λ -ALMOST STATISTICAL CONVERGENCE OF ORDER α

BY

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Abstract. In this paper, we introduce the concept λ -almost statistical convergence of order α . Also some relations between λ -almost statistical convergence of order α and strong $(\hat{V}, \lambda)$ -almost summability of order α are given. Furthermore some relations between the space $[\hat{V}_p^\alpha, \lambda, f]$ and $\hat{S}_\lambda^\alpha$ are examined.

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1. Introduction

The idea of statistical convergence was given by ZYGMUND [27] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by STEINHAUS [26] and FAST [7] and later reintroduced by SCHOENBERG [25] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by FRIDY [8], CONNOR [5], SAVAŞ [24], MURSALEEN [17], MURSALEEN and ALOTAIBI [18], MILLER and ORHAN [16], RATH and TRIPATHY [21], SALAT [23] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Čech compactification of the natural numbers. Moreover, statistical convergence is

closely related to the concept of convergence in probability (for details, see [20]).

Let w be the set of all sequences of real numbers and ℓ_∞ , c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup |x_k|$, where $k \in \mathbb{N} = \{1, 2, \dots\}$, the set of positive integers. A linear functional $\mathcal{L}$ on ℓ_∞ is said to be a Banach limit (see [1]) if it has the properties:

- i) $\mathcal{L}(x) \geq 0$ if $x \geq 0$ (i.e. $x_n \geq 0$ for all n);
- ii) $\mathcal{L}(e) = 1$, where $e = (1, 1, \dots)$;
- iii) $\mathcal{L}(Dx) = \mathcal{L}(x)$,

where D is the shift operator defined by $(Dx_n) = (x_{n+1})$.

Let $\mathfrak{B}$ be the set of all Banach limits on ℓ_∞ . A bounded sequence x is said to be almost convergent to a number L if $\mathcal{L}(x) = L$ for all $\mathcal{L} \in \mathfrak{B}$. The set of all almost convergent sequences will be denoted by $\hat{c}$. LORENTZ [12] proved that $x = (x_k) \in \hat{c}$ if and only if $\lim_n \frac{1}{n} \sum_{k=1}^n x_{k+m}$ exists, uniformly in m .

Several authors including LORENTZ [12], DURAN [6], KING [10], ÇOLAK and ÇAKAR [4] have studied almost convergent sequences. MADDOX [13] has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^n |x_{k+m} - L| = 0, \quad \text{uniformly in } m.$$

It can be shown that the sequence $x = (1, 0, 1, 0, 1, 0, \dots)$ is strongly almost convergent to $\frac{1}{2}$. By $[\hat{c}]$ we denote the space of all strongly almost convergent sequences. It is easy to see that $c \subset [\hat{c}] \subset \hat{c} \subset \ell_\infty$ and the inclusions are strict, for example the sequence $x = (x_k) = ((-1)^k)$ almost convergent but not strongly almost convergent.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$, $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$, where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S - \lim x = L$ or $x_k \rightarrow L(S)$ and S denotes the set of all statistically convergent sequences. It is note that statistically convergent a sequence need not be bounded. For this, consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} \sqrt{k}, & k = n^2, n = 1, 2, \dots \\ 3, & k \neq n^2 \end{cases}.$$

Then $S - \lim x = 3$, but $x \notin \ell_\infty$.

The order of statistical convergence of a sequence of numbers was given by GADJIEV and ORHAN in [9] and after then statistical convergence of order α and strong p -Cesàro summability of order α studied by ÇOLAK [3].

The statistical convergence of order α is defined as follows. Let $0 < \alpha \leq 1$ be given. The sequence (x_k) is said to be statistically convergent of order α if there is a complex number L such that $\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$, for every $\varepsilon > 0$, in which case we say that x is statistically convergent of order α , to L . In this case we write $S^\alpha - \lim x_k = L$. The set of all statistically convergent sequences of order α will be denoted by S^α . It is easy to see that every convergent sequence is statistically convergent of order α , that is $c \subset S^\alpha$ for each $0 < \alpha \leq 1$, but converse does not hold. For example the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} \frac{1}{\sqrt{k}}, & k = n^2, n = 1, 2, \dots \\ 1, & k \neq n^2 \end{cases}$$

is statistically convergent of order α with $S^\alpha - \lim x_k = 1$ for $\alpha > \frac{1}{3}$, but is not convergent.

The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ and $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ) -summable to a number L (see [11]) if $t_n(x) \rightarrow L$ as $n \rightarrow \infty$. For example the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & \text{if } k \text{ is odd} \\ -1, & \text{if } k \text{ is even} \end{cases}$$

is (V, λ) -summable to 0 for $\lambda = (n)$.

We write

$$\begin{aligned} [C, 1] &= \left\{ x = (x_k) : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |x_k - L| = 0 \text{ for some } L \right\}, \\ [V, \lambda] &= \left\{ x = (x_k) : \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0 \text{ for some } L \right\}, \end{aligned}$$

for the sets of sequences $x = (x_k)$ which are strongly Cesàro summable and strongly (V, λ) -summable respectively.

The notion of λ -statistical convergence was introduced by MURSALEEN [17] and the concept of almost λ -statistical convergence was studied by SAVAŞ [24]. Recently λ -statistical convergence was generalized by ÇOLAK and BEKTAŞ [2].

2. Main results

In this section we give the main results of the paper. In Theorem 2.4 we give the inclusion relations between the sets of λ -almost statistical convergent sequences of order α for different α 's, and so that the inclusion relations between the set of λ -almost statistical convergent sequences of order α and the set of λ -almost statistical convergent sequences. In Theorem 2.8 we give the relationship between the strong almost $[\hat{V}_p^\alpha, \lambda]$ -summability of order α and the strong almost $[\hat{V}_p^\beta, \lambda]$ -summability of order β . In Theorem 2.11 we give the relationship between the strong almost $[\hat{V}_p^\alpha, \lambda]$ -summability of order α and the λ -almost statistical convergence of order β .

Definition 2.1. Let the sequence $\lambda = (\lambda_n)$ of real numbers be defined as above and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k) \in w$ is said to be λ -almost statistically convergent of order α if there is a complex number L such that $\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}| = 0$, uniformly in m , where $I_n = [n - \lambda_n + 1, n]$ and λ_n^α denote the α^{th} power $(\lambda_n)^\alpha$ of λ_n , that is $\lambda^\alpha = (\lambda_n^\alpha) = (\lambda_1^\alpha, \lambda_2^\alpha, \dots, \lambda_n^\alpha, \dots)$. In this case we write $\hat{S}_\lambda^\alpha - \lim x_k = L$. The set of all λ -almost statistically convergent sequences of order α will be denoted by $\hat{S}_\lambda^\alpha$. For example the sequence $x = (x_k)$ defined by

$$x_{k+m} = \begin{cases} k+m, & k+m = n^2 \\ 0, & k+m \neq n^2 \end{cases}, \quad n = 1, 2, \dots$$

is λ -almost statistically convergent sequences of order α to 0, for $\frac{1}{2} < \alpha \leq 1$ and $\lambda = (n)$.

The λ -almost statistical convergence of order α is same with the λ -almost statistical convergence, that is $\hat{S}_\lambda^\alpha = \hat{S}_\lambda$ for $\alpha = 1$. The λ -almost statistical convergence of order α is well defined for $0 < \alpha \leq 1$. But it is not well defined for $\alpha > 1$ in general. For this let $x = (x_k)$ be defined as

follows:

$$x_{k+m} = \begin{cases} 1, & \text{if } k+m \text{ even} \\ 0, & \text{if } k+m \text{ odd} \end{cases}.$$

Then both

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_{k+m} - 1| \geq \varepsilon\}| = \lim_{n \rightarrow \infty} \frac{\lambda_n}{2\lambda_n^\alpha} = 0,$$

uniformly in m and

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_{k+m} - 0| \geq \varepsilon\}| = \lim_{n \rightarrow \infty} \frac{\lambda_n}{2\lambda_n^\alpha} = 0, \text{ uniformly in } m,$$

for $\alpha > 1$, so that $x = (x_k)$, λ -almost statistically converges of order α , both to 1 and 0, i.e. $\hat{S}_\lambda^\alpha - \lim x_k = 1$ and $\hat{S}_\lambda^\alpha - \lim x_k = 0$. But this is impossible.

Theorem 2.2. *Let $0 < \alpha \leq 1$ and $x = (x_k)$, $y = (y_k)$ be sequences of complex numbers.*

- (i) *If $\hat{S}_\lambda^\alpha - \lim x_k = x_0$ and $c \in \mathbb{C}$, then $\hat{S}_\lambda^\alpha - \lim(cx_k) = cx_0$;*
- (ii) *If $\hat{S}_\lambda^\alpha - \lim x_k = x_0$ and $\hat{S}_\lambda^\alpha - \lim y_k = y_0$, then $\hat{S}_\lambda^\alpha - \lim(x_k + y_k) = x_0 + y_0$.*

Proof. (i) It is clear in case $c = 0$. Suppose that $c \neq 0$ then the proof of (i) follows from

$$\frac{1}{\lambda_n^\alpha} |\{k \in I_n : |cx_{k+m} - cx_0| \geq \varepsilon\}| = \frac{1}{\lambda_n^\alpha} \left| \{k \in I_n : |x_{k+m} - x_0| \geq \frac{\varepsilon}{|c|}\} \right|$$

and that of (ii) follows from

$$\begin{aligned} & \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_{k+m} + y_{k+m} - (x_0 + y_0)| \geq \varepsilon\}| \\ & \leq \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : |x_{k+m} - x_0| \geq \frac{\varepsilon}{2} \right\} \right| + \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : |y_{k+m} - y_0| \geq \frac{\varepsilon}{2} \right\} \right|. \end{aligned}$$

If we take $\lambda_n = n^\alpha$ for $0 < \alpha < 1$ in the definition of $\hat{S}_\lambda$ then we obtain $\hat{S}^\alpha \subseteq \hat{S}_\lambda$. That is why $\hat{S}_\lambda^\alpha$ is different from $\hat{S}_\lambda$ defined in [24] in general. If $\lambda_n = n^\alpha$ with $\alpha = 1$, that is $\lambda_n = n$ then $\hat{S}^\alpha = \hat{S}_\lambda = \hat{S}$, that is the

almost statistical convergence of order α , λ -almost statistical convergence and almost statistical convergence are coincide in case $\lambda_n = n^\alpha$ with $\alpha=1$. $\square$

Definition 2.3. Let the sequence $\lambda = (\lambda_n)$ be as above, $\alpha \in (0, 1]$ be any real number and let p be a positive real number. A sequence x is said to be strongly $(\hat{V}, \lambda)$ -almost summable of order α , if there is a complex number L such that

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |x_{k+m} - L|^p = 0 \text{ uniformly in } m,$$

where $I_n = [n - \lambda_n + 1, n]$. The strong (V, λ) -almost summability of order α reduces to the strong $(\hat{V}, \lambda)$ -summability for $\alpha = 1$. The set of all strongly $(\hat{V}, \lambda)$ -summable sequences of order α will be denoted by $[\hat{V}_p^\alpha, \lambda]$.

Theorem 2.4. If $0 < \alpha \leq \beta \leq 1$ then $\hat{S}_\lambda^\alpha \subseteq \hat{S}_\lambda^\beta$.

Proof. If $0 < \alpha \leq \beta \leq 1$ then we may write

$$\frac{1}{\lambda_n^\beta} |\{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}| \leq \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}|,$$

for every $\varepsilon > 0$ and this gives that $\hat{S}_\lambda^\alpha \subseteq \hat{S}_\lambda^\beta$. $\square$

If we take $\beta = 1$ in Theorem 2.4 then we obtain the following result.

Corollary 2.5. If a sequence is λ -almost statistically convergent of order α , to L , then it is λ -almost statistically convergent to L , that is $\hat{S}_\lambda^\alpha \subseteq \hat{S}_\lambda$ for each $\alpha \in (0, 1]$.

From Theorem 2.4 we have the following results and the proof is easy.

Corollary 2.6. (i) $\hat{S}_\lambda^\alpha = \hat{S}_\lambda^\beta \iff \alpha = \beta$.

(ii) $\hat{S}_\lambda^\alpha = \hat{S}_\lambda \iff \alpha = 1$.

Theorem 2.7. $\hat{S} \subseteq \hat{S}_\lambda^\alpha$ if

$$(1) \quad \lim_{n \rightarrow \infty} \inf \frac{\lambda_n^\alpha}{n} > 0.$$

Proof. For given $\varepsilon > 0$ we have $\{k \leq n : |x_{k+m} - L| \geq \varepsilon\} \supset \{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}$. Therefore we may write

$$\begin{aligned} \frac{1}{n} |\{k \leq n : |x_{k+m} - L| \geq \varepsilon\}| &\geq \frac{1}{n} |\{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}| \\ &\geq \frac{\lambda_n^\alpha}{n} \cdot \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}|. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using (1), we get $x_k \rightarrow L(\hat{S}) \implies x_k \rightarrow L(\hat{S}_\lambda^\alpha)$. $\square$

Theorem 2.8. Let $0 < \alpha \leq \beta \leq 1$ and p be a positive real number. Then $[\hat{V}_p^\alpha, \lambda] \subseteq [\hat{V}_p^\beta, \lambda]$.

Proof. Let $x = (x_k) \in [\hat{V}_p^\alpha, \lambda]$. Then given α and β such that $0 < \alpha \leq \beta \leq 1$ and a positive real number p , we may write

$$\frac{1}{\lambda_n^\beta} \sum_{k \in I_n} |x_{k+m} - L|^p \leq \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |x_{k+m} - L|^p$$

and this gives that $[\hat{V}_p^\alpha, \lambda] \subseteq [\hat{V}_p^\beta, \lambda]$. $\square$

The following result is a consequence of Theorem 2.8.

Corollary 2.9. For each $\alpha \in (0, 1]$ and $0 < p < \infty$, we have $[\hat{V}_p^\alpha, \lambda] \subseteq [\hat{V}_p, \lambda]$.

Theorem 2.10. Let $0 < \alpha \leq 1$ and $0 < p < q < \infty$. Then $[\hat{V}_q^\alpha, \lambda] \subset [\hat{V}_p^\alpha, \lambda]$.

Proof is seen from Hölder inequality.

Theorem 2.11. Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$. If a sequence is strongly $(\hat{V}, \lambda)$ -almost summable of order α , to L , then it is λ -almost statistically convergent of order β , to L , i.e. $[\hat{V}_p^\alpha, \lambda] \subset \hat{S}_\lambda^\beta$.

Proof. For any sequence $x = (x_k)$ and $\varepsilon > 0$, we have

$$\begin{aligned} \sum_{k \in I_n} |x_{k+m} - L|^p &= \sum_{\substack{k \in I_n \\ |x_{k+m} - L| \geq \varepsilon}} |x_{k+m} - L|^p + \sum_{\substack{k \in I_n \\ |x_{k+m} - L| < \varepsilon}} |x_{k+m} - L|^p \\ &\geq \sum_{\substack{k \in I_n \\ |x_{k+m} - L| \geq \varepsilon}} |x_{k+m} - L|^p \geq |\{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}| \cdot \varepsilon^p \end{aligned}$$

and so that

$$\begin{aligned} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |x_{k+m} - L|^p &\geq \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}| \cdot \varepsilon^p \\ &\geq \frac{1}{\lambda_n^\beta} |\{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}| \cdot \varepsilon^p. \end{aligned}$$

From this it follows that if $x = (x_k)$ is strongly $(\hat{V}, \lambda)$ -almost summable of order α , to L , then it is λ -almost statistically convergent of order β , to L . $\square$

If we take $\beta = \alpha$ in Theorem 2.11 we obtain the following result.

Corollary 2.12. *Let α be a fixed real number such that $0 < \alpha \leq 1$ and $0 < p < \infty$. If a sequence is strongly $(\hat{V}, \lambda)$ -almost summable of order α , to L , then it is λ -almost statistically convergent of order α , to L , i.e. $[\hat{V}_p^\alpha, \lambda] \subset \hat{S}_\lambda^\alpha$.*

Corollary 2.13. *Let $0 < \alpha \leq 1$ and p be a positive real number. Then $[\hat{V}_p^\alpha, \lambda] \subset \hat{S}_\lambda$ and the inclusion is strict if $0 < \alpha < 1$.*

Proof. From Corollary 2.5 and Corollary 2.12 we have $[\hat{V}_p^\alpha, \lambda] \subset \hat{S}_\lambda$. $\square$

3. Modulus function and statistical convergence

The notion of a modulus was introduced by NAKANO [19]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

- i) $f(x) = 0$ if and only if $x = 0$;
- ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$;
- iii) f is increasing;
- iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. A modulus may be bounded or unbounded. For example, $f(x) = x^p$, ($0 < p \leq 1$) is unbounded and $f(x) = \frac{x}{1+x}$ is bounded. MADDUX [14], RUCKLE [22], MALKOWSKY and SAVAŞ [15] used a modulus function to construct some sequence spaces.

Definition 3.1. Let f be a modulus function, $p = (p_k)$ be a sequence of strictly positive real numbers and let $\alpha \in (0, 1]$ be any real number. Now we define

$$[\hat{V}_p^\alpha, \lambda, f] = \{x = (x_k) : \lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} [f(|x_{k+m} - L|)]^{p_k} = 0 \text{ uniformly in } m, \text{ for some } L\}.$$

In the following theorems we shall assume that the sequence $p = (p_k)$ is bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$.

Theorem 3.2. Let $\alpha, \beta \in (0, 1]$ be any real numbers such that $\alpha \leq \beta$ and f be a modulus function. Then $[\hat{V}_p^\alpha, \lambda, f] \subset \hat{S}_\lambda^\beta$.

Proof. Let $x \in [\hat{V}_p^\alpha, \lambda, f]$ and let $\varepsilon > 0$ be given. Then since $\lambda_n^\alpha \leq \lambda_n^\beta$ for each n we may write

$$\begin{aligned} & \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} [f(|x_{k+m} - L|)]^{p_k} \\ &= \frac{1}{\lambda_n^\alpha} \left[\sum_{\substack{k \in I_n \\ |x_{k+m} - L| \geq \varepsilon}} [f(|x_{k+m} - L|)]^{p_k} + \sum_{\substack{k \in I_n \\ |x_{k+m} - L| < \varepsilon}} [f(|x_{k+m} - L|)]^{p_k} \right] \\ &\geq \frac{1}{\lambda_n^\beta} \left[\sum_{\substack{k \in I_n \\ |x_{k+m} - L| \geq \varepsilon}} [f(|x_{k+m} - L|)]^{p_k} + \sum_{\substack{k \in I_n \\ |x_{k+m} - L| < \varepsilon}} [f(|x_{k+m} - L|)]^{p_k} \right] \\ &\geq \frac{1}{\lambda_n^\beta} \sum_{\substack{k \in I_n \\ |x_{k+m} - L| \geq \varepsilon}} [f(\varepsilon)]^{p_k} \geq \frac{1}{\lambda_n^\beta} \sum_{\substack{k \in I_n \\ |x_{k+m} - L| \geq \varepsilon}} \min([f(\varepsilon)]^h, [f(\varepsilon)]^H) \\ &\geq \frac{1}{\lambda_n^\beta} |\{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}| \min([f(\varepsilon)]^h, [f(\varepsilon)]^H). \end{aligned}$$

Since $x \in [\hat{V}_p^\alpha, \lambda, f]$, the left hand side of the above inequality tends to zero as $n \rightarrow \infty$ uniformly in m . Since $\min([f(\varepsilon)]^h, [f(\varepsilon)]^H) > 0$ the right hand side of above inequality tends to zero as $n \rightarrow \infty$ uniformly in m . This implies that $[\hat{V}_p^\alpha, \lambda, f] \subset \hat{S}_\lambda^\beta$. $\square$

Theorem 3.3. *Let $\alpha, \beta \in (0, 1]$ be any real numbers such that $\alpha \leq \beta$. If the modulus f is bounded then $\hat{S}_\lambda^\alpha \subset [\hat{V}_p^\beta, \lambda, f]$.*

Proof. Let $x \in \hat{S}_\lambda^\alpha$ and suppose that f is bounded. Let $\varepsilon > 0$ be given and $\sum_1$ and $\sum_2$ denote the sums over $k \in I_n$, $|x_{k+m} - L| \geq \varepsilon$ and $k \in I_n$, $|x_{k+m} - L| < \varepsilon$ respectively. Since f is bounded there exists an integer K such that $f(x) \leq K$, for all $x \geq 0$. Then since $\lambda_n^\alpha \leq \lambda_n^\beta$ for each n we may write

$$\begin{aligned} \frac{1}{\lambda_n^\beta} \sum_{k \in I_n} [f(|x_{k+m} - L|)]^{p_k} &\leq \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} [f(|x_{k+m} - L|)]^{p_k} \\ &= \frac{1}{\lambda_n^\alpha} \left(\sum_1 [f(|x_{k+m} - L|)]^{p_k} + \sum_2 [f(|x_{k+m} - L|)]^{p_k} \right) \\ &\leq \frac{1}{\lambda_n^\alpha} \sum_1 \max(K^h, K^H) + \frac{1}{\lambda_n^\alpha} \sum_2 [f(\varepsilon)]^{p_k} \\ &\leq \max(K^h, K^H) \frac{1}{\lambda_n^\alpha} |\{k \leq n : |x_{k+m} - L| \geq \varepsilon\}| + \max(f(\varepsilon)^h, f(\varepsilon)^H). \end{aligned}$$

Since $x \in \hat{S}_\lambda^\alpha$ the first term of the right hand side of above inequality tends to zero as $n \rightarrow \infty$ uniformly in m , and the second term can be made as small as desired. Therefore the left hand side of above inequality tends to zero as $n \rightarrow \infty$ uniformly in m . Hence $x \in [\hat{V}_p^\beta, \lambda, f]$. $\square$

Theorem 3.4. *Let $\alpha \in (0, 1]$ be any real number. Then $\hat{S}_\lambda^\alpha = [\hat{V}_p^\alpha, \lambda, f]$ if and only if f is bounded.*

Proof. Let f be bounded. By Theorem 3.2 and Theorem 3.3 we have $\hat{S}_\lambda^\alpha = [\hat{V}_p^\alpha, \lambda, f]$.

Conversely assume that f is unbounded. Using the same technique as in proof of Theorem 3 in [15] we may find a sequence $x \in \hat{S}_\lambda^\alpha - [\hat{V}_p^\alpha, \lambda, f]$. This completes the proof. $\square$

If we take $\alpha = 1$ in Theorem 3.4 we obtain Theorem 3 in [15] as the following.

Corollary 3.5. *The $\hat{S}_\lambda = [\hat{V}_p, \lambda, f]$ equality holds if and only if f is bounded.*

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