A NEW APPLICATION OF ALMOST INCREASING SEQUENCES

BY

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Abstract. BOR has proved a main theorem dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series. In this paper, we have generalized this theorem to the $\varphi - |A, p_n|_k$ summability factors, under weaker conditions by using an almost increasing sequence instead of a positive monotonic non-decreasing sequence.

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1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with the partial sums (s_n) . Let (p_n) be a sequence of positive numbers such that

$$(1) \quad P_n = \sum_{v=0}^n p_v \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

$$(2) \quad \sigma_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v$$

defines the sequence (σ_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n) , generated by the sequence of coefficients (p_n) (see [5]). The series $\sum a_n$ is said

to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [2])

$$(3) \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty.$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$(4) \quad A_n(s) = \sum_{v=0}^n a_{nv} s_v, \quad n = 0, 1, \dots$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [7])

$$(5) \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\bar{\Delta} A_n(s)|^k < \infty,$$

where $\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s)$. Let (φ_n) be any sequence of positive real numbers. We say that the series $\sum a_n$ is summable $\varphi - |A, p_n|_k$, $k \geq 1$, if

$$(6) \quad \sum_{n=1}^{\infty} \varphi_n^{k-1} |\bar{\Delta} A_n(s)|^k < \infty.$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |A, p_n|_k$ summability reduces to $|A, p_n|_k$ summability. Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|\bar{N}, p_n|_k$ summability. Furthermore, if we take $\varphi_n = n$ and $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n , then $\varphi - |A, p_n|_k$ summability reduces to $|C, 1|_k$ summability (see [4]).

Before stating the main theorem we must first introduce some further notations.

Given a normal matrix $A = (a_{nv})$, we associate two lower semi matrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$(7) \quad \bar{a}_{nv} = \sum_{i=v}^n a_{ni}, \quad n, v = 0, 1, \dots$$

$$(8) \quad \hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \dots$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$(9) \quad A_n(s) = \sum_{v=0}^n a_{nv} s_v = \sum_{v=0}^n \bar{a}_{nv} a_v,$$

$$(10) \quad \bar{\Delta} A_n(s) = \sum_{v=0}^n \hat{a}_{nv} a_v.$$

2. Known result

BOR [3] has proved the following theorem for $|\bar{N}, p_n|_k$ summability method.

Theorem A. *Let (p_n) be a sequence of positive numbers such that*

$$(11) \quad P_n = O(np_n) \quad \text{as } n \rightarrow \infty.$$

If (X_n) is a positive monotonic non-decreasing sequence such that

$$(12) \quad \lambda_m X_m = O(1) \quad \text{as } m \rightarrow \infty,$$

$$(13) \quad \sum_{n=1}^m n X_n |\Delta^2 \lambda_n| = O(1),$$

$$(14) \quad \sum_{n=1}^m \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad \text{as } m \rightarrow \infty,$$

where $t_n = \frac{1}{n+1} \sum_{v=1}^n v a_v$ then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k, k \geq 1$.

3. The main result

The aim of this paper is to generalize Theorem A to $\varphi - |A, p_n|_k$ summability. Now we shall prove the following theorem.

Theorem. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$(15) \quad \bar{a}_{n0} = 1, \quad n = 0, 1, \dots,$$

$$(16) \quad a_{n-1,v} \geq a_{nv}, \quad \text{for } n \geq v + 1,$$

$$(17) \quad a_{nn} = O\left(\frac{p_n}{P_n}\right),$$

$$(18) \quad |\hat{a}_{n,v+1}| = O(v |\Delta_v \hat{a}_{nv}|).$$

Let (X_n) be an almost increasing sequence and $(\frac{\varphi_n p_n}{P_n})$ be a non-increasing sequence. If conditions (11)-(13) of Theorem A and

$$(19) \quad \sum_{n=1}^m \varphi_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |t_n|^k = O(X_m) \quad \text{as } m \rightarrow \infty,$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |A, p_n|_k$, $k \geq 1$.

It should be noted that if we take (X_n) as a positive monotonic non-decreasing sequence, $a_{nv} = \frac{p_v}{P_n}$ and $\varphi_n = \frac{P_n}{p_n}$ in this theorem, then we get Theorem A. In this case, condition (19) reduces to condition (14) and the condition " $(\frac{\varphi_n p_n}{P_n})$ is a non-increasing sequence" is automatically satisfied. We require the following lemma for the proof of the theorem.

Lemma ([6]). *Under the conditions on (X_n) and (λ_n) which are taken in the statement of our theorem, then we have the following:*

$$(20) \quad nX_n |\Delta \lambda_n| = O(1) \quad \text{as } n \rightarrow \infty,$$

$$(21) \quad \sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty.$$

4. Proof of the theorem

Let (T_n) denotes A-transform of the series $\sum a_n \lambda_n$. Then, by (9) and (10), we have

$$\bar{\Delta} T_n = \sum_{v=1}^n \hat{a}_{nv} a_v \lambda_v = \sum_{v=1}^n \frac{\hat{a}_{nv} \lambda_v}{v} v a_v.$$

Applying Abel's transformation to this sum, we get

$$\begin{aligned} \bar{\Delta} T_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} \lambda_v}{v} \right) \sum_{r=1}^v r a_r + \frac{\hat{a}_{nn} \lambda_n}{n} \sum_{r=1}^n r a_r \\ &= \frac{n+1}{n} a_{nn} \lambda_n t_n + \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v (\hat{a}_{nv}) \lambda_v t_v \\ &\quad + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta \lambda_v t_v + \sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} \lambda_{v+1} t_v \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say.} \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that $\sum_{n=1}^{\infty} \varphi_n^{k-1} |T_{n,r}|^k < \infty$, for $r = 1, 2, 3, 4$. Firstly, we have that

$$\begin{aligned}
\sum_{n=1}^m \varphi_n^{k-1} |T_{n,1}|^k &= \sum_{n=1}^m \varphi_n^{k-1} \left| \frac{n+1}{n} a_{nn} \lambda_n t_n \right|^k \\
&= O(1) \sum_{n=1}^m \varphi_n^{k-1} |\lambda_n|^{k-1} |\lambda_n| |t_n|^k \left(\frac{p_n}{P_n} \right)^k = O(1) \sum_{n=1}^m \varphi_n^{k-1} |\lambda_n| |t_n|^k \left(\frac{p_n}{P_n} \right)^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{r=1}^n \varphi_r^{k-1} \left(\frac{p_r}{P_r} \right)^k |t_r|^k + O(1) |\lambda_m| \sum_{v=1}^m \varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^k |t_v|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| X_n + O(1) |\lambda_m| X_m \\
&= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and lemma.

Applying Hölder's inequality with indices k and k' , where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} |T_{n,2}|^k &= \sum_{n=2}^{m+1} \varphi_n^{k-1} \left| \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v(\hat{a}_{nv}) \lambda_v t_v \right|^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |t_v|^k \right) \times \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |t_v|^k \right) \\
&= O(1) \sum_{v=1}^m |\lambda_v|^{k-1} |\lambda_v| |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^m |\lambda_v| |t_v|^k \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} \sum_{n=v+1}^{m+1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^m |\lambda_v| |t_v|^k \varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^k \\
&= O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and lemma.

Now, since $v|\Delta\lambda_v| = O\left(\frac{1}{X_v}\right) = O(1)$, by (20), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} |T_{n,3}|^k &= \sum_{n=2}^{m+1} \varphi_n^{k-1} \left| \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta\lambda_v t_v \right|^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta\lambda_v| |t_v|^k \right) \times \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta\lambda_v| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} v |\Delta_v \hat{a}_{nv}| |\Delta\lambda_v| |t_v|^k \right) \times \left(\sum_{v=1}^{n-1} v |\Delta_v \hat{a}_{nv}| |\Delta\lambda_v| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} v |\Delta_v \hat{a}_{nv}| |\Delta\lambda_v| |t_v|^k \right) \\
&= O(1) \sum_{v=1}^m v |\Delta\lambda_v| |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\Delta_v \hat{a}_{nv}| \\
&= O(1) \sum_{v=1}^m v |\Delta\lambda_v| |t_v|^k \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} \sum_{n=v+1}^{m+1} |\Delta_v \hat{a}_{nv}| \\
&= O(1) \sum_{v=1}^m v |\Delta\lambda_v| |t_v|^k \varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v|\Delta\lambda_v|)| \sum_{r=1}^v \varphi_r^{k-1} \left(\frac{p_r}{P_r} \right)^k |t_r|^k \\
&\quad + O(1) m |\Delta\lambda_m| \sum_{v=1}^m \varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^k |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} v X_v |\Delta^2 \lambda_v| + O(1) \sum_{v=1}^{m-1} X_v |\Delta\lambda_{v+1}| + O(1) m |\Delta\lambda_m| X_m \\
&= O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and lemma.

Finally, using the fact $P_n = O(np_n)$, by (11), as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1} \varphi_n^{k-1} |T_{n,4}|^k = \sum_{n=2}^{m+1} \varphi_n^{k-1} \left| \sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} \lambda_{v+1} t_v \right|^k$$

$$\begin{aligned}
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} \frac{1}{v} |\hat{a}_{n,v+1}| |\lambda_{v+1}|^k |t_v|^k \right) \times \left(\sum_{v=1}^{n-1} \frac{1}{v} |\hat{a}_{n,v+1}| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} \frac{1}{v} |\hat{a}_{n,v+1}| |\lambda_{v+1}|^k |t_v|^k \right) \times \left(\sum_{v=1}^{n-1} |\Delta_v \hat{a}_{nv}| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} \frac{1}{v} |\hat{a}_{n,v+1}| |\lambda_{v+1}|^k |t_v|^k \right) \\
&= O(1) \sum_{v=1}^m \frac{1}{v} |\lambda_{v+1}|^{k-1} |\lambda_{v+1}| |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^m \frac{1}{v} |\lambda_{v+1}| |t_v|^k \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} \sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^m \frac{1}{v} |\lambda_{v+1}| |t_v|^k \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} \\
&= O(1) \sum_{v=1}^m |\lambda_{v+1}| |t_v|^k \varphi_v^{k-1} \left(\frac{P_v}{v p_v} \right) \left(\frac{p_v}{P_v} \right)^k \\
&= O(1) \sum_{v=1}^m |\lambda_{v+1}| |t_v|^k \varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^k \\
&= O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and lemma. This completes the proof of the theorem. $\square$

Corollary 1. *If we take $\varphi_n = \frac{P_n}{p_n}$, then we get a result concerning the $|A, p_n|_k$ summability.*

Corollary 2. *If we take $a_{nv} = \frac{p_v}{P_n}$, then we have another a result dealing with $|\bar{N}, p_n, \varphi_n|_k$ summability.*

Corollary 3. *If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n , then we get a result dealing with $|C, 1, \varphi_n|_k$ summability.*

Corollary 4. *If we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n , then we get a result for $|C, 1|_k$ summability.*

Corollary 5. *If we take $k = 1$ and $a_{nv} = \frac{p_v}{P_n}$, then we get a result for $|\bar{N}, p_n|$ summability and in this case the condition " $(\frac{\varphi_n p_n}{P_n})$ is a non-increasing sequence" is not needed.*

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