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CONSISTENT CONJECTURAL VARIATIONS EQUILIBRIUM IN A SEMI-MIXED OLIGOPOLY WITH DISCONTINUOUS DEMAND

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Abstract. *In this paper we study a semi-mixed oligopoly model within the framework of conjectural variation. The semi-mixed structure refers to the presence of a semi-public producer that maximizes the convex combination of domestic social surplus and its net profit. The parameter of this convex combination is called socialization level. Each producer conjectures the variations of the market-clearing price in response to its own production's variations. We extend the previously studied models to the more general case of oligopoly, considering the case when the consumers' demand function is not necessarily differentiable nor continuous. By introducing the notions of exterior and interior equilibriums, we proved the existence and uniqueness theorems for the conjectural variations equilibrium and the equilibrium state known as consistent. After that, we analyzed the behavior of the market's consistent equilibrium state under the changes in the consumers' demand. Finally, we considered the particular case when the consumers' demand is an affine (linear) function to compare the consistent equilibrium against the Cournot and perfect competition equilibriums. Based on this analysis, we formulated an optimality criterion for the socialization level and provided the necessary conditions for its existence.*
Keywords: Game theory, Mixed oligopoly, Consistent conjectural variations, Discontinuous demand

1. Introduction.

The study of the behavior of the agents in a mixed oligopoly (in which, together with the private/foreign firms that maximize their net profit, also compete the state companies, known as "special agents", using another utility function) becomes more and more popular in recent times. For example, in [1-4], there is an agent who tries to maximize domestic social surplus. In [5-8], the models studied include a producer who maximizes the income-per-worker function. [9, 10] deal with the third type of semi-mixed oligopoly, in which the special agent maximizes the convex combination of the domestic social surplus and its net profit. The mixed oligopolies attract the interest of economists, in particular, because of their importance for the economy of the European countries, as

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well as Canada and Japan (see, e.g., [1]). The mixed oligopolies are often met in the transitional economies of the Eastern European countries and the countries of the Soviet Union, where the competition between the partially state-owned and private companies is usual in many branches, especially in the financial areas. According to [2, 4], in many cases, the state controls a sufficient part of the stocks of the privatized companies, but one can also find companies with a mixed structure (private and social).

In the majority of the above-mentioned publications, the mixed oligopolies are studied in the frameworks of the classical models by Cournot, Hotelling, or Stackelberg (see, e.g., [2, 3]). The concept of equilibrium by Nash is based on the idea that each player can change its strategy while the rest of the agents in the game do not. However, another more general concept equilibrium, named conjectural variations, was proposed by a series of authors (see, e.g., Bowley [11] and Frisch [12]). This concept assumes that the model's agents select the best strategy, considering that the other players can answer by also changing their strategies. In our previous works [13-17] we already considered the oligopoly model with conjectural variations, in which the degree of influence of each agent is quantified with special parameters (the influence coefficients).

The economists widely use the various forms of the conjectural variations equilibrium (CVE) to predict the results of the non-cooperative nature in oligopoly markets. The previous researchers of the CVE mainly considered the games between two players (see, [18]), and one of the main concepts in such investigations was the notion of conjecture. The variational conjecture r_{ij} is used to describe the i-th player's conjecture about the j-th player's reaction to an infinitesimal variation of the i-th player's strategy. Such a construction defines the concept of conjectured reaction functions. Having constructed the rivals' reaction functions, every player optimizes its (conjectured) utility function, forming in this way its best response function to the conjectures. The equilibrium occurs in the case when no player is interested in declining its strategy.

Such an equilibrium in conjectures (CVE) is often called consistent or "rational" when, for every player, its best response function and the function of its conjectured reaction, coincide. However, such a comparison, which is possible when the game is between two players, runs into an impossible obstacle in the game of three and more players (see, [18]) because every best response function must coincide with every conjectured reaction function for every other player.

One of the possible ways to overcome this obstacle is the model's structure proposed by Bulavsky [19], in which every player makes its conjectures about the variational response of some integral indicator (for example, the market price of the product) as a reaction to the infinitesimal variations of its supply volume, instead of conjecturing the individual reaction functions of its rivals. Knowing the analogous conjectures of the rest of the players, every agent may apply a certain verification procedure to verifying if its conjecture is consistent with the conjectures of the other players. It is quite natural to call the conjectural variations equilibrium as consistent if all the conjectures successfully pass this verification procedure.

In our previous works [20-23], the results from [19] were extended to the mixed and semi-mixed oligopolies where the consumers' demand function was assumed to be continuously differentiable. Analogously to our other publications [13-17], where the classical oligopoly model is considered, we looked for the conjectural variations equilibrium, but using the ideas from [19]. In contrast, in [24], we managed to relax the latter model to the case when the consumers' demand function is not necessarily continuous, but only for the mixed oligopoly case.

For this paper, we extended the results of our previous publications [20-24] by studying the consistent conjectural variations equilibrium for a semi-mixed oligopoly model, in which the special agent maximizes the convex combination of domestic social surplus and its net profit, and where the consumers' demand function is not necessarily continuous (in our previous works we considered either, a semi-mixed oligopoly where the consumers' demand is continuously differentiable, or a mixed-oligopoly where the consumers' demand is not necessarily continuous; the mixed oligopoly being a particular case of the semi-mixed oligopoly). Moreover, we considered the particular case when the consumers' demand is an affine function and also extended the results from [22, 23] by formulating an optimality criterion for the socialization level of the semi-public company, for the general case of oligopoly where the producers have quadratic cost functions (in our previous works we considered only the case of duopoly or the case of oligopoly when every private producer had the same cost function).

Note: The proofs of the results (lemmas, theorems, and corollaries) presented in this paper were exported as supplementary materials.

2. **Model Specification.** In this section, we extend the results from [24] to the case when the special agent maximizes the convex combination of domestic social surplus and its net profit. Thus, let us consider an oligopoly market of one homogeneous good with $n+1$ producers, $n \in \mathbb{N} = \{1, 2, \dots\}$, among which the special agent is identified by the index $i = 0$, while the other agents i, $i \in \{1, \dots, n\}$, are the private companies who maximize solely their net profits. Every company/producer i, $i \in \{0, 1, \dots, n\}$, has its cost function $f_i(q_i)$, where $q_i \geq 0$ is its production volume sent to the market.

Similarly to [24], we consider here the demand of two kinds: the passive demand given by the demand function $G(p)$, and the active demand $D \geq 0$ that does not depend on the price $p > 0$. The passive demand function $G(p)$ is assumed to be non-negative and non-increasing. Regarding the smoothness of the passive demand function, we assume that it is continuously differentiable everywhere with the possible exception of a finite number of points, in which it can be non-differentiable or even discontinuous. Let us denote the left-hand limit of $G(p)$ at the point p by $G(p-)$ and the right-hand limit by $G(p+)$. Then, we can describe the equilibrium between demand and supply for a given price p by the following balance inequalities

$$G(p+) + D \leq \sum_{i=0}^{n} q_i \leq G(p-) + D. \tag{1}$$

The properties of the passive demand and the cost functions are described by the following assumptions.

Assumption 2.1. *The demand function $G(p)$ is defined for all $p > 0$, being non-negative, non-increasing, and piecewise continuously differentiable. The number of points where the demand function is not continuously differentiable is finite and, at those points, both $G(p)$ and its derivative $G'(p)$ have (finite) left-hand limits, $G(p-)$ and $G'(p-)$ respectively, as well as right-hand limits, $G(p+)$ and $G'(p+)$ respectively.*

Assumption 2.2. *For each i, $i \in \{0, 1, \dots, n\}$, the cost function $f_i(q_i)$ is quadratic, strictly increasing, and strictly convex, with $f_i(0) = 0$, i.e.,*

$$f_i(q_i) = \frac{1}{2} a_i q_i^2 + b_i q_i, \tag{2}$$

where $a_i, b_i > 0$. In addition, we assume that

$$b_0 \leq \max_{i \in \{1, \dots, n\}} \{b_i\}. \tag{3}$$

The producers $i \in \{1, \ldots, n\}$ are private firms that choose their production volume $q_i \geq 0$ to maximize their net profit given by the function

$$\pi_i(p, q_i) = pq_i - f_i(q_i). \tag{4}$$

On the other hand, the producer $i = 0$ (called public) is a semi-public company that selects its production volume $q_0 \geq 0$ to maximize the convex combination of social surplus and its net profit, given by the function

$$S(\beta, p, q_0, q_1, \ldots, q_n) = \beta \left(\int_0^{\sum_{i=0}^{n} q_i} p(x)dx - p\sum_{i=1}^{n} q_i - f_0(q_0) \right) + (1-\beta)(pq_0 - f_0(q_0)), \tag{5}$$

where (following the ideas from [9, 10]) $\beta \in (0, 1]$ is a parameter that we call *socialization level.*

Now, if we accept that each agent of the oligopoly assumes that the variations in its production can affect the market-clearing price p, then, the first order necessary conditions for their production quantities q_i will take the following form.

For the private firms $i \in \{1, \ldots, n\}$

$$\frac{\partial \pi_i}{\partial q_i} = p + q_i \frac{\partial p}{\partial q_i} - f_i'(q_i) \begin{cases} = 0, & \text{if } q_i > 0, \\ \leq 0, & \text{if } q_i = 0, \end{cases} \tag{6}$$

and for the semi-public company $i = 0$

$$\frac{\partial S}{\partial q_0} = p + \left[(1-\beta)q_0 - \beta \sum_{i=1}^{n} q_i \right] \frac{\partial p}{\partial q_0} - f_0'(q_0) \begin{cases} = 0, & \text{if } q_0 > 0, \\ \leq 0, & \text{if } q_0 = 0. \end{cases} \tag{7}$$

From Formulas (6) and (7), we see that in order to describe the behavior of the producers (both private and public), it is enough to evaluate the derivatives

$$\frac{\partial p}{\partial q_i} = -\nu_i, \quad i \in \{0, 1, \ldots, n\}, \tag{8}$$

where the minus sign is introduced to work with the non-negative values of ν_i, which we call the i-th producer's influence coefficient.

In addition, to use the first-order necessary conditions (6) and (7) as sufficient, we need the i-th producer's objective function to be (at least, locally) concave. The latter is achieved if we assume that the conjectured influence coefficients ν_i, $i \in \{0, 1, \ldots, n\}$, are non-negative constants. Under this assumption, the conjectured (local) dependence of the private firms' profit π_i upon the variations in their production volumes $(\eta_i - q_i)$ has the form

$$\hat{\pi}_i(\eta_i) = [p - \nu_i(\eta_i - q_i)]\eta_i - f_i(\eta_i), \tag{9}$$

which is a quadratic and concave function with respect to η_i. Thus, the first-order necessary (and now sufficient) conditions for the equilibrium's production volumes $\eta_i = q_i$, $i \in \{1, \ldots, n\}$, to be optimal are given by

$$\begin{cases} p = \nu_i q_i + a_i q_i + b_i, & \text{if } q_i > 0, \\ p \leq b_i, & \text{if } q_i = 0. \end{cases} \tag{10}$$

Similarly, the semi-public company's conjectured (local) dependence of its objective function S upon the variations in its production volume $(\eta_0 - q_0)$ has the form

$$\hat{S}(\eta_0) = \beta \left\{ \int_0^{\eta_0 + \sum_{i=1}^n q_i} p(x)dx - [p - \nu_0(\eta_0 - q_0)] \sum_{i=1}^n q_i - f_0(\eta_0) \right\} + (1-\beta)\{[p - \nu_0(\eta_0 - q_0)]\eta_0 - f_0(\eta_0)\}, \tag{11}$$

which is a concave function with respect to η_0, so the necessary and sufficient condition for the equilibrium's production volume $\eta_0 = q_0$ to be optimal are as follows

$$\begin{cases} p = \nu_0 \left[(1-\beta)q_0 - \beta \sum_{i=1}^n q_i \right] + a_0 q_0 + b_0, & \text{if } q_0 > 0, \\ p \le -\beta \nu_0 \sum_{i=1}^n q_i + b_0, & \text{if } q_0 = 0. \end{cases} \tag{12}$$

In this paper, we use the approach first proposed in [19] and further developed in [20-24], in which the forecast parameters for the equilibrium state are determined together with the price p and the production volumes q_i, $i \in \{0, 1, \ldots, n\}$, based upon a special verification procedure that will be described in Section 4. In this case, the influence coefficients ν_i, $i \in \{0, 1, \ldots, n\}$, are the numerical parameters determined only for the equilibrium state. As in the works mentioned above, we will call such equilibrium state as *interior*. However, before we introduce the concept of interior equilibrium, we first need to define another concept of equilibrium that we call *exterior*.

3. Exterior Equilibrium: Existence, Uniqueness, and One-Sided Derivatives.

Definition 3.1. *The vector $(p, q_0, q_1, \ldots, q_n)$ is called* exterior equilibrium *for the influence coefficients $\nu_i \ge 0$, $i \in \{0, 1, \ldots, n\}$, if the market is balanced, i.e., condition* (1) *is satisfied, and for each producer the optimality conditions hold, i.e., the relationships* (10) *for all i, $i \in \{1, \ldots, n\}$, and* (12)*, are valid.*

From now on, we will consider only the case when the list of really producing agents does not change (i.e., it is independent of the values of the influence coefficients ν_i). To guarantee this, we introduce an additional condition.

Assumption 3.1. *For the price value $p_0 = \max_{i \in \{1,\ldots,n\}} \{b_i\}$, the following inequality is valid:*

$$\sum_{i=0}^n \frac{p_0 - b_i}{a_i} < G(p_0+). \tag{13}$$

If the equilibrium price p satisfies that $p > p_0$, then, we can guarantee that none of the producers will leave the market.

Lemma 3.1. *Let Assumptions 2.1 and 2.2 hold. If the vector $(p, q_0, q_1, \ldots, q_n)$ is the exterior equilibrium for the given influence coefficients, then, $p > p_0$ if and only if $q_i > 0$ for all $i \in \{0, 1, \ldots, n\}$.*[1]

[1]The proofs of the lemmas, theorems and corollaries presented in this paper were exported as supplementary materials.

Assumptions 2.1, 2.2, and 3.1 guarantee that the exterior equilibrium exists uniquely for any set of the influence coefficients $\nu_i \geq 0$, $i \in \{1, \ldots, n\}$, and $\nu_0 \in [0, \overline{\nu}_0)$, where

$$\overline{\nu}_0 = \begin{cases} \frac{a_0\left(G(p_0+) - \sum_{i=0}^{n} \frac{p_0 - b_i}{a_i}\right)}{\sum_{i=1}^{n} \frac{p_0 - b_i}{a_i} - (1-\beta)G(p_0+)}, & \text{if } \sum_{i=1}^{n} \frac{p_0 - b_i}{a_i} > \max\left\{\frac{1-\beta}{\beta}\left(\frac{p_0 - b_0}{a_0}\right), (1-\beta)G(p_0+)\right\}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{14}$$

This result is established in the following theorem.

Theorem 3.1. *Under Assumptions 2.1, 2.2, and 3.1, for any $\nu_0 \in [0, \overline{\nu}_0)$, $\nu_i \geq 0$, $i \in \{1, \ldots, n\}$, $D \geq 0$, and $\beta \in (0, 1]$, there exists uniquely the exterior equilibrium $(p^{\text{ex}}, q_0^{\text{ex}}, q_1^{\text{ex}}, \ldots, q_n^{\text{ex}})$, where the functions $p^{\text{ex}} = p^{\text{ex}}(\beta, \nu_0, \nu_1, \ldots, \nu_n, D)$, and $q_i^{\text{ex}} = q_i^{\text{ex}}(\beta, \nu_0, \nu_1, \ldots, \nu_n, D)$, $i \in \{0, 1, \ldots, n\}$, are continuous with respect to ν_i, $i \in \{0, 1, \ldots, n\}$, D, and β. Moreover, $p^{\text{ex}} > p_0$, $q_i^{\text{ex}} > 0$ for all $i \in \{0, 1, \ldots, n\}$, and the equilibrium price has left and right-hand partial derivatives with respect to D given by*

$$\frac{\partial p^{\text{ex}}}{\partial D^-} = \begin{cases} \frac{1}{\frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \sum_{i=0}^{n} \frac{1}{\nu_i + a_i} - G'(p^{\text{ex}}-)}, & \text{if } \sum_{i=0}^{n} q_i^{\text{ex}} = G(p^{\text{ex}}-) + D, \\ 0, & \text{if } \sum_{i=0}^{n} q_i^{\text{ex}} < G(p^{\text{ex}}-) + D, \end{cases} \tag{15}$$

and

$$\frac{\partial p^{\text{ex}}}{\partial D^+} = \begin{cases} \frac{1}{\frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \sum_{i=0}^{n} \frac{1}{\nu_i + a_i} - G'(p^{\text{ex}}+)}, & \text{if } \sum_{i=0}^{n} q_i^{\text{ex}} = G(p^{\text{ex}}+) + D, \\ 0, & \text{if } \sum_{i=0}^{n} q_i^{\text{ex}} > G(p^{\text{ex}}+) + D, \end{cases} \tag{16}$$

respectively.

If we consider every point p where the demand function $G(p)$ is discontinuous, we can connect the points $(p, G(p-))$ and $(p, G(p+))$, in the graph of $G(p)$, with a vertical line to obtain a curve $\mathcal{L}$, which we call the (passive) *demand curve*. Each point (p, G) of the demand curve $\mathcal{L}$ satisfies the relationships $G(p+) \leq G \leq G(p-)$, and the exterior equilibrium $(p^{\text{ex}}, q_0^{\text{ex}}, q_1^{\text{ex}}, \ldots, q_n^{\text{ex}})$ defines the point $(p^{\text{ex}}, G^{\text{ex}}) \in \mathcal{L}$ such that $G^{\text{ex}} + D = \sum_{i=0}^{n} q_i^{\text{ex}}$. Now, at the points (p, G) belonging to the vertical line segments of the curve $\mathcal{L}$, we define the left-hand derivative $G'(p-) = -\infty$ if $G < G(p-)$ and the right-hand derivative $G'(p+) = -\infty$ if $G > G(p+)$. Thus, Formulas (15) and (16) from Theorem 3.1 can be rewritten uniformly as follows:

$$\begin{aligned} \frac{\partial p^{\text{ex}}}{\partial D^{\pm}} &= \frac{1}{\frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \sum_{i=0}^{n} \frac{1}{\nu_i + a_i} - G'(p^{\text{ex}}\pm)} \\ &= \frac{1}{\frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \sum_{i=1}^{n} \frac{1}{\nu_i + a_i} - G'(p^{\text{ex}}\pm)}. \end{aligned} \tag{17}$$

Therefore, at the equilibrium state $(p^{\text{ex}}, q_0^{\text{ex}}, q_1^{\text{ex}}, \ldots, q_n^{\text{ex}})$, the relationship $\frac{\partial p^{\text{ex}}}{\partial D^-} \neq \frac{\partial p^{\text{ex}}}{\partial D^+}$ can happen only if $G'(p^{\text{ex}}-) \neq G'(p^{\text{ex}}+)$, i.e., if the point $(p^{\text{ex}}, G^{\text{ex}})$ is a sharp corner of the demand curve $\mathcal{L}$. Moreover, if $\frac{\partial p^{\text{ex}}}{\partial D^-} < \frac{\partial p^{\text{ex}}}{\partial D^+}$, then, the corner point is convex, whereas if $\frac{\partial p^{\text{ex}}}{\partial D^-} > \frac{\partial p^{\text{ex}}}{\partial D^+}$, then, the corner point is concave.

Now, with the help of the exterior equilibrium and Formula (17) obtained from Theorem 3.1, we can deduce Bulavsky's verification procedure (called *consistency criterion*) to define the consistency of the influence coefficients which yield the consistent (interior) equilibrium.

4. **Consistency Criterion and Interior Equilibrium.** To define the concept of interior equilibrium we have to consider the procedure of verification of the influence coefficients introduced in [19]. Assume that we have the exterior equilibrium $(p^{\text{ex}}, q_0^{\text{ex}}, q_1^{\text{ex}}, \ldots, q_n^{\text{ex}})$ that occurred for certain values of ν_i, $i \in \{0, 1, \ldots, n\}$, D, and β. Suppose that one of the producers, for example, producer k, wants to determine its influence coefficient ν_k. To do that, the producer k abstains from maximizing its objective function and starts making small variations in its production volume q_k^{ex}. Mathematically, this is equivalent to assuming, that producer k temporally quits the market, so its production has to be subtracted from the consumers' demand, specifically, from the active demand D. Thus, for the new active demand $D_k = D - q_k$, the infinitesimal variations in q_k^{ex} imply the infinitesimal variations in D_k (but in the opposite direction). In this situation, the producer k can estimate the left and right-hand derivatives of the equilibrium price with respect to the new active demand D_k as follows

$$\frac{\partial p^{\text{ex}}}{\partial D_k^{\pm}} = \frac{\partial p^{\text{ex}}}{\partial (D - q_k)^{\pm}} = -\frac{\partial p^{\text{ex}}}{\partial q_k^{\mp}}, \tag{18}$$

i.e., the left and right-hand limits of its influence coefficient.

By Theorem 3.1, we can apply Formula (17) to calculating the derivatives $\frac{\partial p^{\text{ex}}}{\partial D_k^{\pm}}$, however, we need to remember that the producer k has temporarily left the market, so we have to modify Formula (17) by excluding the terms with the index $i = k$. Hence, we obtain the following criterion.

Definition 4.1 (Consistency Criterion)**.** *At the exterior equilibrium* $(p^{\text{ex}}, q_0^{\text{ex}}, q_1^{\text{ex}}, \ldots, q_n^{\text{ex}})$, *the influence coefficients* ν_i, $i \in \{0, 1, \ldots, n\}$, *are referred to as* consistent *if for every* i, $i \in \{0, 1, \ldots, n\}$, *there exists* r_i *such that*

$$\min\left\{G'(p^{\text{ex}}-), G'(p^{\text{ex}}+)\right\} \le r_i \le \max\left\{G'(p^{\text{ex}}-), G'(p^{\text{ex}}+)\right\}, \tag{19}$$

and the following equalities hold:

$$\nu_0 = \frac{1}{\displaystyle\sum_{i=1}^{n} \frac{1}{\nu_i + a_i} - r_0}, \tag{20}$$

$$\nu_i = \frac{1}{\dfrac{1}{(1-\beta)\nu_0 + a_0} + \dfrac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \displaystyle\sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{1}{\nu_j + a_j} - r_i}, \quad i \in \{1, \ldots, n\}. \tag{21}$$

In the above definition, the values $G'(p^{\text{ex}}-)$ and $G'(p^{\text{ex}}+)$ are calculated at the point $(p, G) = \left(p^{\text{ex}}, \sum_{i=0}^{n} q_i^{\text{ex}} - D\right)$ on the demand curve $\mathcal{L}$. Thus, if $G'(p^{\text{ex}}-) = -\infty$ or $G'(p^{\text{ex}}+) = -\infty$, it could be possible that $r_i = -\infty$ for some $i \in \{0, 1, \ldots, n\}$, in which case the corresponding consistent influence coefficient is $\nu_i = 0$.

Now, making use of the consistency criterion, we can define the concept of interior equilibrium.

Definition 4.2. *The vector* $(p^{\text{ex}}, q_0^{\text{ex}}, q_1^{\text{ex}}, \ldots, q_n^{\text{ex}}, \nu_0, \nu_1, \ldots, \nu_n)$ *is called* interior equilibrium *if, for the given influence coefficients* ν_i, $i \in \{0, 1, \ldots, n\}$, *the vector* $(p^{\text{ex}}, q_0^{\text{ex}}, q_1^{\text{ex}}, \ldots, q_n^{\text{ex}})$ *is the exterior equilibrium, and the Consistency Criterion 4.1 is satisfied, i.e., there exist values* r_i, $i \in \{0, 1, \ldots, n\}$, *such that the relationships* (19)-(21) *are valid. If, in* *addition, $r_i = r_j$ for all $i, j \in \{0, 1, \ldots, n\}$, then, the interior equilibrium is called* strong interior equilibrium.

If the interior equilibrium corresponds to a smooth point on the demand curve $\mathcal{L}$, then, all the values r_i, $i \in \{0, 1, \ldots, n\}$, must be the same. Only if the interior equilibrium corresponds to a vertex, i.e., if $\frac{\partial p^{\text{ex}}}{\partial D^-} \neq \frac{\partial p^{\text{ex}}}{\partial D^+}$, the Consistency Criterion 4.1 allows different values for r_i.

Theorem 4.1. *Under Assumptions 2.1, 2.2, and 3.1, for any $D \geq 0$ and $\beta \in (0, 1]$, there exists the strong interior equilibrium $(p^*, q_0^*, q_1^*, \ldots, q_n^*, \nu_0^*, \nu_1^*, \ldots, \nu_n^*)$.*

In the next two sections, we apply the results obtained above to analyzing the behavior of the market's interior equilibrium. In Section 5 we consider the consumers' demand to be a discontinuous function, while in Section 6 we consider the consumers' demand to be an affine function.

5. Structure of Demand and Equilibrium.

This section aims to study how the variations in the demand structure affect the equilibrium's price and supplies. We are interested in the strong interior equilibrium, because of which we will only investigate the behavior of the solutions to the consistency criterion's system of equations in the following form:

$$\nu_0 = \frac{1}{\sum\limits_{i=1}^{n} \dfrac{1}{\nu_i + a_i} - r}, \tag{22}$$

$$\nu_i = \frac{1}{\dfrac{1}{(1-\beta)\nu_0 + a_0} + \dfrac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \dfrac{1}{\nu_j + a_j} - r}, \quad i \in \{1, \ldots, n\}, \tag{23}$$

where $r \in [-\infty, 0]$.

For $r = -\infty$, the system of Equations (22) and (23) has a unique solution $\nu_i = 0$, $i \in \{0, 1, \ldots, n\}$. For the other values $r \in (-\infty, 0]$, the following assertion is true.

Lemma 5.1. *For any $r \in (-\infty, 0]$ and $\beta \in (0, 1]$, there exists a unique solution $\nu_i(r, \beta)$, $i \in \{0, 1, \ldots, n\}$, for the system of Equations* (22) *and* (23)*, which is continuously differentiable with respect to r and β. Moreover,*

$$\lim_{r \to -\infty} \nu_i(r, \beta) = 0, \quad i \in \{0, 1, \ldots, n\}. \tag{24}$$

The optimality conditions (10), for $q_i > 0$, $i \in \{1, \ldots, n\}$, and (12), for $q_0 > 0$, yield the functions

$$q_i(p, \nu_i) = \frac{p - b_i}{\nu_i + a_i}, \quad i \in \{1, \ldots, n\}, \tag{25}$$

$$q_0(p, \beta, \nu_0, \nu_1, \ldots, \nu_n) = \frac{p - b_0 + \beta \nu_0 \sum\limits_{i=1}^{n} q_i(p, \nu_i)}{(1-\beta)\nu_0 + a_0}, \tag{26}$$

respectively.

Therefore, for each $r \in [-\infty, 0]$ and $\beta \in (0, 1]$, after finding the solution $\nu_i = \nu_i(r, \beta)$, $i \in \{0, 1, \ldots, n\}$, for (22) and (23), we can construct the total supply function

$$\begin{aligned} Q(p, r, \beta) &= q_0(p, \beta, \nu_0, \nu_1, \ldots, \nu_n) + \sum_{i=1}^{n} q_i(p, \nu_i) \\ &= \frac{p - b_0}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \sum_{i=1}^{n} \frac{p - b_i}{\nu_i + a_i}. \end{aligned} \tag{27}$$

Since the values $\nu_i = \nu_i(r, \beta)$, $i \in \{0, 1, \ldots, n\}$, depend solely upon r and β, the total supply function given by (27) represents a straight line for all $p \in (p_0, +\infty)$, whose slope is equal to

$$\frac{\partial Q}{\partial p} = \frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \sum_{i=1}^{n} \frac{1}{\nu_i + a_i}. \tag{28}$$

5.1. **Example 1.** Consider the following illustrative example. The socialization level is $\beta = 1$, the active demand is $D_0 = 1000$, and the passive demand is the step function

$$G(p) = \begin{cases} 1000, & \text{if } p \le 23, \\ 0, & \text{if } p > 23, \end{cases} \tag{29}$$

so we are considering that the demand makes a jump.

The semi-public company's production costs are given by the quadratic function

$$f_0(q_0) = 0.1q_0^2 + 1.5q_0. \tag{30}$$

We consider $n = 2$ private firms which have the same quadratic production costs given by the function

$$f_i(q_i) = 0.01q_i^2 + 2.5q_i, \quad i \in \{1, 2\}. \tag{31}$$

For the fixed value $\beta = 1$, each value of $r \in [-\infty, 0]$ defines a total supply function $Q(p, r, 1)$ which is linear with respect to $p > p_0 = \max\{1.5, 2.5, 2.5\} = 2.5$. Using Formula (28) we can compute the slope of $Q(p, r, 1)$ for each value of r, which (for this example) is strictly decreasing; thus, the straight line $Q(p, r, 1)$ will rotate clockwise as the value of r increases from $-\infty$ to zero. This is shown in Figure 1.

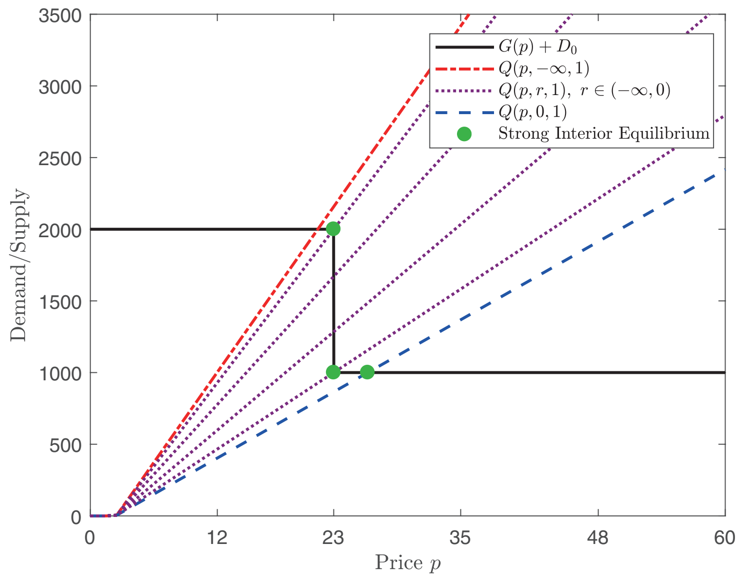

FIGURE 1. Demand and supply as functions of p for Example 1

In the graphs of Figure 1, we can see the straight lines depicting the total supply functions $Q(p, r, 1)$ for various values of r, including the 2 extreme cases $Q(p, -\infty, 1)$ and $Q(p, 0, 1)$. The points $(p, G + D) = (23, 2000)$, $(p, G + D) = (23, 1000)$, and $(p, G + D) \approx (26.22, 1000)$, satisfy the condition

$$\min\left\{G'(p-), G'(p+)\right\} \le r \le \max\left\{G'(p-), G'(p+)\right\}, \tag{32}$$

so they represent 3 different strong interior equilibriums, showing that the uniqueness of the interior equilibrium is not guaranteed. We will refer to these interior equilibriums in that order.

Now, let the economy stay in the first (strong) interior equilibrium at $(p, G + D) = (23, 2000)$, and suppose that the active demand D starts growing up from its original value $D_0 = 1000$, elevating the demand curve as shown in Figure 2.

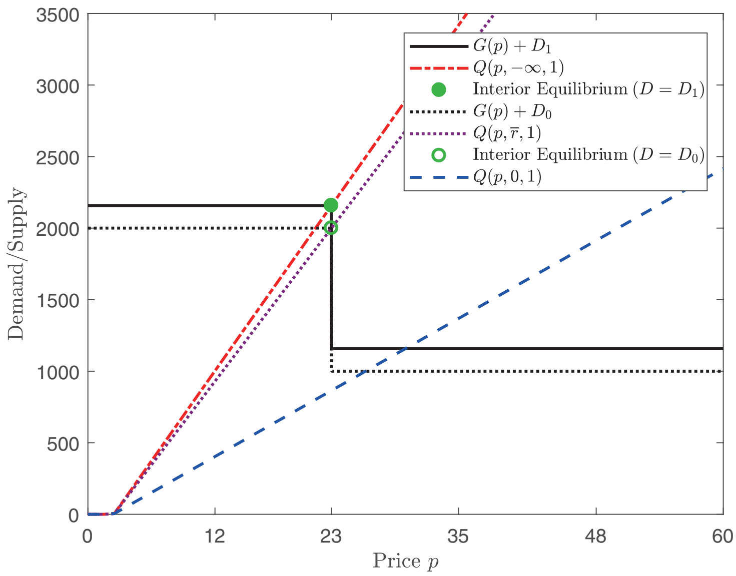

FIGURE 2. Behavior of the (strong) interior equilibrium when the active demand changes from D_0 to D_1

From Figure 2 we see that, while D increases from D_0 to D_1, the first interior equilibrium changes continuously, keeping the price intact at the value $p_1 = 23$ and completely satisfying the demand $G(p) + D$. The supply increases under the same price at the cost of diminishing the equilibrium's influence coefficients (as a consequence of the value r decreasing). The next stage of the process is reflected in Figure 3.

From Figure 3 we can see that, if the active demand belongs to the interval (D_1, D_2), the first interior equilibrium is fixed at $(p, G + D) = (23, 2157.5)$. In this interval, neither the market price nor the total supply depend on the active demand D, but there is a deficit as a consequence of the total demand $G(p) + D$ not being satisfied. This deficit will grow up to the volume of the passive demand $G(p_1)$.

The third stage happens after the active demand crosses the value $D_2 = 2157.5$ as shown in Figure 4.

From Figure 4 we see that, when the active demand reaches the value D_2, the first and second interior equilibriums meet at the point $(p, G + D) = (23, 2157.5)$. Any further increments of the active demand D beyond D_2 will cause these two interior equilibriums to disappear, forcing the economy to jump to the remaining (third) interior equilibrium, which will result in the market price jumping from $p_1 = 23$ to $p_2 > 53.81$. At this new state, the passive demand falls to zero, and even though the total supply has not dropped, the price becomes higher.

Now, assume that the active demand starts to decrease. The events will not go back the same way. The first and second interior equilibriums will appear again but the economy will stay at the third interior equilibrium belonging to the straight line $Q(p, 0, 1)$ until the active demand falls to the value $D_0 \approx 864.96$, as shown in Figure 5.

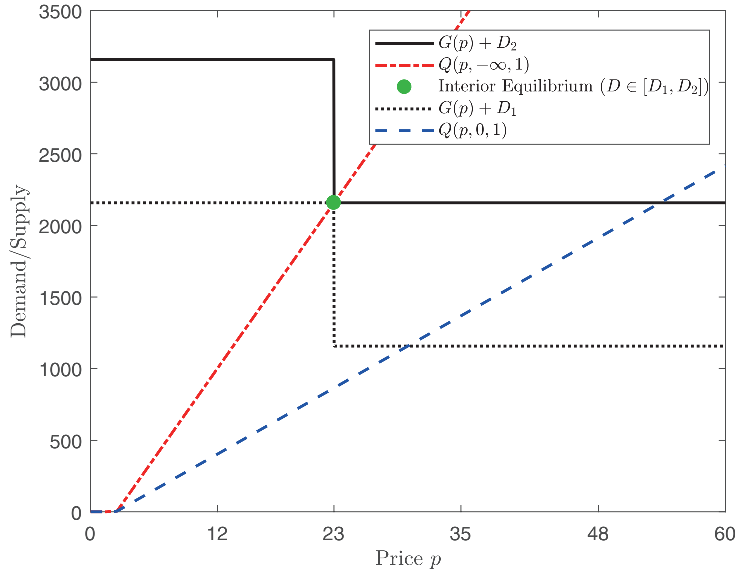

FIGURE 3. Behavior of the interior equilibrium when the active demand changes from D_1 to D_2

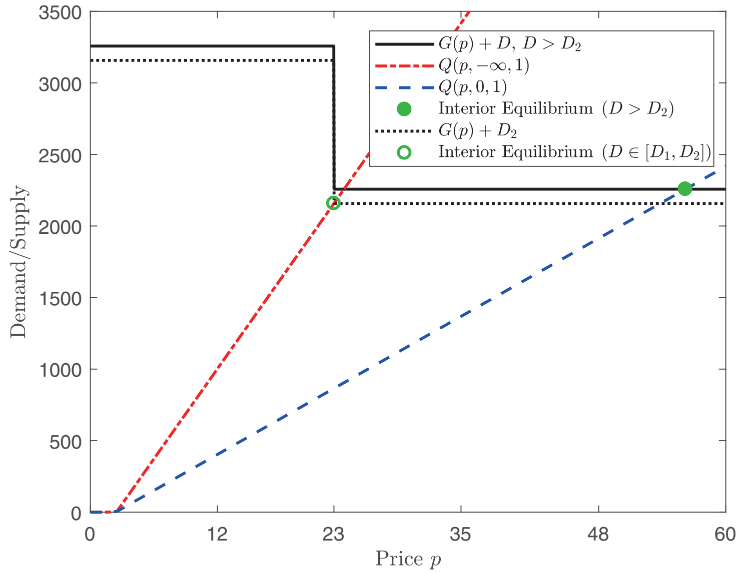

FIGURE 4. Behavior of the interior equilibrium when the active demand increases beyond D_2

From Figure 5 we can see that, even if the active demand comes back to its initial value by decreasing from D_2, the economy will stay at the third interior equilibrium, so the price will be higher than its initial value $p_1 = 23$ and only the active demand will be satisfied.

If the active demand continues to decrease below D_0, the second and third interior equilibriums will disappear and the economy will jump back to the first interior equilibrium, as shown in Figure 6.

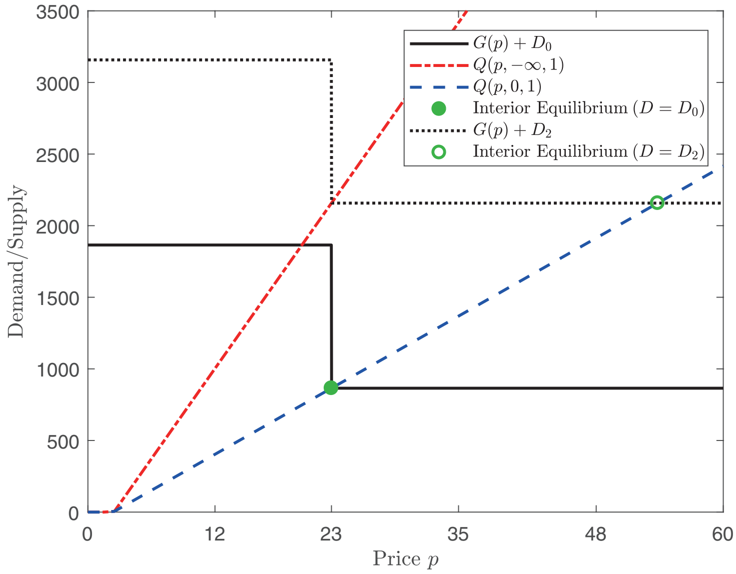

Figure 5. Behavior of the interior equilibrium when the active demand changes from D_2 to D_0

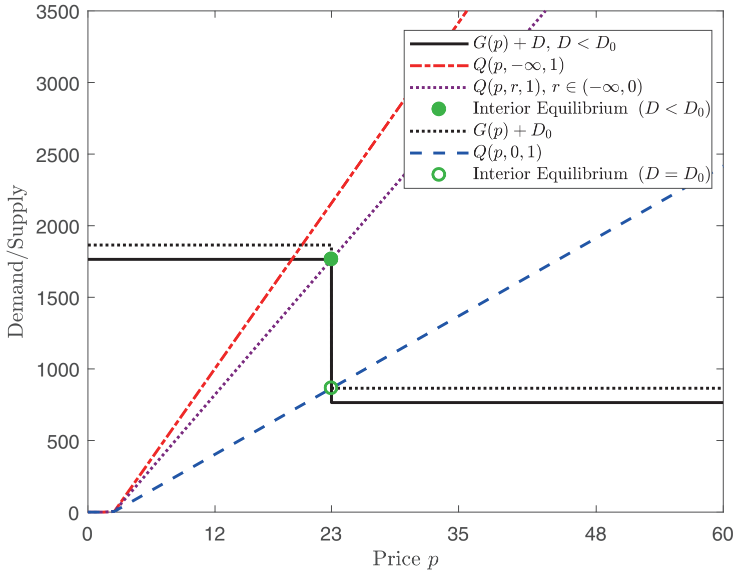

Figure 6. Behavior of the interior equilibrium when the active demand falls below D_0

From Figure 6 we see that, after the economy jumps back to the first interior equilibrium, the price will stay at $p_1 = 23$, but the total supply will grow up to the value $G(p)+D$, satisfying both the passive and the active demands.

If the demand curve has two or more steps, the picture will be similar. The main difference will be that the process described above would be repeated as many times as the number of steps between the straight lines $Q(p, -\infty, 1)$ and $Q(p, 0, 1)$. Finally, we note that, in a more realistic case, when the demand curve has somewhat smoothed steps, the process will be almost the same.

5.2. **Example 2.** Consider the same example again, but now the value of the active demand is fixed at $D = 0$, and we replace the passive demand $G(p)$ with the following function

$$G(p) = \begin{cases} 1000, & \text{if } p \leq 23, \\ 500, & \text{if } 23 < p \leq 35, \\ 0, & \text{if } p > 35. \end{cases} \tag{33}$$

This time, the active demand will not change, but the structure of the passive demand will, depicting the possible changes in the ability to buy the product of a certain group of consumers. We are going to illustrate this process considering two groups of consumers. Again, the demand curve is a very simple step-function, but now with two steps as shown in Figure 7.

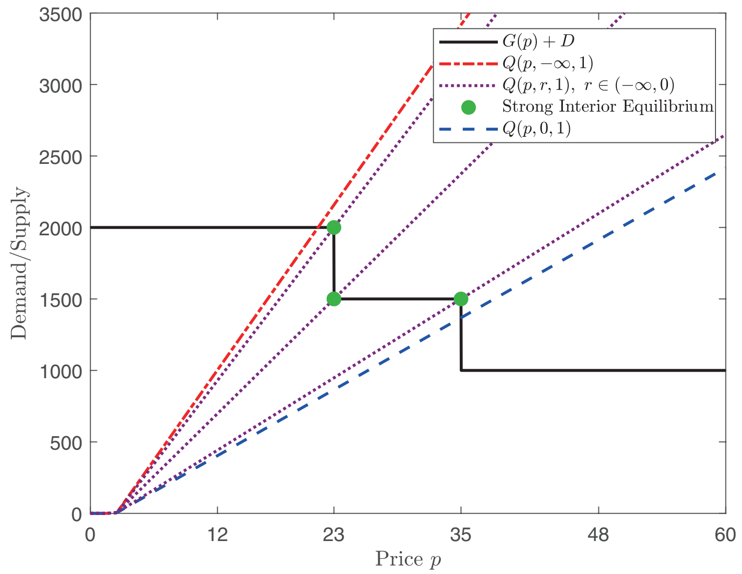

FIGURE 7. Demand and supply as functions of p for Example 2

In the graph of Figure 7, the price $p_1 = 23$ is critical for a group of consumers with a fixed passive demand of $G_1 = 500$, whereas the price $p_2 = 35$ is critical for a richer group of consumers with a fixed passive demand of $G_2 = 500$, in addition to active demand with the value $D = 1000$, thus, the structure of $G(p)$ defined by (33).

Again, let the economy stay in the first interior equilibrium at $(p, G + D) = (23, 2000)$, and let us assume that the consumer's ability of the first (poorer) group drops. This process can be modeled by decreasing the value of the critical price p_1, as shown in Figure 8.

From Figure 8, we see the first stage of this process. The market price will be the same as the critical price p_1 for the poorer consumers until the critical price drops down to the value $p_1 = 21.5$ and the production process reaches perfect competition.

At the second stage, when the critical price p_1 for the first group of consumers belongs to the interval $(16.74, 21.5)$, the market price continues to drop together with the critical price, but the total supply begins to decrease, creating a deficit which will increase as the price (and the critical price) approaches $p = p_1 \approx 16.74$.

If the consumers' ability to buy the product continues to drop, the economy will jump to the third interior equilibrium at $(p, G + D) = (35, 1500)$, causing the market price to jump from $p = 16.74$ to $p = 35$, as shown in Figure 9.

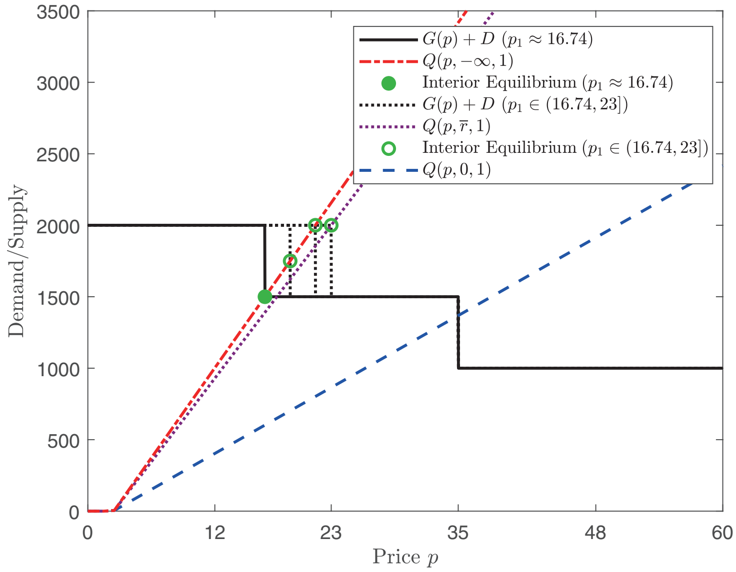

FIGURE 8. Behavior of the interior equilibrium when the critical price p_1 changes from $p_1 = 23$ to $p_1 \approx 16.74$

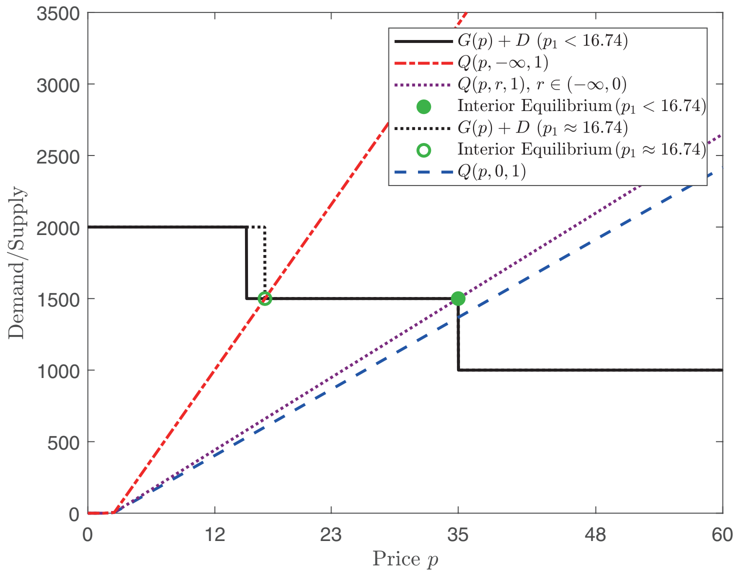

FIGURE 9. Behavior of the interior equilibrium when the critical price p_1 drops beyond $p_1 \approx 16.74$

From Figure 9, we see that, as the equilibrium state jumps, the poorer group of consumers leaves the market completely, liquidating the deficit in the demand.

In this new equilibrium state, even if the consumers' ability (to buy the product) of the first group is restored, the economy will not return to the initial state. A continuous return is possible if the consumers' ability of the first group increases up to that of the second group as shown in Figure 10.

As a conclusion, we note the following fact. The perfect competition regime appears (for both examples) in the vertical parts of the demand curve. If the steps are smooth, but with a high slope (i.e., almost vertical), the perfect competition regime does not appear

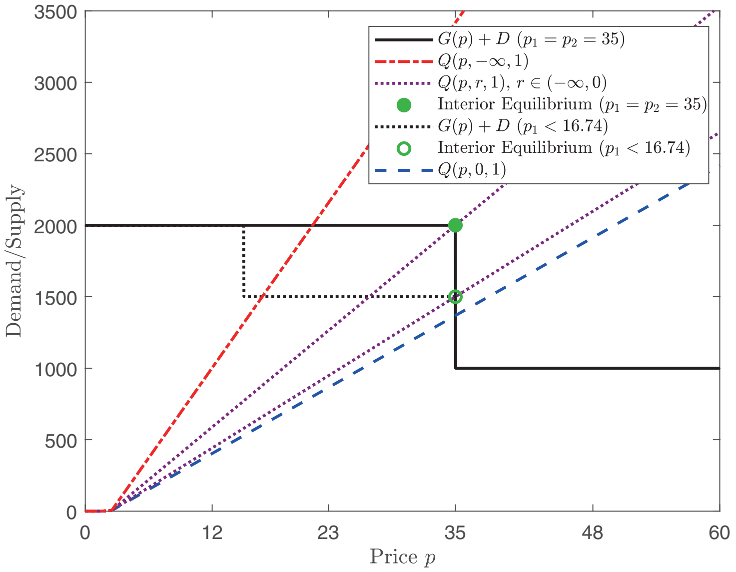

FIGURE 10. Behavior of the interior equilibrium when the critical price p_1 changes from $p_1 < 16.74$ to $p_1 = p_2 = 35$

for strictly convex quadratic cost functions. Its role is played instead by a close production regime with small influence coefficients, defined by the steepness of the demand curve (in its almost vertical parts).

6. **A Particular Case: Affine Demand Function, Identical Private Producers and Quadratic Cost Functions.** In this section, we consider a particular case that guarantees the uniqueness of the interior equilibrium to conduct a comparative analysis between the latter with the exterior equilibriums corresponding to the Cournot and the perfect competition conjectures.

Then, let us consider that the active demand is zero and the passive demand is piecewise linear.

In this case, Assumption 2.1 is restated as follows.

Assumption 6.1. *The passive demand is given by the piecewise function*

$$G(p) = \begin{cases} -Kp + T & \text{if } 0 < p < \dfrac{T}{K}, \\ 0 & \text{if } p \geq \dfrac{T}{K}, \end{cases} \tag{34}$$

where $K > 0$ *and* $T > 0$.

By the proof of Theorem 3.1 and the structure of $G(p)$, we know that the exterior equilibrium's price p^{ex}, which is obtained by the intersection of the total volume $Q(p) = \sum_{i=0}^{n} q_i(p)$ and the demand $G(p)$, must lie within the open interval $\left(p_0, \frac{T}{K}\right)$, and then, we can consider the demand function to be the affine function

$$G(p) = -Kp + T > 0, \tag{35}$$

and rewrite the balance Equation (1) as

$$\sum_{i=0}^{n} q_i = G(p). \tag{36}$$

In the following subsections, we proceed to analyze the behavior of the 3 equilibriums kind: consistent, Cournot, and perfect competition, as functions of the socialization level β.

6.1. **Analysis of consistent equilibrium.** For this particular case, the consistency criterion's definition is reformulated as follows.

Definition 6.1 (Consistency Criterion for the Particular Case)**.** *The influence coefficients $\nu_i \geq 0$, $i \in \{0, 1, \ldots, n\}$, are called consistent for the corresponding exterior equilibrium $(p^{\text{ex}}, q_0^{\text{ex}}, q_1^{\text{ex}}, \ldots, q_n^{\text{ex}})$, if the following equalities are valid:*

$$\nu_0 = \frac{1}{\sum\limits_{i=1}^{n} \frac{1}{\nu_i + a_i} + K}, \tag{37}$$

and

$$\nu_i = \frac{1}{\frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{\nu_j + a_j} + K}, \quad i \in \{1, \ldots, n\}. \tag{38}$$

Theorem 6.1. *Under Assumptions 2.2, 3.1, and 6.1, for every $\beta \in (0, 1]$ there exists uniquely the interior equilibrium $(p^*, q_0^*, q_1^*, \ldots, q_n^*, \nu_0^*, \nu_1^*, \ldots, \nu_n^*)$.*

As a consequence of Theorem 6.1, we have that for every $\beta \in (0, 1]$ there exists uniquely the interior (consistent) equilibrium, which defines the functions $p^* = p^*(\beta)$, $q_i^* = q_i^*(\beta)$, and $\nu_i^* = \nu_i^*(\beta)$, $i \in \{0, 1, \ldots, n\}$, as well as the private firms' profit functions $\pi_i^*(\beta) = \pi_i(p^*(\beta), q_i^*(\beta))$, $i \in \{1, \ldots, n\}$.

Theorem 6.2. *The interior equilibrium's functions $p^*(\beta)$, $q_i^*(\beta)$, $\nu_i^*(\beta)$, $i \in \{0, 1, \ldots, n\}$, and the private firms' profit functions $\pi_i^*(\beta)$, $i \in \{1, \ldots, n\}$, are continuously differentiable with respect to $\beta \in (0, 1]$. Moreover, the functions $p^*(\beta)$ and $\nu_i^*(\beta)$, $i \in \{0, 1, \ldots, n\}$, are strictly decreasing for all $\beta \in (0, 1]$.*

6.2. **Analysis of Cournot equilibrium.** In oligopoly models, the classical Cournot conjecture is understood by the following identities

$$\omega_i = \frac{\partial G}{\partial q_i} = 1, \quad \forall i \in \{0, 1, \ldots, n\}. \tag{39}$$

Within the framework studied in this paper, the latter identities, given by (39), yield the following influence coefficients

$$\nu_i = -\frac{\partial p}{\partial q_i} = -\frac{\omega_i}{G'(p)} = -\frac{1}{G'(p)}, \quad \forall i \in \{0, 1, \ldots, n\}. \tag{40}$$

Hence, the classical Cournot conjecture for this particular case is given by

$$\nu_i^c = \frac{1}{K}, \quad \forall i \in \{0, 1, \ldots, n\}. \tag{41}$$

As a consequence of Theorem 3.1, we have that for every $\beta \in (0, 1]$, there exists uniquely the exterior equilibrium for the Cournot conjectures $\nu_i^c = \frac{1}{K}$, $i \in \{0, 1, \ldots, n\}$, which defines the functions $p^c = p^c(\beta)$ and $q_i^c = q_i^c(\beta)$, $i \in \{0, 1, \ldots, n\}$.

We can easily see that the Cournot equilibrium is a different equilibrium state from the interior equilibrium since the consistency criterion's system of Equations (37) and (38) do not hold. Indeed,

$$\frac{1}{\sum\limits_{i=1}^{n} \frac{1}{\nu_i^c + a_i} + K} < \frac{1}{K} = \nu_0^c, \tag{42}$$

and

$$\frac{1}{\dfrac{1}{(1-\beta)\nu_0^c+a_0}+\dfrac{\nu_0^c+a_0}{(1-\beta)\nu_0^c+a_0}\sum_{\substack{j=1\\j\neq i}}^{n}\dfrac{1}{\nu_j^c+a_j}+K}<\frac{1}{K}=\nu_i^c,\quad i\in\{1,\dots,n\}. \tag{43}$$

Similarly as before, for every $\beta\in(0,1]$, the Cournot (exterior) equilibrium defines the private firms' profit functions $\pi_i^c(\beta)=\pi_i(p^c(\beta),q_i^c(\beta))$, $i\in\{1,\dots,n\}$.

Theorem 6.3. *Under Assumptions 2.2, 3.1 and 6.1, the exterior equilibrium's functions* $p^c(\beta)$, $q_i^c(\beta)$, $i\in\{0,1,\dots,n\}$, *and the private firms' profit functions* $\pi_i^c(\beta)$, $i\in\{1,\dots,n\}$, *are continuously differentiable with respect to* $\beta\in(0,1]$. *Moreover, the function* $p^c(\beta)$ *is strictly decreasing for all* $\beta\in(0,1]$.

6.3. **Analysis of perfect competition equilibrium.** In oligopoly models, the perfect competition conjecture is understood by the following identities

$$\omega_i=\frac{\partial G}{\partial q_i}=0,\quad \forall i\in\{0,1,\dots,n\}. \tag{44}$$

Within the framework studied in this paper, the latter identities, given by (44), yield the following influence coefficients

$$\nu_i=-\frac{\partial p}{\partial q_i}=-\frac{\omega_i}{G'(p)}=0,\quad \forall i\in\{0,1,\dots,n\}. \tag{45}$$

Hence, the perfect competition conjecture for this particular case is given by

$$\nu_i^t=0,\quad \forall i\in\{0,1,\dots,n\}. \tag{46}$$

As a consequence of Theorem 3.1, we have that for every $\beta\in(0,1]$, there exists uniquely the exterior equilibrium for the perfect competition conjectures $\nu_i^t=0$, $i\in\{0,1,\dots,n\}$, which defines the functions $p^t=p^t(\beta)$ and $q_i^t=q_i^t(\beta)$, $i\in\{0,1,\dots,n\}$.

We can easily see that the perfect competition equilibrium is also a different equilibrium state from the interior equilibrium since the consistency criterion's system of Equations (37) and (38) does not hold. Indeed,

$$\frac{1}{\displaystyle\sum_{i=1}^{n}\frac{1}{\nu_i^t+a_i}+K}>0=\nu_0^t, \tag{47}$$

and

$$\frac{1}{\dfrac{1}{(1-\beta)\nu_0^t+a_0}+\dfrac{\nu_0^t+a_0}{(1-\beta)\nu_0^t+a_0}\sum_{\substack{j=1\\j\neq i}}^{n}\dfrac{1}{\nu_j^t+a_j}+K}>0=\nu_i^t,\quad i\in\{1,\dots,n\}. \tag{48}$$

Once again, for every $\beta\in(0,1]$, the perfect competition (exterior) equilibrium defines the private firms' profit functions $\pi_i^t(\beta)=\pi_i(p^t(\beta),q_i^t(\beta))$, $i\in\{1,\dots,n\}$.

Theorem 6.4. *Under Assumptions 2.2, 3.1 and 6.1, the exterior equilibrium's functions* $p^t(\beta)$, $q_i^t(\beta)$, $i\in\{0,1,\dots,n\}$, *and the private firms' profit functions* $\pi_i^t(\beta)$, $i\in\{1,\dots,n\}$, *are constant with respect to* $\beta\in(0,1]$.

6.4. **Comparative analysis.** To define an optimality criterion for the socialization level β, in this section we compare the equilibrium price and private firms' profits for the consistent, Cournot, and perfect competition, conjectures.

Theorem 6.5. *Under Assumptions 2.2, 3.1, and 6.1, the following inequalities hold:*

$$\lim_{\beta \downarrow 0} p^c(\beta) > \lim_{\beta \downarrow 0} p^*(\beta) > p^t. \tag{49}$$

In [22], it was proven, for the duopoly case, that the inequality $p^c(\beta) > p^*(\beta)$ is always satisfied for any $\beta \in (0, 1]$. However, as shown in [23], for the case of oligopoly the latter inequality might not hold when $\beta \uparrow 1$, depending on the model's parameters.

Theorem 6.6. *Under Assumptions 2.2, 3.1, and 6.1, for any $\beta \in (0, 1]$, if $\sum_{i=1}^n \pi_i^c(\beta) \geq \sum_{i=1}^n \pi_i^*(\beta)$, then, it is satisfied that $p^*(\beta) < p^c(\beta)$.*

Corollary 6.1. *Suppose that Assumptions 2.2, 3.1, and 6.1 are true. If the relationships*

$$\lim_{\beta \downarrow 0} \sum_{i=1}^n \pi_i^c(\beta) > \lim_{\beta \downarrow 0} \sum_{i=1}^n \pi_i^*(\beta) \text{ and } \sum_{i=1}^n \pi_i^c(1) < \sum_{i=1}^n \pi_i^*(1) \tag{50}$$

are valid, then, there exists the value $\hat{\beta} \in (0, 1)$ such that $\sum_{i=1}^n \pi_i^c\left(\hat{\beta}\right) = \sum_{i=1}^n \pi_i^*\left(\hat{\beta}\right)$ and $p^*\left(\hat{\beta}\right) < p^c\left(\hat{\beta}\right)$.

Now, we say that the semi-public company is socially responsible and makes use of subsidy policies to pay a monetary compensation to the consumers for the high price that appears when the market is at the Cournot equilibrium state; otherwise, it has to economically motivate the private firms to choose the consistent conjectural variations behavior instead of the Cournot conjecture. However, if relationship (50) holds, there exists the value $\hat{\beta}$ (from Corollary 6.1) such that the accumulated net profit of every private firm is the same in both, the consistent and the Cournot equilibriums, in which case, the semi-public company can persuade the private producers to use the consistent strategies (instead of the Cournot conjectures) so that the market price will not be as high (which is a consequence of Theorem 6.6). Hence, the semi-public company can fulfill its social responsibility without paying any kind of subsidies, thus, keeping its budget safe. Therefore, we understand this socialization level $\hat{\beta}$ as optimal.

Definition 6.2. *If the conditions from Corollary 6.1 hold, the value of the parameter $\hat{\beta} \in (0, 1)$ such that $\sum_{i=1}^n \pi_i^c\left(\hat{\beta}\right) = \sum_{i=1}^n \pi_i^*\left(\hat{\beta}\right)$, is called* optimal socialization level.

Even though the existence of the optimal socialization level was proven for the semi-mixed duopoly in [22] without the need of the condition (50), for the more general oligopoly the situation when $\sum_{i=1}^n \pi_i^c(\beta) > \sum_{i=1}^n \pi_i^*(\beta)$ for every $\beta \in (0, 1]$ can happen (as shown in [23]) if the value a_0 is much greater than the other values a_i, which means that the expenses of the public company are much greater than the expenses of the private firms (i.e., the semi-public company is weaker than the private firms), in which case, the optimal socialization level will not exist. For these situations, we would have to redefine the optimality criterion for the socialization level, which is part of our future works.

7. **Conclusions.** In this paper, we extended the previously studied mixed oligopoly models within the conjectural variations equilibrium framework to the more general case when the consumers' demand function (or its derivative) is not necessarily continuous, while the cost functions of the producers are quadratic. We provided results for the existence and uniqueness of the conjectural variations (exterior) equilibrium for any feasible set of

conjectures. Then, we introduced the consistency criterion to define the consistency of the exterior equilibrium state, which is known as consistent conjectural variations (interior) equilibrium and proved its existence.

After that, we examined the behavior of the (strong) interior equilibrium under the changes of the active and passive demands in two experiments, describing the similarities with the actual behavior of the market in real-life situations.

Finally, we considered the particular case of the semi-mixed oligopoly when the active demand is zero and the active demand is an affine function to guarantee the uniqueness of the consistent conjectural variations (interior) equilibrium and conducted a comparative analysis between the latter with the Cournot and perfect competition equilibriums. With the results obtained in this analysis, we formulated an optimality criterion for the semi-public company's socialization level β and proved its existence (with the additional condition that the CCVE model can generate better profits for the private firms).

In our future work, we are planning to examine the qualitative behavior of the model's functions when the Cournot model is always better for the private firms to redefine the optimality of the socialization level.

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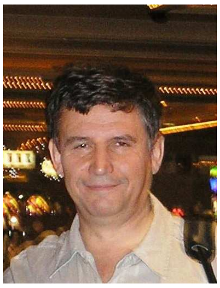

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