – *On set-valued fuzzy integrals*, Fuzzy Sets and Systems, 56 (1993), 237–241.
24. WANG, Z.; KLIR, G.J.; WANG, W. – *Monotone set functions defined by Choquet integral*, Fuzzy Sets and Systems, 81 (1996), 241–250.

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