A CLASS OF REACTION-DIFFUSION SYSTEMS WITH NONLOCAL INITIAL CONDITIONS*

BY

MONICA-DANA BURLICĂ and DANIELA ROȘU

Abstract. We consider an abstract nonlinear multi-valued reaction-diffusion system with delay and, using some compactness arguments coupled with metric fixed point techniques, we prove some sufficient conditions for the existence of at least one C^0 -solution.

Mathematics Subject Classification 2010: 34G25, 34K30, 35K55, 35K57, 47H06.

Key words: differential delay evolution systems, nonlocal delay initial condition, metric fixed point arguments, non-resonance condition, nonlinear reaction-diffusion systems.

1. Introduction

The goal of this paper is to provide an existence result for C^0 -solutions to the following class of nonlinear delay differential reaction-diffusion systems with nonlocal initial conditions:

$$(1.1) \quad \begin{cases} u'(t) \in Au(t) + F(t, u_t, v_t), & t \in \mathbb{R}_+, \\ v'(t) \in Bv(t) + g(t), & t \in \mathbb{R}_+, \\ g(t) \in G(t, u_t, v_t), & t \in \mathbb{R}_+, \\ u(t) = p(u, v)(t), & t \in [-\tau, 0], \\ v(t) = q(u, v)(t), & t \in [-\tau, 0]. \end{cases}$$

Here X and Y are Banach spaces, $A : D(A) \subseteq X \rightsquigarrow X$ and $B : D(B) \subseteq Y \rightsquigarrow Y$ are the infinitesimal generators of two nonlinear semigroups of contractions, $\tau > 0$ is arbitrary fixed, $F : \mathbb{R}_+ \times C([- \tau, 0]; \overline{D(A)}) \times C([- \tau, 0];$

*This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0052.

$\overline{D(B)} \rightarrow X$ is a continuous function while $G : \mathbb{R}_+ \times C([- \tau, 0]; \overline{D(A)}) \times C([- \tau, 0]; \overline{D(B)}) \rightsquigarrow Y$ is a strongly-weakly u.s.c. multi-function with nonempty, convex and weakly compact values. The nonlocal functions $p : C_b([- \tau, +\infty); \overline{D(A)}) \times C_b([- \tau, +\infty); \overline{D(B)}) \rightarrow C([- \tau, 0]; \overline{D(A)})$ and $q : C_b([- \tau, +\infty); X) \times C_b([- \tau, +\infty); \overline{D(B)}) \rightarrow C([- \tau, 0]; \overline{D(B)})$ are non-expansive.

If X is a Banach space and a is a real fixed number, we denote by $C_b([a, +\infty); X)$ the space of X -valued bounded and continuous functions on $[a, +\infty)$, with the sup-norm $\|\cdot\|_{C_b([a, +\infty); X)} = \sup\{\|u(t)\|, t \in [a, +\infty)\}$.

The closed subset in $C_b([a, +\infty); X)$ consisting of all elements $u \in C_b([a, +\infty); X)$ satisfying $u(t) \in \overline{D(A)}$, for each $t \in [a, +\infty)$, is denoted by $C_b([a, +\infty); \overline{D(A)})$. If $[a, b]$ is a real compact interval we denote by $C([a, b]; X)$ the space of all continuous functions from $[a, b]$ to X endowed with the sup-norm $\|\cdot\|_{C([a, b]; X)}$ and by $C([a, b]; \overline{D(A)})$ the subset which contains the functions $u \in C([a, b]; X)$ with $u(t) \in \overline{D(A)}$ for each $t \in [a, b]$.

If $u \in C([- \tau, +\infty); X)$ and $t \in [0, +\infty)$, the delayed function $u_t \in C([- \tau, 0]; X)$ is defined by $u_t(s) := u(t + s)$ for each $s \in [- \tau, 0]$. Similarly, we define $C_b([a, +\infty); Y)$, $C_b([a, +\infty); \overline{D(B)})$, $C([a, b]; \overline{D(B)})$ and the function v_t .

In the present paper we prove an existence result for a class of nonlinear multi-valued reaction-diffusion system with delay and nonlocal initial conditions. For other results concerning reaction-diffusion systems we mention: BURLICĂ [5], BURLICĂ and ROȘU [6], [7], DÍAZ and VRABIE [12], NECULA and VRABIE [17], ROȘU [19], [20] and the references therein. For (non-delayed) evolution equations subjected to nonlocal initial conditions see PAICU and VRABIE [18] and the references therein. For delay evolution equations with nonlocal initial data see BURLICĂ and ROȘU [8] and VRABIE [22], [23] and [25]. The single valued system was considered by BURLICĂ, ROȘU and VRABIE [10]. Our paper is inspired from BURLICĂ, ROȘU and VRABIE [10] and based on the result of BURLICĂ and ROȘU [8] and VRABIE [23] and our main Theorem is adapted to the abstract multi-valued reaction-diffusion systems.

The paper is divided into 5 sections. In Section 2 we recall some useful results throughout the paper. In Section 3 we formulate the main result, i.e. Theorem 3.1. In Section 4 we give some auxiliary lemmas and we prove the main result. In the last Section 5 we give an example referring to the some nonlinear reaction-diffusion system with delay, subjected to nonlocal

initial conditions: one of the unknown function is subjected to a periodic condition and the other one to a mean condition.

2. Preliminaries

We recall some basic concepts and results concerning m -dissipative operators and nonlinear evolution equations in Banach spaces. For other definitions, results and details regarding these topics, we refer the reader to BARBU [3] and VRABIE [21]. For more details concerning functional differential equations with delay we refer to Hale [16].

Let X be a Banach space with norm $\|\cdot\|$. An operator $A : D(A) \subseteq X \rightsquigarrow X$ is called *dissipative* if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, we have $[x_1 - x_2, y_2 - y_1]_+ \geq 0$. Here, $[x, y]_h := \frac{1}{h}(\|x + hy\| - \|x\|)$ and

$$[x, y]_+ := \lim_{h \downarrow 0} [x, y]_h = \inf\{[x, y]_h; h > 0\},$$

for each $x, y \in X$. We remark that $|[x, y]_+| \leq \|y\|$ for each $x, y \in X$. For other properties of the mapping $(x, y) \mapsto [x, y]_+$, see BARBU [3, Proposition 3.7, p. 100].

The operator $A : D(A) \subseteq X \rightsquigarrow X$ is called *m -dissipative* if it is dissipative and the range of $I - \lambda A$ is $R(I - \lambda A) = X$, for each $\lambda > 0$.

We consider the evolution equation

$$(2.1) \quad u'(t) \in Au(t) + f(t),$$

where $f \in L^1(a, b; X)$. A continuous function $u \in C([a, b]; \overline{D(A)})$ is called a C^0 -solution, or *integral solution* of (2.1) on $[a, b]$, if it satisfies:

$$\|u(t) - x\| \leq \|u(s) - x\| + \int_s^t [u(\tau) - x, f(\tau) + y]_+ d\tau$$

for each $x \in D(A)$, $y \in Ax$ and $a \leq s \leq t \leq b$.

Theorem 2.1. *Let $\omega > 0$ and let $A : D(A) \subseteq X \rightsquigarrow X$ be an m -dissipative operator such that $A + \omega I$ is dissipative. Then, for each $\xi \in \overline{D(A)}$ and $f \in L^1(a, b; X)$, there exists a unique C^0 -solution of (2.1) on $[a, b]$ which satisfies $u(a) = \xi$. If $f, g \in L^1(a, b; X)$ and u, v are two C^0 -solutions of (2.1) corresponding to f and g respectively, then:*

$$(2.2) \quad \|u(t) - v(t)\| \leq e^{-\omega(t-s)} \|u(s) - v(s)\| + \int_s^t e^{-\omega(t-\theta)} \|f(\theta) - g(\theta)\| d\theta$$

for each $a \leq s \leq t \leq b$.

In particular, if $x \in D(A)$ and $y \in Ax$, we have

$$(2.3) \quad \|u(t) - x\| \leq e^{-\omega(t-s)} \|u(s) - x\| + \int_s^t e^{-\omega(t-\theta)} \|f(\theta) + y\| d\theta$$

for each $a \leq s \leq t \leq b$.

See BARBU [3, Theorem 4.1, p. 128].

For $\xi \in \overline{D(A)}$, $f \in L^1(a, b; X)$ and $\tau \in [a, b]$, we denote by $u(\cdot, \tau, \xi, f)$ the unique C^0 -solution $u : [\tau, b] \rightarrow \overline{D(A)}$, of the problem (2.1) which satisfies the initial condition $u(\tau) = \xi$. The *semigroup* generated by A on $\overline{D(A)}$ is denoted by $\{S(t) : \overline{D(A)} \rightarrow \overline{D(A)}, t \geq 0\}$ and is defined by $S(t)\xi = u(t, 0, \xi, 0)$ for each $\xi \in \overline{D(A)}$ and $t \geq 0$. We say that the operator A generates a *compact semigroup* if, for each $t > 0$, the operator $S(t)$ is a compact one.

Definition 2.1. The m -dissipative operator $A : D(A) \subseteq X \rightsquigarrow X$ is called of *complete continuous type* if for each $(f_n)_n$ in $L^1(a, b; X)$ with $\lim_n f_n = f$ weakly in $L^1(a, b; X)$ and $(u_n)_n$ in $C([a, b]; \overline{D(A)})$, with u_n a C^0 -solution on $[a, b]$ of $u'_n(t) \in Au_n(t) + f_n(t)$, $t \in [a, b]$ and $\lim_n u_n = u$ strongly in $C([a, b]; X)$, it follows that u is a C^0 -solution on $[a, b]$ of the limit problem $u'(t) \in Au(t) + f(t)$, $t \in [a, b]$.

If A generates a compact semigroup and the topological dual of X is uniformly convex or A is linear, then A is of complete continuous type. See VRABIE [21, Corollary 2.3.1, p. 49]. The nonlinear diffusion operator $\Delta\theta$ in $L^1(\Omega)$ is an example of a m -dissipative operator which generates a compact semigroup and is of complete continuous type in a Banach space whose dual is not uniformly convex. See Theorem 5.1.

Definition 2.2. A subset $\mathcal{F} \subseteq L^1(a, b; X)$ is called *uniformly integrable* if, for each $\gamma > 0$ there exists $\delta(\gamma) > 0$ such that

$$\int_E \|f(s)\| ds \leq \gamma$$

for each $f \in \mathcal{F}$ and each measurable subset $E \subseteq [a, b]$ whose Lebesgue measure satisfies $\lambda(E) < \delta(\gamma)$.

Remark 2.1. One may easily check out that each uniformly integrable subset in $L^1(a, b; X)$ is norm bounded in $L^1(a, b; X)$. We also remark that if $\mathcal{F} \subseteq L^1(a, b; X)$ is bounded in $L^p(a, b; X)$ for some $p \in (1, +\infty]$, then $\mathcal{F}$ is uniformly integrable.

Theorem 2.2. *Let $A : D(A) \subseteq X \rightsquigarrow X$ be m -dissipative operator which generates a compact semigroup. Let $B \subseteq \overline{D(A)}$ be bounded and let $\mathcal{F}$ be uniformly integrable in $L^1(a, b; X)$. Then, for each $c \in (a, b)$, the C^0 -solutions set $\{u(\cdot, a, \xi, f); \xi \in B, f \in \mathcal{F}\}$ is relatively compact in $C([c, b]; X)$. If, in addition, B is relatively compact in X , then the C^0 -solutions set is relatively compact even in $C([a, b]; X)$.*

See BARAS [2] and VRABIE [21, Theorems 2.3.2 and 2.3.3, p. 46-47].

Now, we recall a version of a general fixed point theorem, for multi-functions, due to GLICKSBERG [15] in a locally convex space.

Theorem 2.3. *Let K be a nonempty, convex and compact set in a separated locally convex space and let $Q : K \rightsquigarrow K$ be a nonempty, closed and convex valued multi-function with closed graph. Then Q has at least one fixed point, i.e. there exists $f \in K$ such that $f \in Q(f)$.*

3. The main result

Let $a \in \mathbb{R}$. On the space $C_b([a, +\infty); Y)$ we consider the family of semi-norms $\{\|\cdot\|_k; k \in \mathbb{N}, k > a\}$, defined by $\|v\|_k = \|v\|_{C([a, k]; Y)}$ which generates a locally convex and separated topology. This space will be denoted by $\widetilde{C}_b([a, +\infty); Y)$.

We also denote by $\widetilde{L}^1(0, +\infty; Y)$ the space $L_{\text{loc}}^1(0, +\infty; Y)$ endowed with the family of semi-norms $\{\|\cdot\|_{k, x^*}; x^* \in L^1(0, k; Y)^*, k = 1, 2, \dots\}$, defined by $\|g\|_{k, x^*} = |x^*(g)|$. This space is also locally convex and separated. Moreover, the convergence in $\widetilde{L}^1(0, +\infty; Y)$ is nothing but the weak convergence in $L^1(0, k; Y)$ for $k = 1, 2, \dots$.

In the sequel we need the next hypotheses:

(H_A) the operator $A : D(A) \subseteq X \rightsquigarrow X$ is m -dissipative, $0 \in D(A)$, $0 \in A0$ and there exists $\omega > 0$ such that $A + \omega I$ is dissipative;

(H_B) the operator $B : D(B) \subseteq Y \rightsquigarrow Y$ satisfies:

(B_1) B is m -dissipative, $0 \in D(B)$, $0 \in B0$ and there exists $\gamma > 0$ such that $B + \gamma I$ is dissipative;

- (B₂) B generates a compact semigroup;
 (B₃) B is of complete continuous type;
 (H_F) the function $F : \mathbb{R}_+ \times C([- \tau, 0]; \overline{D(A)}) \times C([- \tau, 0]; \overline{D(B)}) \rightarrow X$ is continuous and satisfies:

(F₁) there exists $\ell > 0$ such that

$$\|F(t, u, v) - F(t, \tilde{u}, \tilde{v})\| \leq \ell [\|u - \tilde{u}\|_{C([- \tau, 0]; X)} + \|v - \tilde{v}\|_{C([- \tau, 0]; Y)}]$$

for each $t \in \mathbb{R}_+$, each $u, \tilde{u} \in C([- \tau, 0]; \overline{D(A)})$ and each $v, \tilde{v} \in C([- \tau, 0]; \overline{D(B)})$;

(F₂) there exists $m > 0$ such that $\|F(t, 0, 0)\| \leq m$ for each $t \in [0, +\infty)$;

(H_G) the multi-function $G : \mathbb{R}_+ \times C([- \tau, 0]; \overline{D(A)}) \times C([- \tau, 0]; \overline{D(B)}) \rightsquigarrow Y$ has nonempty, convex and weakly compact values and it is strongly-weakly u.s.c.;

(G₁) with $\ell > 0$ and $m > 0$ given by (F₁) and (F₂), we have

$$\|y\| \leq \ell [\|u\|_{C([- \tau, 0]; X)} + \|v\|_{C([- \tau, 0]; Y)}] + m$$

for each $u \in C([- \tau, 0]; \overline{D(A)})$, each $v \in C([- \tau, 0]; \overline{D(B)})$, each $y \in G(t, u, v)$ and each $t \geq 0$;

(H_c) the constants $\ell > 0$, $\omega > 0$ and $\gamma > 0$ satisfy the nonresonance condition $\ell < \frac{\omega\gamma}{\omega + \gamma}$;

(H_p) $p : C_b([- \tau, +\infty); \overline{D(A)}) \times C_b([- \tau, +\infty); \overline{D(B)}) \rightarrow C([- \tau, 0]; \overline{D(A)})$ is continuous from the space $\tilde{C}_b([- \tau, +\infty); \overline{D(A)}) \times \tilde{C}_b([- \tau, +\infty); \overline{D(B)})$ to $C([- \tau, 0]; \overline{D(A)})$ and satisfies:

(p₁) for each $u \in C_b([- \tau, +\infty); \overline{D(A)})$ and each $v \in C_b([- \tau, +\infty); \overline{D(B)})$, we have $\|p(u, v)\|_{C([- \tau, 0]; X)} \leq \|u\|_{C_b([0, +\infty); X)}$;

(p₂) there exists $a > 0$ such that for each $u, \tilde{u} \in C_b([- \tau, +\infty); \overline{D(A)})$ and each $v, \tilde{v} \in C_b([- \tau, +\infty); \overline{D(B)})$, we have

$$\begin{aligned} & \|p(u, v) - p(\tilde{u}, \tilde{v})\|_{C([- \tau, 0]; X)} \\ & \leq \max\{\|u - \tilde{u}\|_{C_b([a, +\infty); X)}, \|v - \tilde{v}\|_{C_b([0, +\infty); Y)}\}; \end{aligned}$$

(H_q) $q : C_b([-\tau, +\infty); X) \times C_b([-\tau, +\infty); \overline{D(B)}) \rightarrow C([-\tau, 0]; \overline{D(B)})$ is continuous from $\tilde{C}_b([-\tau, +\infty); \overline{D(A)}) \times \tilde{C}_b([-\tau, +\infty); \overline{D(B)})$ to $C([-\tau, 0]; \overline{D(B)})$ and satisfies:

- (q_1) for each $u \in C_b([-\tau, +\infty); X)$ and each $v \in C_b([-\tau, +\infty); \overline{D(B)})$, we have $\|q(u, v)\|_{C([-\tau, 0]; Y)} \leq \|v\|_{C_b([0, +\infty); Y)}$;
- (q_2) for each $u, \tilde{u} \in C_b([-\tau, +\infty); X)$ and $v, \tilde{v} \in C_b([-\tau, +\infty); \overline{D(B)})$, we have $\|q(u, v) - q(\tilde{u}, \tilde{v})\|_{C([-\tau, 0]; Y)} \leq \max\{\|u - \tilde{u}\|_{C_b([0, +\infty); X)}, \|v - \tilde{v}\|_{C_b([0, +\infty); Y)}\}$;
- (q_3) for each bounded set $\mathcal{U}$ in $C_b([-\tau, +\infty); X)$ and each bounded $\mathcal{V}$ in $C_b([-\tau, +\infty); \overline{D(B)})$ which is relatively compact in $\tilde{C}_b([0, +\infty); Y)$ for each $\delta \in (0, +\infty)$, the set $q(\mathcal{U}, \mathcal{V})$ is relatively compact in the space $C([-\tau, 0]; Y)$.

Our main result is:

Theorem 3.1. *If (H_A) , (H_B) , (H_F) , (H_G) , (H_p) , (H_q) and (H_c) are satisfied, then (1.1) has at least one C^0 -solution, $(u, v) \in C_b([-\tau, +\infty); \overline{D(A)}) \times C_b([-\tau, +\infty); \overline{D(B)})$, satisfying*

$$(3.1) \quad \begin{cases} \|u\|_{C_b([-\tau, +\infty); X)} \leq \frac{m}{\omega_m - \ell} \\ \|v\|_{C_b([-\tau, +\infty); Y)} \leq \frac{m}{\gamma - \ell}. \end{cases}$$

4. Proof of Theorem 3.1

Let $\varepsilon > 0$, let $(f, g) \in C_b([0, +\infty); X) \times L^\infty(0, +\infty; Y)$ and let us consider the system

$$(4.1) \quad \begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbb{R}_+, \\ v'(t) \in Bv(t) + g(t), & t \in \mathbb{R}_+, \\ u(t) = (1 - \varepsilon)p(u, v)(t), & t \in [-\tau, 0], \\ v(t) = (1 - \varepsilon)q(u, v)(t), & t \in [-\tau, 0]. \end{cases}$$

Lemma 4.1. *If the hypotheses (H_A) , (B_1) , (p_1) , (p_2) , (q_1) and (q_2) are satisfied, then the system (4.1) has a unique C^0 -solution $(u, v) \in C_b([-\tau, \infty); \overline{D(A)}) \times C_b([-\tau, \infty); \overline{D(B)})$.*

If, in addition, $\|f\|_\infty \leq m$ and $\|g\|_\infty \leq m$, then we have

$$(4.2) \quad \begin{cases} \|u\|_{C_b([-\tau, +\infty); X)} \leq \frac{m}{\omega} \\ \|v\|_{C_b([-\tau, +\infty); Y)} \leq \frac{m}{\gamma}. \end{cases}$$

For the proof, see BURLICĂ, ROȘU and VRABIE [10, Lemma 5.1].

Idea of the proof of Theorem 3.1. Let us consider the approximate problem

$$(4.3) \quad \begin{cases} u'(t) \in Au(t) + F(t, u_t, v_t), & t \in \mathbb{R}_+, \\ v'(t) \in Bv(t) + g(t), & t \in \mathbb{R}_+, \\ g(t) \in \chi_{[0, 1/\varepsilon]}(t)G(t, u_t, v_t), & t \in \mathbb{R}_+, \\ u(t) = (1 - \varepsilon)p(u, v)(t), & t \in [-\tau, 0], \\ v(t) = (1 - \varepsilon)q(u, v)(t), & t \in [-\tau, 0], \end{cases}$$

as well as the following two auxiliary systems

$$(4.4) \quad \begin{cases} \tilde{v}'(t) \in B\tilde{v}(t) + g(t), & t \in [0, +\infty), \\ \tilde{v}(t) = (1 - \varepsilon)q(u, \tilde{v})(t), & t \in [-\tau, 0] \end{cases}$$

and

$$(4.5) \quad \begin{cases} \tilde{u}'(t) \in A\tilde{u}(t) + F(t, \tilde{u}_t, \tilde{v}_t), & t \in [0, +\infty), \\ \tilde{u}(t) = (1 - \varepsilon)p(\tilde{u}, \tilde{v})(t), & t \in [-\tau, 0], \end{cases}$$

where $\varepsilon \in (0, 1)$ is arbitrary fixed. We will use a fixed point device described below.

Fix an arbitrary $(u, g) \in C_b([-\tau, +\infty); X) \times L^\infty(0, +\infty; Y)$. Using VRABIE [24, Lemma 4.5], the problem (4.4) has a unique C^0 -solution $\tilde{v} \in C_b([-\tau, +\infty); \overline{D(B)})$. Next, by BURLICĂ and ROȘU [8, Theorem 3.1], the problem (4.5) has a unique C^0 -solution $\tilde{u} \in C_b([-\tau, +\infty); \overline{D(A)})$.

Now we define the multifunction

$$\Gamma_\varepsilon : C_b([-\tau, +\infty); X) \times L^\infty(0, +\infty; Y) \rightsquigarrow C_b([-\tau, +\infty); X) \times L^\infty(0, +\infty; Y)$$

by

$$(4.6) \quad \Gamma_\varepsilon(u, g) := \left\{ (\tilde{u}, \tilde{g}); \tilde{g} \in \tilde{L}^1(0, +\infty; Y), \tilde{g}(t) \in G_\varepsilon(t, \tilde{u}_t, \tilde{v}_t) \right. \\ \left. \text{a.e. for } t \in [0, +\infty) \right\},$$

for each $(u, g) \in C_b([- \tau, +\infty); X) \times L^\infty(0, +\infty; Y)$, where

$$G_\varepsilon(t, u, v) = \chi_{[0, 1/\varepsilon]}(t)G(t, u, v),$$

for each $(t, u, v) \in [0, +\infty) \times C([- \tau, 0]; \overline{D(A)}) \times C([- \tau, 0]; \overline{D(B)})$, and $\chi_{[0, 1/\varepsilon]}$ is the characteristic function of $[0, 1/\varepsilon]$. Obviously (4.3) has a C^0 -solution if and only if the multi-function Γ_ε has a fixed point on a suitably defined set.

Finally, we consider a family $\{(u_\varepsilon, v_\varepsilon); \varepsilon \in (0, 1)\}$ of C^0 -solutions for the problem (4.3) and we show that we can pick up a sequence $((u_{\varepsilon_n}, v_{\varepsilon_n}))_n$, such that $\lim_n \varepsilon_n = 0$, $\lim_n (u_{\varepsilon_n}, v_{\varepsilon_n}) = (u, v)$ in $C_b([- \tau, +\infty); X) \times C_b([- \tau, +\infty); Y)$ and (u, v) is a C^0 -solution of (1.1).

We begin by showing that we can suitably define a nonempty, convex and compact set $\mathcal{K}_\varepsilon$ in the product space $C_b([- \tau, +\infty); X) \times L^\infty(0, +\infty; Y)$ such that Γ_ε maps $\mathcal{K}_\varepsilon$ into itself and has sequentially closed graph with respect to the strong topology on $C_b([- \tau, +\infty); X)$ and the locally convex topology on $\tilde{L}^1(0, +\infty; Y)$.

We will do this with the help of the next lemmas.

Lemma 4.2. *Let (H_A) , (B_1) , (B_2) in (H_B) , (H_F) , (H_p) , (H_q) and (H_c) be satisfied, and let*

$$(4.7) \quad r = \frac{m\omega\gamma}{\omega\gamma - \ell(\omega + \gamma)}.$$

Then, for each $(u, g) \in C_b([- \tau, +\infty); X) \times L^\infty(0, +\infty; Y)$ satisfying

$$(4.8) \quad \begin{cases} \|u\|_{C_b([- \tau, +\infty); X)} \leq \frac{r}{\omega} \\ \|g\|_{L^\infty(0, +\infty; Y)} \leq r, \end{cases}$$

the pair $(\tilde{u}, \tilde{v})$, where $\tilde{v}$ is the unique C^0 -solution of (4.4) and $\tilde{u}$ the unique C^0 -solution of (4.5), satisfies

$$(4.9) \quad \begin{cases} \|\tilde{u}\|_{C_b([- \tau, +\infty); X)} \leq \frac{r}{\omega} \\ \|\tilde{v}\|_{C_b([- \tau, +\infty); Y)} \leq \frac{r}{\gamma}, \end{cases}$$

and

$$(4.10) \quad \|F(t, \tilde{u}_t, \tilde{v}_t)\| \leq r,$$

for each $t \in [0, +\infty)$.

For the proof, see BURLICĂ, ROȘU and VRABIE [10, Lemma 5.2].

The lemma below is a continuity with respect to the data result and it was proved by BURLICĂ, ROȘU and VRABIE [9].

Lemma 4.3. *Let $\{F_n : \mathbb{R}_+ \times C([- \tau, 0]; \overline{D(A)}) \rightarrow X; n \in \mathbb{N}\}$ be a family of continuous functions satisfying:*

(h₁) *there exists $\ell > 0$ such that $\|F_n(t, x) - F_n(t, y)\| \leq \ell \|x - y\|_{C([- \tau, 0]; X)}$ for each $n \in \mathbb{N}$, each $t \in [0, +\infty)$ and $x, y \in C([- \tau, 0]; \overline{D(A)})$;*

(h₂) *there exists $m > 0$ such that $\|F_n(t, 0)\| \leq m$ for each $n \in \mathbb{N}$ and each $t \in [0, +\infty)$;*

(h₃) *$\lim_n F_n(t, x) = F(t, x)$ uniformly for $t \in [0, +\infty)$ (for t in bounded intervals in $[0, +\infty)$) and x in bounded subsets in $C([- \tau, 0]; \overline{D(A)})$.*

Let $\{p_n : C_b([- \tau, +\infty); \overline{D(A)}) \rightarrow C([- \tau, 0]; \overline{D(A)})$; $n \in \mathbb{N}\}$ be a family of functions satisfying:

(h₄) *for each $n \in \mathbb{N}$ and $u \in C_b([- \tau, +\infty); \overline{D(A)})$, we have $\|p_n(u)\|_{C([- \tau, 0]; X)} \leq \|u\|_{C_b([0, +\infty); X)}$;*

(h₅) *there exists $a > 0$ such that for each $n \in \mathbb{N}$ and $u, \tilde{u} \in C_b([- \tau, +\infty); \overline{D(A)})$, we have $\|p_n(u) - p_n(\tilde{u})\|_{C([- \tau, 0]; X)} \leq \|u - \tilde{u}\|_{C_b([a, +\infty); X)}$;*

(h₆) *$\lim_n p_n(u) = p(u)$ uniformly for u in bounded subsets in $C_b([- \tau, +\infty); \overline{D(A)})$ (and p is continuous from $\tilde{C}_b([- \tau, +\infty); \overline{D(A)})$ to $C([- \tau, 0]; \overline{D(A)})$).*

Let us assume further that A satisfies (H_A) and $\ell < \omega$ holds true. Let $(u_n)_n$ be the sequence of C^0 -solutions of the problem

$$(4.11) \quad \begin{cases} u'_n(t) \in Au_n(t) + F_n(t, u_n), & t \in [0, +\infty), \\ u_n(t) = p_n(u_n)(t), & t \in [-\tau, 0], \end{cases}$$

whose existence and uniqueness is ensured by BURLICĂ and ROȘU [8, Theorem 3.1]. Then

$$\lim_n u_n = u$$

in $C_b([- \tau, +\infty); X)$ (in $\tilde{C}_b([- \tau, +\infty); X)$), where u is the C^0 -solution of the limiting problem

$$(4.12) \quad \begin{cases} u'(t) \in Au(t) + F(t, u_t), & t \in [0, +\infty), \\ u(t) = p(u)(t), & t \in [-\tau, 0]. \end{cases}$$

Lemma 4.4. *Let us suppose that the hypotheses (H_A) , (H_B) , (H_F) , (H_G) , (H_p) , (H_q) and (H_c) are satisfied, and let $r > 0$ be given by (4.7). Let $\rho = r/\omega$ and $K_\varepsilon := K_\rho \times K_r$, where K_ρ is the closed ball with center 0 and radius ρ in $C_b([-\tau, +\infty); X)$ and K_r is the closed ball with center 0 and radius r in $L^\infty(0, +\infty; Y)$ multiplied by $\chi_{[0, 1/\varepsilon]}$. Then the operator Γ_ε defined by (4.6) maps K_ε into itself and Γ_ε has sequentially closed graph with respect to the norm topology on $C_b([-\tau, +\infty); X)$ and the locally convex topology on $\tilde{L}^1(0, +\infty; Y)$.*

Proof. The operator

$$\Gamma_\varepsilon : C_b([-\tau, +\infty); X) \times L^\infty(0, +\infty; Y) \rightsquigarrow C_b([-\tau, +\infty); X) \times L^\infty(0, +\infty; Y)$$

is defined by

$$\Gamma_\varepsilon(u, g) := \{(\tilde{u}, \tilde{g}) : \tilde{g} \in \tilde{L}^1(0, +\infty; Y), \tilde{g}(t) \in G_\varepsilon(t, \tilde{u}_t, \tilde{v}_t) \text{ a.e. for } t \in [0, +\infty)\}$$

where $G_\varepsilon(t, \tilde{u}_t, \tilde{v}_t) = \chi_{[0, 1/\varepsilon]}(t)G(t, \tilde{u}_t, \tilde{v}_t)$ and $\tilde{u}$ is the unique C^0 -solution of the problem (4.5) where $\tilde{v}$ is the unique C^0 -solution of the problem (4.4). If $(u, g) \in K_\varepsilon$, from Lemma 4.2 we deduce that the pair $(\tilde{u}, \tilde{v}) \in C_b([-\tau, \infty); \overline{D(A)}) \times C_b([-\tau, \infty); \overline{D(B)})$ satisfies

$$(4.13) \quad \|\tilde{u}\|_{C_b([-\tau, +\infty); Y)} \leq \frac{r}{\omega}, \quad \|\tilde{v}\|_{C_b([-\tau, +\infty); Y)} \leq \frac{r}{\gamma}.$$

An appeal to (G_1) shows that Γ_ε maps K_ε into itself.

To prove that Γ_ε has sequentially closed graph with respect to the norm topology on $C_b([-\tau, +\infty); X)$ and the locally convex topology on $\tilde{L}^1(0, +\infty; Y)$, let $((u_n, g_n))_n$ be an arbitrary sequence in K_ε and $(\tilde{u}_n, \tilde{g}_n) \in \Gamma_\varepsilon(u_n, g_n)$ for each $n \in \mathbb{N}$. That means there exists $\tilde{v}_n \in C_b([-\tau, \infty); \overline{D(B)})$, the unique C^0 -solution for the problem

$$(4.14) \quad \begin{cases} \tilde{v}'_n(t) \in B\tilde{v}_n(t) + g_n(t), & t \in [0, +\infty), \\ \tilde{v}_n(t) = (1 - \varepsilon)q(u_n, \tilde{v}_n)(t), & t \in [-\tau, 0], \end{cases}$$

and $\tilde{u}_n \in C_b([-\tau, \infty); \overline{D(A)})$, the unique C^0 -solution for the problem

$$(4.15) \quad \begin{cases} \tilde{u}'_n(t) \in A\tilde{u}_n(t) + F(t, \tilde{u}_{n_t}, \tilde{v}_{n_t}), & t \in [0, +\infty), \\ \tilde{u}_n(t) = (1 - \varepsilon)p(\tilde{u}_n, \tilde{v}_n)(t), & t \in [-\tau, 0], \end{cases}$$

for $n \in \mathbb{N}$. By the definition of Γ_ε , $\tilde{g}_n \in \tilde{L}^1(0, +\infty; Y)$ and $\tilde{g}_n(t) \in G_\varepsilon(t, \tilde{u}_{n_t}, \tilde{v}_{n_t})$ a.e. for $t \in [0, +\infty)$ and for each $n \in \mathbb{N}$. We suppose that

$$(4.16) \quad \lim_n (u_n, g_n) = (u, g) \text{ in } C_b([-\tau, +\infty); X) \times \tilde{L}^1(0, +\infty; Y)$$

and

$$(4.17) \quad \lim_n (\tilde{u}_n, \tilde{g}_n) = (\tilde{u}, \tilde{g}) \text{ in } C_b([-\tau, +\infty); X) \times \tilde{L}^1(0, +\infty; Y).$$

Reasoning as in BURLICĂ, ROȘU and VRABIE [10, Lemma 5.3], the set $\{\tilde{v}_n; n \in \mathbb{N}\}$ is relatively compact in $C_b([-\tau, +\infty); Y)$. Indeed, from (4.13), we get $\|\tilde{v}_n(0)\| \leq \|\tilde{v}_n\|_{C_b([-\tau, +\infty); Y)} \leq \frac{r}{\gamma}$ for each $n \in \mathbb{N}$ and, by Remark 2.1, the set $\{g_n; n \in \mathbb{N}\}$ is uniformly integrable, so we are in the hypotheses of Theorem 2.2, wherefrom we obtain that $\{\tilde{v}_n; n \in \mathbb{N}\}$ is relatively compact in $C([\delta, k]; Y)$ for $k = 1, 2, \dots$ and $\delta \in (0, k)$. Since $\{u_n; n \in \mathbb{N}\}$ is bounded in $C_b([-\tau, +\infty); X)$ and, from (q_3) , $\{\tilde{v}_n(0); n \in \mathbb{N}\}$ is relatively compact in Y , we can apply the second part of Theorem 2.2, wherefrom $\{\tilde{v}_n; n \in \mathbb{N}\}$ is relatively compact in $C_b([0, k]; Y)$ for $k = 1, 2, \dots$. An appeal to condition (q_1) shows that $\{\tilde{v}_n; n \in \mathbb{N}\}$ is relatively compact in $\tilde{C}_b([-\tau, +\infty); Y)$.

That means that there exists $\tilde{v} \in C_b([-\tau, +\infty); Y)$ such that on a subsequence of $(\tilde{v}_n)_n$ —denoted for simplicity by $(\tilde{v}_n)_{n^-}$, we have

$$(4.18) \quad \lim_n \tilde{v}_n = \tilde{v}$$

in $\tilde{C}_b([-\tau, +\infty); Y)$. In fact, we can prove a stronger condition, i.e. the convergence in $C_b([-\tau, +\infty); Y)$.

Since $\tilde{g}_n \in K_r$ which is weakly closed in $\tilde{L}^1(0, +\infty; Y)$ and $\lim_n g_n = g$ weakly in $\tilde{L}^1(0, +\infty; Y)$, we deduce that $g \in K_r$. From this relation combined with (4.18) and with the fact that B is of complete continuous type, we obtain that $\tilde{v}$ is C^0 -solution of the problem $\tilde{v}'(t) \in B\tilde{v}(t) + g(t)$, $t \in [0, +\infty)$. Next, we fix $k_\varepsilon \in \mathbb{N}$, with $k_\varepsilon \geq 1 + 1/\varepsilon$ and an arbitrary constant $\alpha > 0$. The sequence $(\tilde{v}_n)_n$ is convergent to $\tilde{v}$ on $[0, k_\varepsilon]$, so there exists $n_\varepsilon(\alpha) \in \mathbb{N}$ such that $\|\tilde{v}_n(t) - \tilde{v}(t)\| \leq \alpha$ for $n \in \mathbb{N}$, $n \geq n_\varepsilon(\alpha)$ and $t \in [0, k_\varepsilon]$.

For a.e. $s \in [k_\varepsilon, t]$, $\|g_n(s) - g(s)\| = 0$ and, using (2.2), we deduce

$$\begin{aligned} \|\tilde{v}_n(t) - \tilde{v}(t)\| &\leq e^{-\gamma k_\varepsilon} \|\tilde{v}_n(k_\varepsilon) - \tilde{v}(k_\varepsilon)\| + \int_{k_\varepsilon}^t e^{-\gamma(t-s)} \|g_n(s) - g(s)\| ds \\ &\leq \|\tilde{v}_n(k_\varepsilon) - \tilde{v}(k_\varepsilon)\| \leq \alpha \end{aligned}$$

for $n \geq n_\varepsilon(\alpha)$ and $t \geq k_\varepsilon$. From this inequality and taken into account (4.16), we obtain that

$$(4.19) \quad \lim_n \tilde{v}_n = \tilde{v}$$

in $C_b([0, +\infty); Y)$ and so, in $C_b([- \tau, +\infty); Y)$. From (q_2) we deduce that $\tilde{v}(t) = (1 - \varepsilon)q(u, \tilde{v})(t)$ for each $t \in [-\tau, 0]$.

Since for each $n \in \mathbb{N}$, $\tilde{u}_n$ is the C^0 -solution of the problem (4.15) and, by (F_1) , (p_1) and (p_2) , the functions $F_n(t, \cdot) := F(t, \cdot, \tilde{v}_{n_t})$ and $p_n(\cdot) := (1 - \varepsilon)p(\cdot, \tilde{v}_n)$ for $n \in \mathbb{N}$ and $t \in [0, +\infty)$, satisfy Lemma 4.3, we deduce that the limit function $\tilde{u} = \lim_n \tilde{u}_n$ is the unique C^0 -solution of the problem

$$(4.20) \quad \begin{cases} \tilde{u}'(t) \in A\tilde{u}(t) + F(t, \tilde{u}_t, \tilde{v}_t), & t \in [0, +\infty), \\ \tilde{u}(t) = (1 - \varepsilon)p(\tilde{u}, \tilde{v})(t), & t \in [-\tau, 0]. \end{cases}$$

Finally, let us prove that $\tilde{g}(t) \in G_\varepsilon(t, \tilde{u}_t, \tilde{v}_t)$ a.e. $t \in [0, \infty)$. We have $\tilde{g}_n(t) \in G_\varepsilon(t, \tilde{u}_{n_t}, \tilde{v}_{n_t})$ a.e. $t \in [0, k]$ and G_ε is strongly-weakly u.s.c.,

$$\lim_n \tilde{g}_n = \tilde{g} \text{ weakly in } L^1([0, k]; Y),$$

$$\lim_n (\tilde{u}_{n_t}, \tilde{v}_{n_t}) = (\tilde{u}_t, \tilde{v}_t) \text{ in } C([- \tau, 0]; \overline{D(A)}) \times C([- \tau, 0]; \overline{D(B)}) \text{ a.e. } t \in [0, k],$$

for each $k = 1, 2, \dots$. So, we are in the hypotheses of VRABIE [21, Theorem 3.1.2, p. 88], wherefrom we deduce that $\tilde{g}(t) \in G_\varepsilon(t, \tilde{u}_t, \tilde{v}_t)$ a.e. $t \in [0, k]$, for $k = 1, 2, \dots$ and thus a.e. $t \in [0, \infty)$. $\square$

Lemma 4.5. *Let us suppose that the hypotheses (H_A) , (H_B) , (H_F) , (H_G) , (H_p) , (H_q) and (H_c) are satisfied. Then, for each $\varepsilon \in (0, 1)$, the set $\text{conv } \Gamma_\varepsilon(K_\varepsilon)$ is compact in the product space $C_b([- \tau, +\infty); X) \times \tilde{L}^1(0, +\infty; Y)$.*

Proof. We begin by proving that the set $\Gamma_\varepsilon(K_\varepsilon)$ is relatively compact in the product space $C_b([- \tau, +\infty); X) \times \tilde{L}^1(0, +\infty; Y)$. Let $((\tilde{u}_n, \tilde{g}_n))_n$ be an arbitrary sequence in $\Gamma_\varepsilon(K_\varepsilon)$ and $((u_n, g_n))_n \in K_\varepsilon$ such that $(\tilde{u}_n, \tilde{g}_n) \in \Gamma_\varepsilon(u_n, g_n)$ for $n \in \mathbb{N}$. So, $\tilde{v}_n$ is the unique C^0 -solution of the problem

$$\begin{cases} \tilde{v}'_n(t) \in B\tilde{v}_n(t) + g_n(t), & t \in [0, +\infty), \\ \tilde{v}_n(t) = (1 - \varepsilon)q(u_n, \tilde{v}_n)(t), & t \in [-\tau, 0] \end{cases}$$

and $\tilde{u}_n$ is the unique C^0 -solution of the problem

$$\begin{cases} \tilde{u}'_n(t) \in A\tilde{u}_n(t) + F(t, \tilde{u}_{n_t}, \tilde{v}_{n_t}), & t \in [0, +\infty), \\ \tilde{u}_n(t) = (1 - \varepsilon)p(\tilde{u}_n, \tilde{v}_n)(t), & t \in [-\tau, 0], \end{cases}$$

for each $n \in \mathbb{N}$. Since $\{g_n; n \in \mathbb{N}\}$ is bounded in $L^\infty(0, +\infty; Y)$ we deduce that there exists $g \in L^\infty(0, +\infty; Y)$ such that on a subsequence at least, we have $\lim_n g_n = g$ weakly in $\tilde{L}^1(0, \infty; Y)$. Reasoning as in Lemma 4.4, we deduce that there exists $\tilde{v} \in C_b([-\tau, +\infty); Y)$ and a sub-subsequence of $(\tilde{v}_n)_n$ —denoted for simplicity again by $(\tilde{v}_n)_{n^-}$, such that $\lim_n \tilde{v}_n = \tilde{v}$ in $C_b([-\tau, +\infty); Y)$. Let $\tilde{u} \in C_b([-\tau, +\infty); \overline{D(A)})$ the unique C^0 -solution of the problem

$$(4.21) \quad \begin{cases} \tilde{u}'(t) \in A\tilde{u}(t) + F(t, \tilde{u}_t, \tilde{v}_t), & t \in [0, +\infty), \\ \tilde{u}(t) = (1 - \varepsilon)p(\tilde{u}, \tilde{v})(t), & t \in [-\tau, 0], \end{cases}$$

with $\tilde{v}$ as above. Reasoning as in Lemma 4.4, we deduce that on a subsequence at least, we have $\lim_n \tilde{u}_n = \tilde{u}$ in $C_b([-\tau, +\infty); X)$.

Since $\tilde{g}_n(t) \in G_\varepsilon(t, \tilde{u}_{n_t}, \tilde{v}_{n_t})$ a. e. for $t \in [0, +\infty)$ and for each $n \in \mathbb{N}$, by (G_1) , we deduce that, at least on a subsequence—denoted for simplicity by $(\tilde{g}_n)_{n^-}$, we have

$$(4.22) \quad \lim_n \tilde{g}_n = \tilde{g} \text{ weakly in } \tilde{L}^1(0, \infty; Y).$$

Furthermore, because the set K_r is weakly closed in $\tilde{L}^1(0, +\infty; Y)$ we deduce that $\tilde{g} \in K_r$. Let us observe that we have

$$(4.23) \quad \tilde{g}(t) \in G_\varepsilon(t, \tilde{u}_t, \tilde{v}_t)$$

a.e. $t \in [0, \infty)$. Indeed, let us remind that $\tilde{g}_n(t) \in G_\varepsilon(t, \tilde{u}_{n_t}, \tilde{v}_{n_t})$ a.e. $t \in [0, k]$, G_ε is strongly-weakly u.s.c.,

$$\lim_n \tilde{g}_n = \tilde{g} \text{ weakly in } L^1([0, k]; Y),$$

$$\lim_n (\tilde{u}_{n_t}, \tilde{v}_{n_t}) = (\tilde{u}_t, \tilde{v}_t) \text{ in } C([-\tau, 0]; X) \times C([-\tau, 0]; Y) \text{ a.e. } t \in [0, k],$$

for each $k = 1, 2, \dots$. So, by VRABIE [21, Theorem 3.1.2, p. 88], we obtain that $\tilde{g}(t) \in G_\varepsilon(t, \tilde{u}_t, \tilde{v}_t)$ a.e. $t \in [0, k]$, for $k = 1, 2, \dots$ and thus a.e. $t \in [0, \infty)$. We conclude that $\Gamma_\varepsilon(K_\varepsilon)$ is relatively compact in $C_b([-\tau, +\infty); X) \times \tilde{L}^1(0, +\infty; Y)$ and thus, by DUNFORD and SCHWARTZ [13, Theorem 6, p. 416], $\overline{\text{conv } \Gamma_\varepsilon(K_\varepsilon)}$ is compact in $C_b([-\tau, +\infty); X) \times \tilde{L}^1(0, +\infty; Y)$. $\square$

Now, we prove Theorem 3.1.

Proof. Let $\varepsilon \in (0, 1)$ be arbitrary but fixed and let $\mathcal{K}_\varepsilon = \overline{\text{conv } \Gamma_\varepsilon(K_\varepsilon)}$. By Lemma 4.5, it follows that the operator $\Gamma_\varepsilon : \mathcal{K}_\varepsilon \rightsquigarrow \mathcal{K}_\varepsilon$ has convex and

compact values. Moreover, by Lemma 4.4, the graph of Γ_ε is sequentially closed. Since, in a Banach space, the weak closure of a weakly relatively compact set coincides with its weak sequential closure – see EDWARDS [14, Theorem 8.12.1, p. 549] –, using a standard diagonal process, we deduce that the graph of Γ_ε is even closed in $C_b([-\tau, +\infty); X) \times \tilde{L}^1(0, +\infty; Y)$. From Theorem 2.3, we deduce that Γ_ε has at least one fixed point $(u_\varepsilon, g_\varepsilon)$. Clearly this means that the approximate problem (4.3) has at least one solution $(u_\varepsilon, v_\varepsilon)$. For each $\varepsilon \in (0, 1)$, fix such a solution and consider the set $\{(u_\varepsilon, v_\varepsilon); \varepsilon \in (0, 1)\}$.

At this point, reasoning as in the proof of BURLICĂ, ROȘU and VRABIE [10, Theorem 3.1], we deduce that the set $\{(u_\varepsilon, v_\varepsilon); \varepsilon \in (0, 1)\}$ is relatively compact in $\tilde{C}_b([-\tau, +\infty); X) \times \tilde{C}_b([-\tau, +\infty); Y)$. Indeed, let $\varepsilon_n \downarrow 0$ and let $((u_n, v_n))_n$ be a sequence of solutions for

$$\begin{cases} u'_n(t) \in Au_n(t) + F(t, u_{n_t}, v_{n_t}), & t \in \mathbb{R}_+, \\ v'_n(t) \in Bv_n(t) + g_n(t), & t \in \mathbb{R}_+, \\ g_n(t) \in \chi_{[0, 1/\varepsilon_n]}(t)G(t, u_{n_t}, v_{n_t}), & t \in \mathbb{R}_+, \\ u_n(t) = (1 - \varepsilon_n)p(u_n, v_n)(t), & t \in [-\tau, 0], \\ v_n(t) = (1 - \varepsilon_n)q(u_n, v_n)(t), & t \in [-\tau, 0]. \end{cases}$$

Arguing as in Lemma 4.5, we deduce that the set $\{g_n; n \in \mathbb{N}\}$ is weakly relatively compact in $L^1(0, k; Y)$, for $k = 1, 2, \dots$ and the set $\{v_n; n \in \mathbb{N}\}$ is relatively compact in $\tilde{C}_b([-\tau, +\infty); Y)$. So, at least on a subsequence, we have both $\lim_n g_n = g$ weakly in $\tilde{L}^1(0, \infty; Y)$, $\lim_n v_n = v$ in $\tilde{C}_b([-\tau, +\infty); Y)$ and the function v satisfies $v'(t) \in Bv(t) + g(t)$ for each $t \in [0, +\infty)$. Using (H_F) , (p_1) , (p_2) and the continuity property of p we deduce that we can apply Theorem 4.3, $\tilde{C}_b$ -continuity part, with $F_n(t, \cdot) := F(t, \cdot, v_{n_t})$ and $p_n(\cdot) := (1 - \varepsilon)p(\cdot, v_n)$ for $n \in \mathbb{N}$ and $t \in [0, +\infty)$. So, at least on a subsequence, we have $\lim_n u_n = u$ in $\tilde{C}_b([-\tau, +\infty); X)$ where

$$\begin{cases} u'(t) \in Au(t) + F(t, u_t, v_t), & t \in \mathbb{R}_+, \\ u(t) = p(u, v)(t), & t \in [-\tau, 0]. \end{cases}$$

Since G is strongly-weakly u.s.c. and $g_n(t) \in \chi_{[0, 1/\varepsilon_n]}(t)G(t, u_{n_t}, v_{n_t})$ a.e. for $t \in [0, +\infty)$, from VRABIE [21, Theorem 3.1.2, p. 88] we get $g(t) \in G(t, u_t, v_t)$ a.e. $t \in [0, \infty)$. Finally, from the continuity property of q we get $v(t) = q(u, v)(t)$ for each $t \in [-\tau, 0]$ and this completes the proof. $\square$

5. An example

Let Ω be a nonempty, bounded domain in $\mathbb{R}^d$, $d \geq 2$, with C^1 boundary Γ , let $\tau > 0$, $\omega > 0$, let $Q = \mathbb{R}_+ \times \Omega$, $\Sigma = \mathbb{R}_+ \times \Gamma$, $\Omega_\tau = [-\tau, 0] \times \Omega$ and let Δ be the Laplace operator in the sense of distributions over Ω . Let $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ and $\psi : D(\psi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be maximal-monotone operators with $0 \in \varphi(0)$, $0 \in \psi(0)$, let $F : \mathbb{R}_+ \times C([-\tau, 0]; L^1(\Omega)) \times C([-\tau, 0]; L^1(\Omega)) \rightarrow L^1(\Omega)$ be continuous and $g_i : \mathbb{R}_+ \times C([-\tau, 0]; L^1(\Omega)) \times C([-\tau, 0]; L^1(\Omega)) \rightarrow L^1(\Omega)$, $i = 1, 2$, two given functions such that g_1 is l.s.c, g_2 is u.s.c and $g_1(t, u, v)(x) \leq g_2(t, u, v)(x)$, for each $(t, u, v) \in \mathbb{R}_+ \times C([-\tau, 0]; L^1(\Omega)) \times C([-\tau, 0]; L^1(\Omega))$ and a.e. $x \in \Omega$. Let $a > 0$, let μ be a positive σ -finite and complete measure on the class of Borel measurable sets in $[a, +\infty)$, $k \in L^1(a, +\infty; \mu, \mathbb{R})$ be a nonnegative function with $\|k\|_{L^1(a, +\infty; \mu, \mathbb{R})} \leq 1$ and let $W : C([-\tau, 0]; L^1(\Omega)) \rightarrow \mathbb{R}_+$ be nonexpansive with $W(0) = 0$. We consider the following system:

$$(5.1) \quad \begin{cases} \frac{\partial u}{\partial t}(t, x) = \Delta \varphi(u(t, x)) - \omega u(t, x) + F(t, u_t, v_t)(x), & (t, x) \in Q, \\ \frac{\partial v}{\partial t}(t, x) = \Delta \psi(v(t, x)) - \gamma v(t, x) + g(t)(x), & (t, x) \in Q, \\ g(t) \in G(t, u_t, v_t), & t \in \mathbb{R}_+, \\ \varphi(u(t, x)) = 0, \psi(v(t, x)) = 0, & (t, x) \in \Sigma, \\ u(t, x) = \int_a^\infty k(s)W(v(t+s, x))u(t+s, x) d\mu(s), & (t, x) \in \Omega_\tau, \\ v(t, x) = v(t+T, x), & (t, x) \in \Omega_\tau, \end{cases}$$

where

$$G : \mathbb{R}_+ \times C([-\tau, 0]; L^1(\Omega)) \times C([-\tau, 0]; L^1(\Omega)) \rightsquigarrow L^1(\Omega)$$

is defined by

$$G(t, u, v) := \{h \in L^1(\Omega); g_1(t, u, v)(x) \leq h(x) \leq g_2(t, u, v)(x), \text{ a.e. } x \in \Omega\}.$$

If $\theta : D(\theta) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ is monotone with $0 \in \theta(0)$ and $u : \Omega \rightarrow D(\theta)$, we denote by

$$\mathcal{S}_\theta(u) = \{v \in L^1(\Omega); v(x) \in \theta(u(x)), \text{ a.e. for } x \in \Omega\}.$$

The first part of the result below was proved by BREZIS and STRAUSS [4] while the second was proved by BADII, DÍAZ and TESEI [1].

Theorem 5.1. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^d$ with C^1 boundary Γ and let $\theta : D(\theta) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be maximal monotone with $0 \in D(\theta)$ and $0 \in \theta(0)$.*

(i) *Then the operator $\Delta\theta : D(\Delta\theta) \subseteq L^1(\Omega) \rightsquigarrow L^1(\Omega)$, defined by*

$$\begin{cases} D(\Delta\theta) = \{u \in L^1(\Omega); \exists v \in \mathcal{S}_\theta(u) \cap W_0^{1,1}(\Omega), \Delta v \in L^1(\Omega)\} \\ \Delta\theta(u) = \{\Delta v; v \in \mathcal{S}_\theta(u) \cap W_0^{1,1}(\Omega)\} \cap L^1(\Omega) \text{ for } u \in D(\Delta\theta), \end{cases}$$

is m -dissipative on $L^1(\Omega)$.

(ii) *If, in addition, $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $\mathbb{R}$ and C^1 on $\mathbb{R} \setminus \{0\}$ and there exist two constants $C > 0$ and $\alpha > 0$ if $d \leq 2$ and $\alpha > (d-2)/d$ if $d \geq 3$ such that $\theta'(r) \geq C|r|^{\alpha-1}$ for each $r \in \mathbb{R} \setminus \{0\}$, then $\Delta\theta$ generates a compact semigroup.*

For a sufficient condition in order that the semigroup generated by $\Delta\theta$ maps weakly compact sets in $L^1(\Omega)$ into compact sets in $L^1(\Omega)$ for $t > 0$, see DÍAZ and VRABIE [11].

Before proceeding to the statement of the main result of this section, let us define the multifunction

$$G : \mathbb{R}_+ \times C([- \tau, 0]; L^1(\Omega)) \times C([- \tau, 0]; L^1(\Omega)) \rightsquigarrow L^1(\Omega)$$

by $G(t, u, v)(x) := [g_1(t, u, v)(x), g_2(t, u, v)(x)]$, for each $t \in \mathbb{R}_+$, $x \in \Omega$ and $u, v \in C([- \tau, 0]; L^1(\Omega))$.

Theorem 5.2. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^d$, $d \geq 1$, with C^1 boundary Γ and let $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ and $\psi : D(\psi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be maximal monotone operators with $0 \in D(\varphi)$, $0 \in D(\psi)$, $0 \in \varphi(0)$ and $0 \in \psi(0)$. Let $F : \mathbb{R}_+ \times C([- \tau, 0]; L^1(\Omega)) \times C([- \tau, 0]; L^1(\Omega)) \rightarrow L^1(\Omega)$ be continuous, let $g_i : \mathbb{R}_+ \times C([- \tau, 0]; L^1(\Omega)) \times C([- \tau, 0]; L^1(\Omega)) \rightarrow L^1(\Omega)$, $i = 1, 2$, be two given functions such that $g_1(t, u, v)(x) \leq g_2(t, u, v)(x)$, for each $(t, u, v) \in \mathbb{R}_+ \times C([- \tau, 0]; L^1(\Omega)) \times C([- \tau, 0]; L^1(\Omega))$ and a.e. $x \in \Omega$. Let μ be a positive σ -finite and complete measure defined on the class of Borel measurable sets in $[a, +\infty)$, let $k \in L^1(a, +\infty; \mu, \mathbb{R})$ be non-negative and let $W : C([- \tau, 0]; L^1(\Omega)) \rightarrow \mathbb{R}_+$. Let us assume that:*

(h₁) *$\psi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $\mathbb{R}$ and C^1 on $\mathbb{R} \setminus \{0\}$ and there exist two constants $C > 0$ and $\alpha > 0$ if $d \leq 2$ and $\alpha > (d-2)/d$ if $d \geq 3$ such that $\psi'(r) \geq C|r|^{\alpha-1}$ for each $r \in \mathbb{R} \setminus \{0\}$;*

(h₂) there exist $\ell > 0$ and $m > 0$ such that

$$\begin{aligned} \|F(t, u, v) - F(t, \tilde{u}, \tilde{v})\|_{L^1(\Omega)} &\leq \ell[\|u - \tilde{u}\|_{C([- \tau, 0]; L^1(\Omega))} \\ &\quad + \|v - \tilde{v}\|_{C([- \tau, 0]; L^1(\Omega))}], \end{aligned}$$

$$\|y\|_{L^1(\Omega)} \leq \ell[\|u\|_{C([- \tau, 0]; L^1(\Omega))} + \|v\|_{C([- \tau, 0]; L^1(\Omega))}] + m,$$

$$\|F(t, 0, 0)\|_{L^1(\Omega)} \leq m,$$

for each $(t, u, v), (t, \tilde{u}, \tilde{v}) \in \mathbb{R}_+ \times C([- \tau, 0]; L^1(\Omega)) \times C([- \tau, 0]; L^1(\Omega))$
and for each $y \in G(t, u, v)$.

(h₃) g_1 is l.s.c and g_2 is u.s.c;

(h₄) there exist $\ell_1 > 0$ and $m_1 > 0$ such that for $i = 1, 2$ and for each $(t, u, v) \in \mathbb{R}_+ \times C([- \tau, 0]; L^1(\Omega)) \times C([- \tau, 0]; L^1(\Omega))$ we have

$$\|g_i(t, u, v)\| \leq \ell_1[\|u\|_{C([- \tau, 0]; L^1(\Omega))} + \|v\|_{C([- \tau, 0]; L^1(\Omega))}] + m_1;$$

(h₅) $\|k\|_{L^1(a, +\infty; \mu, \mathbb{R})} \leq 1$;

(h₆) $|W(v) - W(\tilde{v})| \leq \|v - \tilde{v}\|_{C([- \tau, 0]; L^1(\Omega))}$, for each $v, \tilde{v} \in C([- \tau, 0]; L^1(\Omega))$;

(h₇) $W(0) = 0$.

Let us assume also that (H_c) is satisfied. Then, (5.1) has at least one C^0 -solution.

Proof. The problem (5.1) can be rewritten as an abstract one of the form (1.1). Since g_1 is l.s.c., g_2 is u.s.c. and both have sublinear growth, we conclude that G is strongly-weakly u.s.c. with nonempty, convex and weakly compact values. So, all hypotheses of the Theorem 3.1 are satisfied. $\square$

REFERENCES

1. BADI, M.; DÍAZ, J. I.; TESEI, A. – *Existence and attractivity for a class of degenerate functional parabolic problems*, Rend. Sem. Mat. Univ. Padova, 78 (1987), 109–124.
2. BARAS, P. – *Compacité de l'opérateur $f \mapsto u$ solution d'une équation non linéaire $(du/dt) + Au \ni f$* , C. R. Acad. Sci. Paris Sér. A-B, 286 (1978), A1113–A1116.

3. BARBU, V. – *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer Monographs in Mathematics, Springer, New York, 2010.
4. BRÉZIS, H.; STRAUSS, W.A. – *Semi-linear second-order elliptic equations in L^1* , J. Math. Soc. Japan, 25 (1973), 565–590.
5. BURLICĂ, M. – *Viability for multi-valued semilinear reaction-diffusion systems*, Ann. Acad. Rom. Sci. Ser. Math. Appl., 2 (2010), 3–24.
6. BURLICĂ, M.; ROȘU, D. – *A viability result for semilinear reaction-diffusion systems*, An. Științ. Univ. "Al.I. Cuza" Iași. Mat. (N.S.), 54 (2008), 361–382.
7. BURLICĂ, M.; ROȘU, D. – *The initial value and the periodic problems for a class of reaction-diffusion systems*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 15 (2008), 427–444.
8. BURLICĂ, M.; ROȘU, D. – *A class of nonlinear delay evolution equations with nonlocal initial conditions*, Proc. Amer. Math. Soc., in print.
9. BURLICĂ, M.; ROȘU, D.; VRABIE, I. I. – *Continuity with respect to the data for a delay evolution equation with nonlocal initial conditions*, Lib. Math. (N.S.), 32 (2012), 39–50.
10. BURLICĂ, M.; ROȘU, D.; VRABIE, I. I. – *Abstract reaction-diffusion systems with nonlocal initial conditions*, submitted.
11. DÍAZ, J. I.; VRABIE, I. I. – *Propriétés de compacité de l'opérateur de Green généralisé pour l'équation des milieux poreux*, C. R. Acad. Sci. Paris Sér. I Math., 309 (1989), 221–223.
12. DÍAZ, J. I.; VRABIE, I. I. – *Existence for reaction-diffusion systems. A compactness method approach*, J. Math. Anal. Appl., 188 (1994), 521–540.
13. DUNFORD, N.; SCHWARTZ, J.T. – *Linear Operators. I, General Theory*, Pure and Applied Mathematics, Vol. 7, Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958.
14. EDWARDS, R. E. – *Functional Analysis. Theory and Applications*, Holt, Rinehart and Winston, New York-Toronto-London, 1965.
15. GLICKSBERG, I. L. – *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc., 3 (1952), 170–174.
16. HALE, J. – *Theory of functional differential equations*, Applied Mathematical Sciences, Vol. 3, Springer-Verlag, New York-Heidelberg, 1977.
17. NECULA, M.; VRABIE, I. I. – *A viability result for a class of fully nonlinear reaction-diffusion systems*, Nonlinear Anal., 69 (2008), 1732–1743.
18. PAICU, A.; VRABIE, I. I. – *A class of nonlinear evolution equations subjected to nonlocal initial conditions*, Nonlinear Anal., 72 (2010), 4091–4100.
19. ROȘU, D. – *Viability for nonlinear multi-valued reaction-diffusion systems*, NoDEA Nonlinear Differential Equations Appl., 17 (2010), 479–496.

20. ROȘU, D. – *Viability for a nonlinear multi-valued system on locally closed graph*, An. Științ. Univ. "Al. I. Cuza" Iași. Mat. (N.S.), 56 (2010), 343–362.
21. VRABIE, I. I. – *Compactness methods for nonlinear evolutions*, Second edition, Pitman Monographs and Surveys in Pure and Applied Mathematics, 75, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1995.
22. VRABIE, I. I. – *Existence for nonlinear evolution inclusions with nonlocal retarded initial conditions*, Nonlinear Anal., 74 (2011), 7047–7060.
23. VRABIE, I. I. – *Existence in the large for nonlinear delay evolution inclusions with nonlocal initial conditions*, J. Funct. Anal., 262 (2012), 1363–1391.
24. VRABIE, I. I. – *Nonlinear retarded evolution equations with nonlocal initial conditions*, Dynam. Systems Appl., 21 (2012), 417–439.
25. VRABIE, I. I. – *Global solutions for nonlinear delay evolution inclusions with nonlocal initial conditions*, Set-Valued Var. Anal., 20 (2012), 477–497.

Received: 15.XII.2012

Accepted: 1.III.2013

Department of Mathematics and Informatics,
"G. Asachi" Technical University,
Iași, 700506,
ROMANIA

monicaburlica@yahoo.com

rosudaniela100@yahoo.com