THE ω^* -OPEN OPEN TOPOLOGY FOR FUNCTION SPACES

BY

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Abstract. In this paper, we have introduce and study the concepts of ω^* -open sets and ω^* -open open topology for function spaces.

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1. Introduction

Due to enormous use and potentiality of continuous functions., there has been a lot of attempts by Eminent Mathematicians to generalize this important concept from different points. The topological aspect of function spaces was first considered by FOX [3]. He introduced compact-open topology on function spaces with the help of compact sets. LAMBRINOS [4], PORTER [7, 8] and other topologists introduced different set-open topologies on function spaces like bounded-open topology, open-open topology etc. FOX [3], ARENS and DUGUNDJI [2] studied extensively the set-open topology and the compact-open topology. The concept of ω -open sets was studied by SUNDARAM and SHEIK JOHN [9]. In this paper, we have introduce and study the concepts of ω^* -open sets and ω^* -open open topology for function spaces.

2. Preliminaries

Definition 2.1 ([5]). A subset S of a topological space (X, τ) is called a semiopen set if $S \subseteq \text{Cl}(\text{Int}(S))$.

Definition 2.2. A subset A of a topological space (X, τ) is called an ω -closed set ([9]) if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semiopen in (X, τ) . The complement of an ω -closed set is called an ω -open set ([9]). The family of all ω -open sets of (X, τ) is denoted by τ^ω .

Definition 2.3. The intersection of all ω -closed sets containing $A \subset X$ is called the ω -closure ([10]) of A and is denoted by $\omega \text{Cl}(E)$. The union of all ω -open sets contained in $A \subset X$ is called the ω -interior ([10]) of A and is denoted by $\omega \text{Int}(E)$.

Definition 2.4 ([1]). Let (X, τ) and (Y, σ) be any two topological spaces. Let $\mathcal{U}$ and $\mathcal{V}$ be collection of subsets of X and Y , respectively. Let $F \subseteq Y^X$ be a collection of functions from X into Y . For $U \in \mathcal{U}$ and $V \in \mathcal{V}$, $(U, V) = \{f \in F : f(U) \subseteq V\}$. Let $S(U, V) = \{(U, V) : U \in \mathcal{U}, V \in \mathcal{V}\}$. If $S(U, V)$ is a subbasis for a topology $\tau(U, V)$ on F , then $\tau(U, V)$ is called a set-set topology.

Definition 2.5 ([1]). A topology τ^* on $F \subseteq Y^X$ is called an admissible topology for F provided the evaluation map $E : (F, \tau^*) \times (X, \tau) \rightarrow (Y, \sigma)$ defined by $E(f, x) = f(x)$ is continuous.

Definition 2.6 ([6]). Let $(G, \circ)$ be a group such that (G, τ) is a topological space, then (G, τ) is a topological group provided the two maps: (1) $m : G \times G \rightarrow G$, defined by $m(g_1, g_2) = g_1 \circ g_2$ and (2) $\Phi : G \rightarrow G$ defined by $\Phi(g) = g^{-1}$ are continuous. If only the first map is continuous then we call (G, τ) a quasi-topological group.

3. ω^* -open open topology for function spaces

Definition 3.1. A subset $M(x)$ of a space X is called an ω -neighbourhood ([10]) of a point $x \in X$ if there exists an ω -open set S such that $x \in S \subseteq M(x)$.

Definition 3.2. Let $\omega(x) = \{A \in \omega O(X) : x \in A\}$ and let $\omega_x = \{A \subseteq X : \text{there exists } \mu \subseteq \omega(x) \text{ such that } \mu \text{ is finite and } \cap \mu \subseteq A\}$. Then ω_x is called the ω -neighbourhood filter at x .

Definition 3.3. For any filter Γ on X , we say that Γ ω -converges to x if and only if Γ is finer than the ω -neighbourhood filter at x .

Definition 3.4. A subset U of X is called an ω^* -open set if whenever a filter Γ ω -converges to x and $x \in U$, and $U \in \Gamma$. The complement of an ω^* -open set is called an ω^* -closed set.

Definition 3.5. The intersection of all ω^* -closed sets containing A is called the ω^* -closure of A denoted by $\text{Cl}_{\omega^*}(A)$. A subset A is ω^* -closed if $A = \text{Cl}_{\omega^*}(A)$. The family of all ω^* -open sets of (X, τ) denoted by τ^{ω^*} . Clearly τ^{ω^*} is a topology on X . In a topological space (X, τ) , it is always true that $\tau \subseteq \tau^{\omega} \subseteq \tau^{\omega^*}$.

Definition 3.6. A point $x \in X$ is said to be an ω^* -interior point of A if there exists an ω^* -open set U containing x such that $U \subseteq A$. The set of all ω^* -interior points of A is said to be ω^* -interior of A and is denoted by $\text{Int}_{\omega^*}(A)$.

Proposition 3.1. For a subset A of a space X , $\text{Int}_{\omega^*}(X \setminus A) = X \setminus \text{Cl}_{\omega^*}(A)$.

Definition 3.7. A net $\{x_\lambda : \lambda \in \Lambda\}$ in X is said to ω^* -converge to $x \in X$ if $\{x_\lambda : \lambda \in \Lambda\}$ is eventually in each ω^* -open set containing x , denoted by $X_\lambda \xrightarrow{\omega^*} x$.

Definition 3.8. Let $\mathcal{U}$ be the collection of all ω^* -open sets in X and $\mathcal{V}$ be the collection of all open sets in Y , then $S_{\omega^{**}} = S(U, V)$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$ is the subbasis for a topology, $\tau_{\omega^{**}}$, on any $F \subseteq Y^X$, is called the ω^* -open open topology.

Proposition 3.2. Let $F \subseteq C(X, Y)$, where $C(X, Y)$ is the set of all continuous functions from the topological space (X, τ) into (Y, σ) . If (Y, σ) is T_i , for $i = 0, 1, 2$; then $(F, T_{\omega^{**}})$ is T_i , for $i = 0, 1, 2$.

Proof. Let us prove the case for $i = 2$. Let $f, g \in F$ be such that $f \neq g$. Then there exist some $x \in X$ such that $f(x) \neq g(x)$. If Y is T_2 , then there exist disjoint open sets U and V in Y such that $f(x) \in U$ and $g(x) \in V$. Since f and g are continuous, there exist open sets M and N in X with $x \in M \cap N$, such that $f(M) \subseteq U$ and $g(N) \subseteq V$. Since each open set is an ω^* -open set, M and N are ω^* -open sets. Hence, $f \in (M, U)$, $g \in (N, V)$ and $(M, U) \cap (N, V) = \emptyset$. Hence $(F, T_{\omega^{**}})$ is T_2 . $\square$

Proposition 3.3. Let (X, τ) and (Y, σ) be any two topological spaces. If $F \subseteq \omega^*C(X, Y)$, then $\tau_{\omega^{**}}$ is admissible for F .

Proof. Let $F \subseteq \omega^*C(X, Y)$. Let $V \in (Y, \sigma)$ and $(f, p) \in E^{-1}(V)$. Then $f(p) \in V$. Since f is ω^* -continuous, there exists an ω^* -open set U such that $p \in U$ and $f(U) \subseteq V$. Hence $(f, p) \in (U, V) \times U$. If $(g, y) \in (U, V) \times U$, then $g(U) \subseteq V$ and $y \in U$. So $g(y) \in V$. Hence $(U, V) \times U \subseteq E^{-1}(V)$. Therefore, $T_{\omega^{**}}$ is admissible for F . $\square$

Definition 3.9. Let (X, τ) and (Y, σ) be any two topological spaces. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be ω^* -homeomorphism if it is a bijection so that the image and the inverse image of ω^* -open sets are ω^* -open. The collection of all ω^* -homeomorphisms from X into Y is denoted by $\omega^*H(X, Y)$.

Remark 3.1. The sets of the form (U, V) , where both U and V are ω^* -open sets in X form a subbasis for $(\omega^*H(X), T_{\omega^{**}})$.

Remark 3.2. $\omega^*H(X)$ with the binary operation $\circ$, composition of functions, and identity element e is a group.

Proposition 3.4. Let (X, τ) be a topological space and G a subgroup of $\omega^*H(X)$. Then $(G, T_{\omega^{**}})$ is a topological group.

Proof. Let X be a topological group and G a subgroup of $\omega^*H(X)$. To prove that the two maps $m : G \times G \rightarrow G$ defined by $m(g_1, g_2) = g_1 \circ g_2$ and $\Phi : G \times G \rightarrow G$ defined by $\Phi(g) = g^{-1}$ are continuous. Let (U, V) be a subbasic open set in $T_{\omega^{**}}$ such that both U and V are ω^* -open sets. Let $(f, g) \in m^{-1}((U, V))$. Then $f \circ g(U) \subseteq V$ and $g(U) \subseteq f^{-1}(V)$. So $(f, g) \in (g(U), V) \times (U, g(U)) \in \tau_{r^{**}} \times \tau_{\omega^{**}}$. But $(g(U), (V)) \times (U, g(U)) \subseteq m^{-1}((U, V))$. Hence m is continuous. Now, the inverse map $\Phi : G \rightarrow G$ is bijective and $\Phi^{-1} = \Phi$. To show that Φ is continuous, it is sufficient to show that Φ is an open map. Let (U, V) be a subbasic open set in $\tau_{r^{**}}$ where U and V are both ω^* -open sets. Since ω^* is a homeomorphism, $\Phi((U, V)) = ((X \setminus V, X \setminus U))$. If C and D are ω^* -closed sets then $\text{Int}_{\omega^*}(C)$ and $\text{Int}_{\omega^*}(D)$ are ω^* -open sets. Since $(X \setminus V), (X \setminus U)$ are ω^* -closed sets, $\text{Int}_{\omega^*}(X \setminus V), \text{Int}_{\omega^*}(X \setminus U)$ are open sets. Since G is a set of ω^* -homeomorphisms, $(X \setminus V, X \setminus U) = (\text{Int}_{\omega^*}(X \setminus V), \text{Int}_{\omega^*}(X \setminus U)) \in \tau_{\omega^{**}}$. Therefore $\Phi(U, V)$ is an open set in $\tau_{\omega^{**}}$. Hence Φ is open. $\square$

Proposition 3.5. Let $\{h_v : v \in v\}$ be a net in the group $\omega^*H(X)$ of self ω^* -homeomorphisms of a topological space (X, τ) . Then $h_v \rightarrow h$ in $\tau_{\omega^{**}}$ if and only if $h_v(x_\lambda) \xrightarrow{\omega} h(x)$ whenever $x_\lambda \xrightarrow{\omega} x$.

Proof. Suppose that $h_v \xrightarrow{\omega^*} h$ in $\tau_{\omega^{*oo}}$ and let $x_\lambda \xrightarrow{\omega^*} x$ in X . Suppose V is ω^* -open and $h(x) \in V$. Then there exists an ω^* -open set U in X containing x such that (U, V) is a subbasic open set in $(\omega^*H(X), \tau_{\omega^{*oo}})$ containing h . Then there exists $v_0 \in v$ such that $h_v \in (U, V)$ for every $v \geq v_0$. That is $h_v(U) \subseteq V$ for every $v \geq v_0$. Since $x_\lambda \xrightarrow{\omega^*} x$ for every ω^* -open set U containing x , there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$, for every $\lambda \geq \lambda_0$. Hence, for every $v \geq v_0, \lambda \geq \lambda_0; h_v(x_\lambda) \in V$. Also $h(x) \in V$. Hence $\{h_v(x_\lambda); v \in v, \lambda \in \Lambda\}$ ω^* -converges to $h(x)$ wherever $x_\lambda \xrightarrow{\omega^*} x$. Conversely, if possible let $h_{\omega^*} \xrightarrow{\omega^*} h$ in $\tau_{\omega^{*oo}}$. Then there exists a neighbourhood (U, V) containing h such that for any $v' \in v$, there exists a $v \in v$ such that $h_v \notin (U, V)$ for $v \geq v'$. That is, $h_v(U) \not\subseteq V$ for $v \geq v'$. From every ω^* -open set U containing x , choose an element x_U with $h_v(x_U) \notin V$. Let ω^*N_x be the ω^* -neighbourhood system at x . Now, $h \in (U, V)$ implies that $h(U) \subseteq V$. Hence, for any $(v', U) \in v \times \omega^*N_x$, there exists a $(v, U) \in v \times \omega^*N_x$ such that $h_v(x_U) \notin h(U)$, for $(v, U) \geq (v', U)$. Hence $\omega^* : h_v(x_U) \not\xrightarrow{\omega^*} h(x)$. Contrapositively, whenever $h_v(x_U) \xrightarrow{\omega^*} h(x)$ for $x_U \xrightarrow{\omega^*} x, h_v \xrightarrow{\omega^*} h$ in $T_{\omega^{*oo}}$. $\square$

Definition 3.10. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be ω^* -continuous if the inverse image of every open set of Y is ω^* -open in X . The set of all ω^* -continuous functions from X into Y is denoted by $\omega^*C(X, Y)$.

Proposition 3.6. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is ω^* -continuous at a point $x \in X$ if and only if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X ω^* -converging to x , then the net $\{f(x_\lambda) : \lambda \in \Lambda\}$ converges to $f(x)$ in Y .

Proof. Assume that f is ω^* -continuous at $x \in X$. Let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X , ω^* -converges to x . Let V be an open set in Y containing $f(x)$. Now, there exists an ω^* -open set U containing x in X such that $f(U) \subseteq V$. Now, $\{x_\lambda : \lambda \in \Lambda\}$ ω^* -converges to x implies that there exists a $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$. Hence, for all $\lambda \geq \lambda_0, f(x_\lambda) \in V$. Therefore, $\{f(x_\lambda) : \lambda \in \Lambda\}$ lies eventually in V , and hence it converges to $f(x)$. Conversely, let f be not ω^* -continuous at x . Then there exists an open set W containing $f(x)$ in Y such that for every ω^* -open set U containing $x \in X$, there exists an element x_U with $x_U \in U$ but $f(x_U) \notin W$. Let ω^*N_x be the ω^* -neighbourhood system at x . So, $\{x_U : U \in \omega^*N_x\}$ is a net in X ω^* -converging to x , but the net $\{f(x_U) : U \in \omega^*N_x\}$ in Y does not lie eventually in W and consequently it cannot converge to $f(x)$. $\square$

Definition 3.11. A net $\{f_\mu : \mu \in M\}$ in $\omega^*C(X, Y)$ ω^* -continuously converges to $f \in \omega^*C(X, Y)$ if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which

ω^* -converges to x in X . The net $\{f_\mu(x_\lambda) : (\lambda, \mu) \in \Lambda \times M\}$ converging to $f(x)$ in Y , where the direction $\geq$ in $\Lambda \times M$ is given as follows : for $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \Lambda \times M$, $(\lambda_1, \mu_1) \geq (\lambda_2, \mu_2)$ if $\lambda_1 \geq \lambda_2$ and $\mu_1 \geq \mu_2$.

Proposition 3.7. *A net $\{f_\mu : \mu \in M\}$ in $\omega^*C(X, Y)$ ω^* -continuously converges to $f \in \omega^*C(X, Y)$ if and only if for every $x \in X$ and for every open neighbourhood V of $f(x)$ in Y , there exist a $\mu_0 \in M$ and a ω^* -open neighbourhood U of x in X such that $f_\mu(U) \subseteq V$ for all $\mu \geq \mu_0$.*

Proof. Let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X ω^* -converging to x and V a neighbourhood of $f(x)$. Suppose that the condition holds. Then there exists an ω^* -open neighbourhood U of x and a $\mu_0 \in M$ such that $f_\mu(U) \subseteq V$ for all $\mu \geq \mu_0$. Since $x_\lambda \xrightarrow{\omega^*} x$, there exists a $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$. Hence, $f_\mu(x_\lambda) \in V$ for all $(\lambda, \mu) \geq (\lambda_0, \mu_0)$. That is $f_\mu(x_\lambda) \rightarrow f(x)$ and thus the net $\{f_\mu : \mu \in M\}$ in $\omega^*C(X, Y)$ ω^* -continuously converges to $f \in \omega^*C(X, Y)$. Conversely, let $x \in X$ and let V be an open neighbourhood of $f(x)$ such that for every $\mu \in M$ and every ω^* -open neighbourhood U of $x \in X$, there exists a $\mu' \geq \mu, \mu' \in M$ such that $f_{\mu'}(U) \not\subseteq V$. Then for every ω^* -open neighbourhood U of x , choose a point $x_U \in U$ such that $f_{\mu'}(x_U) \notin V$. Hence the net $\{x_U : U \in \omega^*N_x\}$ ω^* -converges to x but the net $\{f_\mu(x_U) : (U, \mu) \in (\omega^*N_x \times M)\}$ does not converge to $f(x)$ in Y . $\square$

Definition 3.12. If Λ is directed set, then $\omega^*\overline{\lim}_\Lambda(A_\lambda)$, where $A_\lambda \subseteq X$, is called the ω^* -upper limit of the net $\{A_\lambda : \lambda \in \Lambda\}$ in $P(X)$. That is, the set of all points $x \in X$ such that for every $\lambda_0 \in \Lambda$ and for every ω^* -open neighbourhood U of x in X , there exists a $\lambda \in \Lambda$ for which $\lambda \geq \lambda_0$ and $A_\lambda \cap U \neq \emptyset$.

Proposition 3.8. *If a net $\{f_\mu : \mu \in M\}$ in $\omega^*C(X, Y)$ ω^* -continuously converges to $f \in \omega^*C(X, Y)$, then $\omega^*\overline{\lim}_M(f_\mu^{-1}(K)) \subseteq f^{-1}(K)$ for every closed subset K of Y .*

Proof. Let $\{f_\mu : \mu \in M\}$ be a net in $\omega^*C(X, Y)$ ω^* -continuously converges to f and K be an arbitrary closed subset of Y . Let $x \in \omega^*\overline{\lim}_M(f_\mu^{-1}(K))$ and W an open neighbourhood of $f(x)$ in Y . Since the net $\{f_\mu : \mu \in M\}$ is ω^* -continuously converges to f there exists an ω^* -open neighbourhood V of x in X and a $\mu_0 \in M$ such that $f_\mu(V) \subseteq W$, for all $\mu \geq \mu_0$. Otherwise there exists a $\mu \in M, \mu \geq \mu_0$ such that $V \cap f_\mu^{-1}(K) \neq \emptyset$. Hence, $f_\mu(V) \cap K \subseteq W \cap K \neq \emptyset$. That is, $f(x) \in \text{Cl}(K) = K$. Thus $x \in f^{-1}(K)$. $\square$

Proposition 3.9. *Let $\{f_\mu : \mu \in M\}$ be a net in $\omega^*C(X, Y)$ such that $\omega^*\overline{\lim}_M(f_\mu^{-1}(K)) \subseteq f^{-1}(K)$ for every closed subset K of Y . Then the net $\{f_\mu : \mu \in M\}$ is ω^* -continuously converges to $f \in \omega^*C(X, Y)$.*

Proof. Let $\{f_\mu : \mu \in M\}$ be a net in $\omega^*C(X, Y)$ and $f \in \omega^*C(X, Y)$ be such that the give condition holds for every closed subset K of Y . Let $x \in X$ and W be an open neighbourhood of $f(x)$ in Y . Let $K = Y \setminus W$. Since $x \notin f^{-1}(K)$, then $x \notin \omega^*\overline{\lim}_M(f_\mu^{-1}(K))$. Hence there exists a $\mu_0 \in M$ and an ω^* -open neighbourhood U of x in X such that $f_\mu^{-1}(K) \cap U = \emptyset$ for $\mu \geq \mu_0$. Then $U \subseteq X \setminus f_\mu^{-1}(K) = f_\mu^{-1}(Y \setminus K) \subseteq f_\mu^{-1}(W)$. Thus $f_\mu(U) \subseteq W$ for all $\mu \geq \mu_0$. That is, the net $\{f_\mu : \mu \in M\}$ ω^* -continuously converges to $f \in \omega^*C(X, Y)$. $\square$

REFERENCES

1. ARENS, R. – *Topologies for homeomorphism groups*, Amer. J. Math., 68 (1946), 593–610.
2. ARENS, R.; DUGUNDJI, J. – *Topologies for function spaces*, Pacific J. Math., 1 (1951), 5–31.
3. FOX, R.H. – *On topologies for function spaces*, Bull. Amer. Math. Soc., 51 (1945), 429–432.
4. LAMBRINOS, P.TH. – *The bounded-open topology on function spaces*, Manuscripta Math., 36 (1981/82), 47–66.
5. LEVINE, N. – *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, 70 (1963), 36–41.
6. MURDESHWAR, M.G.; NAIMPALLY, S.A. – *Quasi-uniform topological spaces*, P. Noordhoff Ltd., Groningen, 1966.
7. PORTER, K.F. – *The open-open topology for function spaces*, Internat. J. Math. Math. Sci., 16 (1993), 111–116.
8. PORTER, K.F. – *The regular open-open topology for function spaces*, Internat. J. Math. Math. Sci., 19 (1996), 299–302.
9. SUNDARAM, P.; SHEIK JOHN, M. – *Weakly closed and weakly continuous maps in topological spaces*, Proc. 82nd session of the Indian Science Congress, Calcutta, 1995 (P-49).

10. SHEIK JOHN, M. – *A study on generalizations of closed sets and continuous maps in topological spaces*, Ph. D. Thesis, Bharathiyar University, Coimbatore, India, 2002.

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