A TAUBERIAN THEOREM FOR A GENERAL SUMMABILITY METHOD

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Abstract. We investigate conditions under which M_φ summability implies Abel summability and give the generalized Littlewood Tauberian theorem for M_φ summability method.

Mathematics Subject Classification 2010: 40E05, 40G10.

Key words: Tauberian theorems, power series methods, slow oscillation, Abel summability method, M_φ summability method.

1. Introduction

Let $\mathcal{A}$ denote the space of analytic functions in $0 < x < 1$. To each $f(a) = f(a, x) = \sum_{n=0}^{\infty} a_n x^n$ in $\mathcal{A}$ we associate a series $\sum_{n=0}^{\infty} a_n$ of Taylor coefficients of f . A series $\sum_{n=0}^{\infty} a_n$ is said to be Abel summable to s if $\sum_{n=0}^{\infty} a_n x^n$ converges for $0 < x < 1$ and tends to s as $x \rightarrow 1^-$. Abel's continuity theorem (see [1]) for power series states that if $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{x \rightarrow 1^-} f(a, x) = \sum_{n=0}^{\infty} a_n$. It is known that the converse of this implication is not true in general. However, we have a conditional converse of Abel's continuity theorem known as the generalized Littlewood Tauberian theorem (see [10]) for the Abel summability method asserting that if $\sum_{n=0}^{\infty} a_n$ is Abel summable to s and slowly oscillating, then $\sum_{n=0}^{\infty} a_n = s$. We remind the reader that a series $\sum_{n=0}^{\infty} a_n$ is said to be slowly oscillating (see [7]) if

$$\sum_{k=m+1}^n a_k = o(1), \quad n > m \rightarrow \infty, \quad n/m \rightarrow 1.$$

For a different proof of the generalized Littlewood Tauberian theorem (see [10]), we refer the reader to LANDAU [6], SCHMIDT [7], and ÇANAK [2].

Denote the class of kernels of the integral transforms of functions in $\mathcal{A}$ by Φ . We now need the following properties of functions φ in Φ :

1. There exists a number $\alpha_0 = \alpha_0(\Phi) \in (0, 1)$ such that every $\varphi \in \Phi$ is analytical in $[\alpha_0, 1)$.
2. For every $\varphi \in \Phi$, $\varphi(x) \rightarrow \infty$, $x \rightarrow 1^-$.
3. Each $\varphi \in \Phi$ is zero-free in $[\alpha_0, 1)$.
4. For each $m \geq 1$, $\frac{\varphi_m(x)}{\varphi_{m-1}(x)} = o(1)$, $x \rightarrow 1^-$ where $\varphi_0 = \varphi$ and $\varphi_m(x) = \int_{\alpha_0}^x \varphi_{m-1}(t) dt$.

For every f in $\mathcal{A}$ and $\varphi \in \Phi$ we define $M(f, \varphi) = M(f, \varphi, x) = \frac{\int_{\alpha_0}^x f(t)\varphi(t)dt}{\varphi_1(x)}$ if $x \neq \alpha_0$ and $\lim_{x \rightarrow \alpha_0} M(f, \varphi, x) = f(\alpha_0)$ if $x = \alpha_0$.

A series $\sum_{n=0}^{\infty} a_n$ is said to be M_φ summable to s if

$$(1.1) \quad \lim_{x \rightarrow 1^-} M(f, \varphi, x) = s.$$

Example 1.1. The series $\sum_{n=0}^{\infty} a_n$ whose general term is the Taylor coefficient of the function g defined by $g(x) = f'(\varphi_1(x))$, where f is bounded in $\mathcal{A}$ is M_φ summable.

It is plain that every Abel summable series is M_φ summable, but the converse statement is not always true. The M_φ summability method is regular with respect to the Abel summability method. Indeed, since $\varphi_1'(x) = \varphi(x)$, it easily follows that Abel summability of $\sum_{n=0}^{\infty} a_n$ implies M_φ summability of $\sum_{n=0}^{\infty} a_n$. However, if a function f satisfies the condition

$$\frac{\int_{\alpha_0}^x f'(t)\varphi_1(t) dt}{\varphi_1(x)} = o(1), x \rightarrow 1^-,$$

then the converse statement is also true. This follows from the identity

$$(1.2) \quad f(x) = M(f, \varphi, x) + \frac{\int_{\alpha_0}^x f'(t)\varphi_1(t) dt}{\varphi_1(x)}.$$

An important subclass of Φ is the class of following functions $\psi_m(x) = \frac{1}{(1-x)^m}$, $m = 1, 2, \dots$ and $\alpha_0 = 0$. The function $\psi_m(x) = \frac{1}{(1-x)^m}$ defines a summability method (A, m) which is regular with respect to Abel summability method. For the more information about the M_φ and (A, m) summability methods, we refer to [8]. Recently, a number of authors including

ÇANAK ET AL. [3, 5], ÇANAK [4], and TOTUR [9] have given Tauberian theorems for the (A, m) summability method.

The $(A, 1)$ summability method as a special case of M_φ summability method is of considerable interest. A series $\sum_{n=0}^{\infty} a_n$ is said to be $(A, 1)$ summable to s if

$$(1.3) \quad \lim_{x \rightarrow 1^-} \frac{(1-x)}{x} \int_0^x \frac{f(t)}{(1-t)^2} dt = s.$$

It is clear that Abel summability of $\sum_{n=0}^{\infty} a_n$ implies $(A, 1)$ summability of $\sum_{n=0}^{\infty} a_n$. That the converse is not true in general follows from the series $\sum_{n=0}^{\infty} a_n$ whose general term is the Taylor coefficient of the function f defined by $f(x) = \sin((1-x)^{-1})$ on $0 < x < 1$.

The identity (1.2) becomes

$$f(x) = \frac{(1-x)}{x} \int_0^x \frac{f(t)}{(1-t)^2} dt + \frac{(1-x)}{x} \int_0^x \frac{f'(t)}{1-t} dt$$

for $\varphi(x) = \frac{1}{(1-x)^2}$. Since a series which is Abel summable to s is $(A, 1)$ summable to s , we have

$$(1.4) \quad \frac{(1-x)}{x} \int_0^x \frac{f'(t)}{1-t} dt = o(1), x \rightarrow 1^-.$$

In the case where the condition (1.4) holds, every $(A, 1)$ summable series is Abel summable.

An example of an $(A, 1)$ summable series can be obtained by applying integration by parts to $\frac{(1-x)}{x} \int_0^x \frac{f(t)}{(1-t)^2} dt$.

Example 1.2. If $\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n a_k$ is Abel summable to s , then $\sum_{n=0}^{\infty} a_n$ is $(A, 1)$ summable to 0.

It is natural to ask under which conditions M_φ summability implies Abel summability of $\sum_{n=0}^{\infty} a_n$. In this paper we both answer this question and give the generalized Littlewood Tauberian theorem for M_φ summability method.

2. A Tauberian theorem

By the following theorem, we prove that every M_φ summable series is Abel summable under certain conditions.

Theorem 2.1. *Let $\sum_{n=0}^{\infty} a_n$ be M_{φ} summable to s . If*

$$(2.1) \quad \frac{\int_{\alpha_0}^x f''(t) \varphi_2(t) dt}{\varphi_1(x)} = o(1), x \rightarrow 1^-$$

and f has a zero of order three at α_0 such that $\frac{f(\alpha_0)}{\varphi_2(\alpha_0)}$ is defined to be the limit $\lim_{x \rightarrow \alpha_0} \frac{f(x)}{\varphi_2(x)}$, then $\sum_{n=0}^{\infty} a_n$ is Abel summable to s .

Proof. Applying integration by parts to $M(f, \varphi, x)$, we have

$$(2.2) \quad M(f, \varphi, x) = f(x) - T(f, \varphi, x),$$

where $T(f, \varphi, x) = \frac{\int_{\alpha_0}^x f'(t) \varphi_1(t) dt}{\varphi_1(x)}$. Applying again integration by parts to $T(f, \varphi, x)$, we have

$$(2.3) \quad T(f, \varphi, x) = \frac{\varphi_2(x)}{\varphi_1(x)} f'(x) - \frac{\int_{\alpha_0}^x f''(t) \varphi_2(t) dt}{\varphi_1(x)}.$$

Combining (2.2) with (2.3), we obtain

$$(2.4) \quad M(f, \varphi, x) = g(x) + \frac{\int_{\alpha_0}^x f''(t) \varphi_2(t) dt}{\varphi_1(x)},$$

where $g(x) = f(x) - \frac{\varphi_2(x)}{\varphi_1(x)} f'(x)$. Since $\sum_{n=0}^{\infty} a_n$ is M_{φ} summable to s , we have by (2.1) and (2.4) that

$$(2.5) \quad \lim_{x \rightarrow 1^-} g(x) = \lim_{x \rightarrow 1^-} M(f, \varphi, x) = s.$$

Multiplying $g(x)$ by $\varphi_1(x)$ and dividing by $\varphi_2^2(x)$, we arrive at

$$(2.6) \quad g(x) = -\frac{\varphi_2^2(x)}{\varphi_1(x)} \left(\frac{f(x)}{\varphi_2(x)} \right)'.$$

Since $\frac{f(\alpha_0)}{\varphi_2(\alpha_0)}$ is defined to be the limit $\lim_{x \rightarrow \alpha_0} \frac{f(x)}{\varphi_2(x)}$ at α_0 where f has a zero of order three, we obtain

$$(2.7) \quad f(x) = -\varphi_2(x) \int_{\alpha_0}^x g(t) \frac{\varphi_1(t)}{\varphi_2^2(t)} dt.$$

Since $\lim_{x \rightarrow 1^-} g(x) = s$, we have by L'Hospital's rule that

$$(2.8) \quad \lim_{x \rightarrow 1^-} f(x) = s.$$

This completes the proof. $\square$

Taking $\varphi(x) = \frac{1}{(1-x)^2}$ in Theorem 2.1, we have the following Tauberian theorem for the $(A, 1)$ summability method.

Corollary 2.2. *Let $\sum_{n=0}^{\infty} a_n$ be $(A, 1)$ summable to s . If*

$$(2.9) \quad (1-x) \int_0^x \ln(1-t) f''(t) dt = o(1), x \rightarrow 1^-$$

and f has a zero of order three at 0 such that $\frac{f(x)}{-x-\ln(1-x)}$ at 0 is defined to be the limit $\lim_{x \rightarrow 0} \frac{f(x)}{-x-\ln(1-x)}$, then $\sum_{n=0}^{\infty} a_n$ is Abel summable to s .

We note that Theorem 2.1 can be given for the composition of M_φ and M_ψ defined by

$$(2.10) \quad (M_\psi \circ M_\varphi)(f(a), x) = M(M_\varphi(f(a)), \psi, x) = \frac{\int_{\alpha_0}^x M(f, \varphi, t) \psi(t) dt}{\psi_1(x)},$$

where $\varphi, \psi \in \phi$.

The following theorem is a consequence of the generalized Littlewood Tauberian theorem for Abel summability method.

Theorem 2.3. *Let $\sum_{n=0}^{\infty} a_n$ be M_φ summable to s , (2.1) be satisfied and f have a zero of order three at α_0 such that $\frac{f(\alpha_0)}{\varphi_2(\alpha_0)}$ is defined to be the limit $\lim_{x \rightarrow \alpha_0} \frac{f(x)}{\varphi_2(x)}$. If $\sum_{n=0}^{\infty} a_n$ is slowly oscillating, then $\sum_{n=0}^{\infty} a_n$ converges to s .*

Proof. We have by Theorem 2.1 that $\sum_{n=0}^{\infty} a_n$ is Abel summable to s . Since $\sum_{n=0}^{\infty} a_n$ is slowly oscillating, we have from the generalized Littlewood Tauberian theorem that $\sum_{n=0}^{\infty} a_n$ converges to s . $\square$

If the condition (2.1) is replaced by a stronger condition, we have the following result.

Theorem 2.4. *Let $\sum_{n=0}^{\infty} a_n$ be M_φ summable to s . If*

$$(2.11) \quad f''(x) = o\left(\frac{\varphi(x)}{\varphi_2(x)}\right), x \rightarrow 1^-$$

and $\sum_{n=0}^{\infty} a_n$ is slowly oscillating, then $\sum_{n=0}^{\infty} a_n$ converges to s .

Proof. We easily see that the condition (2.11) implies (2.1). It follows by Theorem 2.1 that $\sum_{n=0}^{\infty} a_n$ is Abel summable to s . Since $\sum_{n=0}^{\infty} a_n$ is slowly oscillating, we have from the generalized Littlewood Tauberian theorem that $\sum_{n=0}^{\infty} a_n$ converges to s . $\square$

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Received: 17.II.2012

Revised: 8.V.2012

Accepted: 10.V.2012

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