A Generalization Of Weibull Distribution

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Abstract

In this paper, a generalization of Weibull distribution (GWD), which includes Weibull and exponential distributions as special cases, has been proposed and investigated. Its moments, hazard rate function and stochastic ordering have been discussed. The method of maximum likelihood estimation has been discussed for estimating its parameters. The goodness of fit of the proposed distribution has been discussed with a real lifetime dataset and the fit has been found quite satisfactory over some well-known lifetime distributions.

Keywords: Exponential distribution, Weibull distribution, Moments, Hazard rate function, Stochastic ordering, Maximum likelihood estimation, Application.

2000 Mathematics Subject Classification: 62E05, 62E99

1. Introduction

The exponential distribution having scale parameter θ is defined by its probability density function (pdf) and cumulative distribution function (cdf)

$$f(y;\theta)=\theta e^{-\theta y};\quad y>0,\theta>0 \tag{1.1}$$

$$F(y;\theta)=1-e^{-\theta y}\ ;y>0,\theta>0, \tag{1.2}$$

Epstein (1958) has detailed study on exponential distribution and its role in life testing. Sato *et al* (1999) obtained a discrete exponential distribution and discussed its properties and applied the distribution to model defect count distribution in semi-conductor deposition equipment and defect count distribution per chips. Gupta and Kundu (1999) have obtained a two-parameter generalized exponential distribution (GED) and discussed its statistical properties, estimation of parameter and applications. Nekoukhou *et at* (2012) obtained a discrete analogue of the generalized exponential distribution and discussed its properties, estimation of parameters and applications. Most of the research works done by different researchers on exponential distributions are available in Ahsanullah and Hamedani (2010).
It should be noted that the two-parameter gamma distribution is the weighted exponential distribution and are defined by its pdf and cdf

$$f(y;\theta,\alpha)=\frac{\theta^{\alpha}}{\Gamma(\alpha)}e^{-\theta y}y^{\alpha-1};\quad y>0,\theta>0,\alpha>0\ , \tag{1.3}$$

$$F\left(y;\theta,\alpha\right)=1-\frac{\Gamma\left(\alpha,\theta y\right)}{\Gamma\left(\alpha\right)}\;;y>0,\theta>0,\alpha>0\;, \tag{1.4}$$

where θ is a scale parameter, α is a shape parameter and the function $\Gamma\left(\alpha,z\right)$ is the upper incomplete gamma function defined as

$$\Gamma\left(\alpha,z\right)=\int_{z}^{\infty}e^{-y}\,y^{\alpha-1}\,dy;\alpha>0,\;z\geq 0 \tag{1.5}$$

Chakraborty and Chakraborty (2012) obtained a discrete analogue of the two-parameter gamma distribution and discussed its properties, parameters estimation and applications.
A two-parameter Weibull distribution (WD) introduced by Weibull (1951) having scale parameter θ and shape parameter β is defined by its pdf and cdf

$$f\left(x;\theta,\beta\right)=\theta\,\beta\,x^{\beta-1}\,e^{-\theta x^{\beta}}\;\;;x>0,\theta>0,\beta>0 \tag{1.6}$$

$$F\left(x;\theta,\beta\right)=1-e^{-\theta x^{\beta}}\;\;;x>0,\theta>0,\beta>0 \tag{1.7}$$

It can be easily shown that the Weibull distribution reduces to classical exponential distribution at $\beta=1$. It should be noted that Weibull distribution is nothing but the power exponential distribution. Taking $x=y^{1/\beta}$ and thus $y=w\left(x\right)=x^{\beta}$ in (1.1), we have

$$g\left(x;\theta,\beta\right)=f\left[w\left(x\right)\right]w'\left(x\right)=\theta e^{-\theta x^{\beta}}\beta\,x^{\beta-1}=\theta\,\beta\,x^{\beta-1}e^{-\theta x^{\beta}}\;, \tag{1.8}$$

which is the pdf of Weibull distribution defined in (1.6). Nakagawa and Osaki (1975) obtained discrete Weibull distribution. Stein and Dattero (1984) introduced a new discrete Weibull distribution. Recently Shanker *et al* (2016) have detailed critical and comparative study on modeling of lifetime data using two-parameter gamma and Weibull distributions and it has been observed that in some datasets gamma gives better fit than Weibull whereas in some datasets Weibull gives better fit than gamma and hence these two distributions are competing each other. Most of the research works done on Weibull distributions are available in Murthy *et al* (2004). The Weibull distribution has been modified and extended to three, four and five parameters by introducing additional parameters to suite to specific set of data by different researchers including Mudholkar and Srivastava (1993), Mudholkar and Kollia 91994), Mudholkar et al (1995), Marshall and Olkin (1997), Lai *et al* (2003), Nadarajah and Kotz (2005), are some among others. Most of the research works done regarding generalizations, extension and modifications of Weibull distributions are available in Lai et al (2011) and Alamalki and Nadarajah (2014). After careful reading of these papers it has been observed that a simple generalization of Weibull distribution can be done which will be different from the previous ones.
The pdf and the cdf of generalized gamma distribution (GGD) introduced by Stacy (1962) having parameters $\theta,\alpha,$ and β is given by

$$f\left(x;\theta,\alpha,\beta\right)=\frac{\beta\,\theta^{\alpha}}{\Gamma\left(\alpha\right)}x^{\beta\alpha-1}\,e^{-\theta x^{\beta}};\;x>0,\theta>0,\alpha>0,\beta>0 \tag{1.9}$$

$$F\left(x;\theta,\alpha,\beta,\right)=1-\frac{\Gamma\left(\alpha,\theta\,x^{\beta}\right)}{\Gamma\left(\alpha\right)};\;\;x>0,\theta>0,\alpha>0,\beta>0 \tag{1.10}$$

where α and β are the shape parameters and θ is the scale parameter, and $\Gamma\left(\alpha,z\right)$ is the upper incomplete gamma function defined in (1.5). Note that Stacy (1962) obtained GGD by taking taking $x=y^{1/\beta}$ and thus $y=w\left(x\right)=x^{\beta}$ in (1.3), and using the approach of obtaining the pdf of Weibull distribution in (1.6).

Clearly the gamma distribution, the Weibull distribution and the exponential distribution are particular cases of (1.9) for $(\beta = 1)$, $(\alpha = 1)$ and $(\alpha = \beta = 1)$ respectively. Detailed discussion about GGD is available in Stacy (1962) and parametric estimation for the GGD has been discussed by Stacy and Mihram (1965). Chakraborty (2015) has obtained a new discrete gamma distribution corresponding to GGD and discussed some of its properties. Note that the research works done on exponential distribution, Weibull distribution, gamma distribution, GGD, their extensions and applications for life testing and modeling are available in Lee and Wang (2003). Recently Shanker and Shukla (2016) have detailed critical and comparative study on modeling of lifetime data using three-parameter GGD and generalized Lindley distribution (GLD) introduced by Zakerzadeh and Dolati (2009) and it has been observed that GGD gives much closer fit than GLD in majority of datasets.
Since Weibull distribution gives better fit than Gamma distribution, Lindley distribution introduced by Lindley (1958), Lognormal distribution and exponential distribution, it is expected and hoped that the generalization of Weibull distribution (GWD) will be a better model than three-parameter GGD and GLD, two-parameter Weibull distribution, Gamma distribution, Lognormal distribution, weighted Lindley distribution (WLD) introduced by Ghitany *et al* (2011), GED and one parameter Lindley and exponential distributions. Keeping this point in mind, an attempt has been made to obtain GWD which includes Weibull distribution and exponential distribution as special cases. Some of its properties including hazard rate function and stochastic ordering have been discussed. The estimation of its parameters has been discussed using maximum likelihood estimation. A real lifetime data has been presented to test the goodness of fit of GWD over GGD, GLD, WLD, GED, Weibull, Gamma, Lognormal, Lindley and exponential distributions and the fit has been found quite satisfactory over these considered distributions.

2. A Generalization of Weibull Distribution

Taking $x = \frac{1}{\alpha} y^{1/\beta}$ and thus $y = w(x) = (\alpha x)^\beta$ in (1.1), and following the approach of obtaining the pdf of Weibull distribution, the pdf of generalization of Weibull distribution (GWD) can be obtained as

$$f(x;\theta,\alpha,\beta) = \alpha\beta\theta(\alpha x)^{\beta-1} e^{-\theta(\alpha x)^\beta};\ x>0,\theta>0,\beta>0,\alpha>0 \tag{2.1}$$

where α, β are shape parameters and θ is the scale parameter. It can be easily verified that Weibull distribution of Weibull (1951) and exponential distribution are particular cases of GWD for $(\alpha = 1)$ and $(\alpha = 1, \beta = 1)$ respectively.

The corresponding cdf of GWD can be obtained as

$$F(x;\theta,\alpha,\beta) = 1 - e^{-\theta(\alpha x)^\beta};\ x>0,\theta>0,\beta>0,\alpha>0 \tag{2.2}$$

Graphs of the pdf and the cdf of GWD are shown in figures 1 and 2 for varying values of the parameters θ, α and β. From figure 1, it is observed that pdf of GWD is taking different shapes including momotonically decreasing, positively skewed, platykurtic, leptokurtic,negatively skewed etc for varying values of parameters.

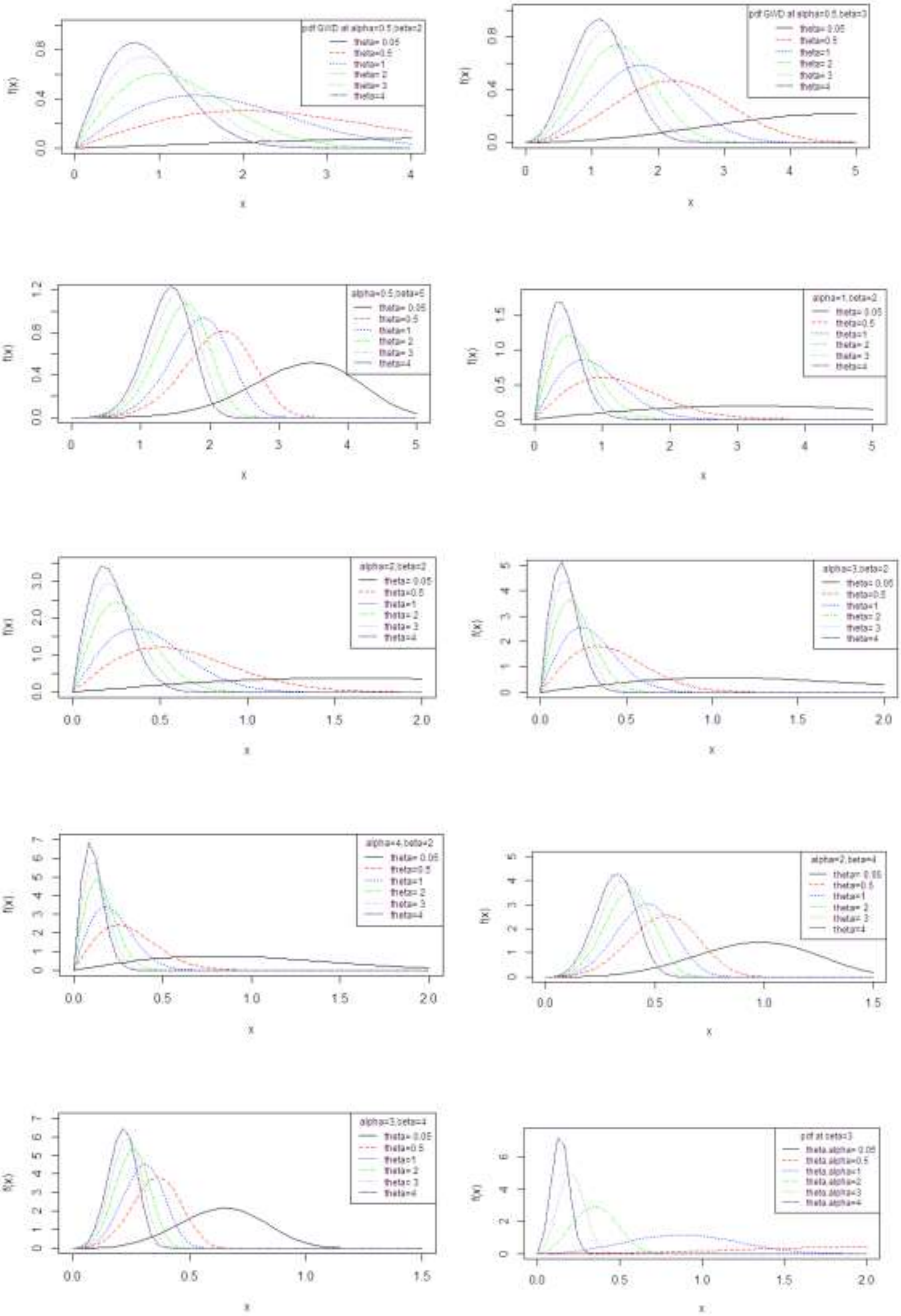

Fig.1. Graphs of pdf of GWD for varying values of parameters θ, α and β

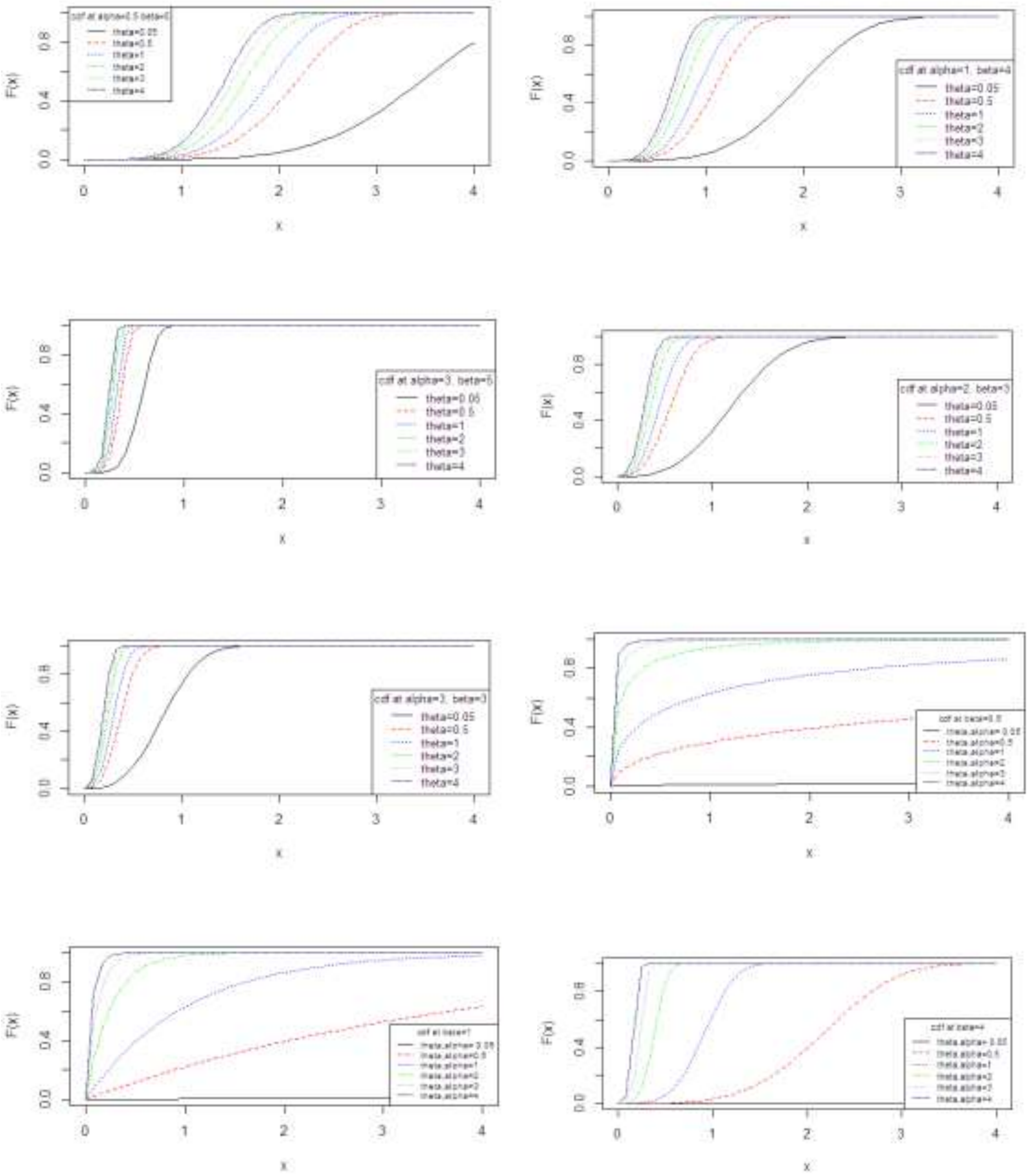

Fig.2. Graphs of cdf of GWD for varying values of parameters θ, α and β.

3. Moments

Using (2.1), the r th moment about origin of GWD can be obtained as

$$\mu_r' = \int_0^\infty x^r \alpha\,\beta\,\theta\left(\alpha x\right)^{\beta-1} e^{-\theta(\alpha x)^\beta}\,dx$$

Taking $u = \theta\left(\alpha\,x\right)^\beta$ and thus $x = \frac{1}{\alpha}\left(\frac{u}{\theta}\right)^{1/\beta}$ and $du = \alpha\,\theta\,\beta\left(\alpha\,x\right)^{\beta-1} dx$, we get

$$\mu_r' = \int_0^\infty \left\{ \frac{1}{\alpha} \left(\frac{u}{\theta} \right)^{1/\beta} \right\}^r e^{-u} du = \frac{1}{\alpha^r} \left(\frac{1}{\theta} \right)^{r/\beta} \int_0^\infty u^{\frac{r}{\beta}+1-1} e^{-u} du = \frac{\Gamma\left(\frac{r}{\beta} + 1 \right)}{\alpha^r \theta^{r/\beta}} ; r = 1, 2, 3, \ldots \tag{3.1}$$

Thus the first two moments about origin and the variance of GWD are obtained as

$$\mu_1' = \frac{\Gamma\left(\frac{1}{\beta} + 1 \right)}{\alpha \, \theta^{1/\beta}} \ , \ \mu_2' = \frac{\Gamma\left(\frac{2}{\beta} + 1 \right)}{\alpha^2 \, \theta^{2/\beta}}$$

and $\mu_2 = \dfrac{\Gamma\left(\frac{2}{\beta} + 1 \right)}{\alpha^2 \, \theta^{2/\beta}} - \left(\dfrac{\Gamma\left(\frac{1}{\beta} + 1 \right)}{\alpha \, \theta^{1/\beta}} \right)^2 = \dfrac{1}{\alpha^2 \, \theta^{2/\beta}} \left[\Gamma\left(\frac{2}{\beta} + 1 \right) - \left\{ \Gamma\left(\frac{1}{\beta} + 1 \right) \right\}^2 \right].$

Similarly substituting $r = 3$ and 4 in (3.1), second and third moments about origin can be obtained. Then using the relationship between central moments and moments about origin, third and fourth central moments can be obtained. The expressions for third and fourth central moments are complicated and very big and hence they are not being presented here.

4. Hazard Rate Function

Suppose X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$ from 1.2 and 1.3. The hazard rate function (also known as the failure rate function) function of X is defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{P\left(X < x + \Delta x \,|\, X > x \right)}{\Delta x} = \frac{f(x)}{1 - F(x)}$$

The corresponding $h(x)$ of GWD can be obtained as

$$h(x;\theta,\alpha,\beta) = \frac{f(x;\theta,\alpha,\beta)}{1 - F(x;\theta,\alpha,\beta)} = \alpha \, \beta \, \theta \left(\alpha \, x \right)^{\beta - 1} ; x > 0, \theta > 0, \alpha > 0, \beta > 0 \tag{4.1}$$

Graphs of $h(x)$ of GWD for varying values of the parameters θ, α and β are shown in figure 3. The $h(x)$ of GWD are monotonically decreasing, constant, monotonically increasing or bathtub shape varying for varying values of parameters θ, α and β.

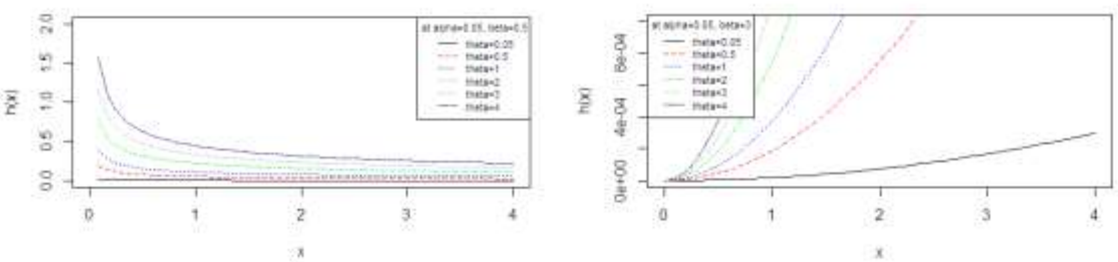

Fig.3. Graphs of $h(x)$ of GWD for varying of values of parameters θ, α and β.

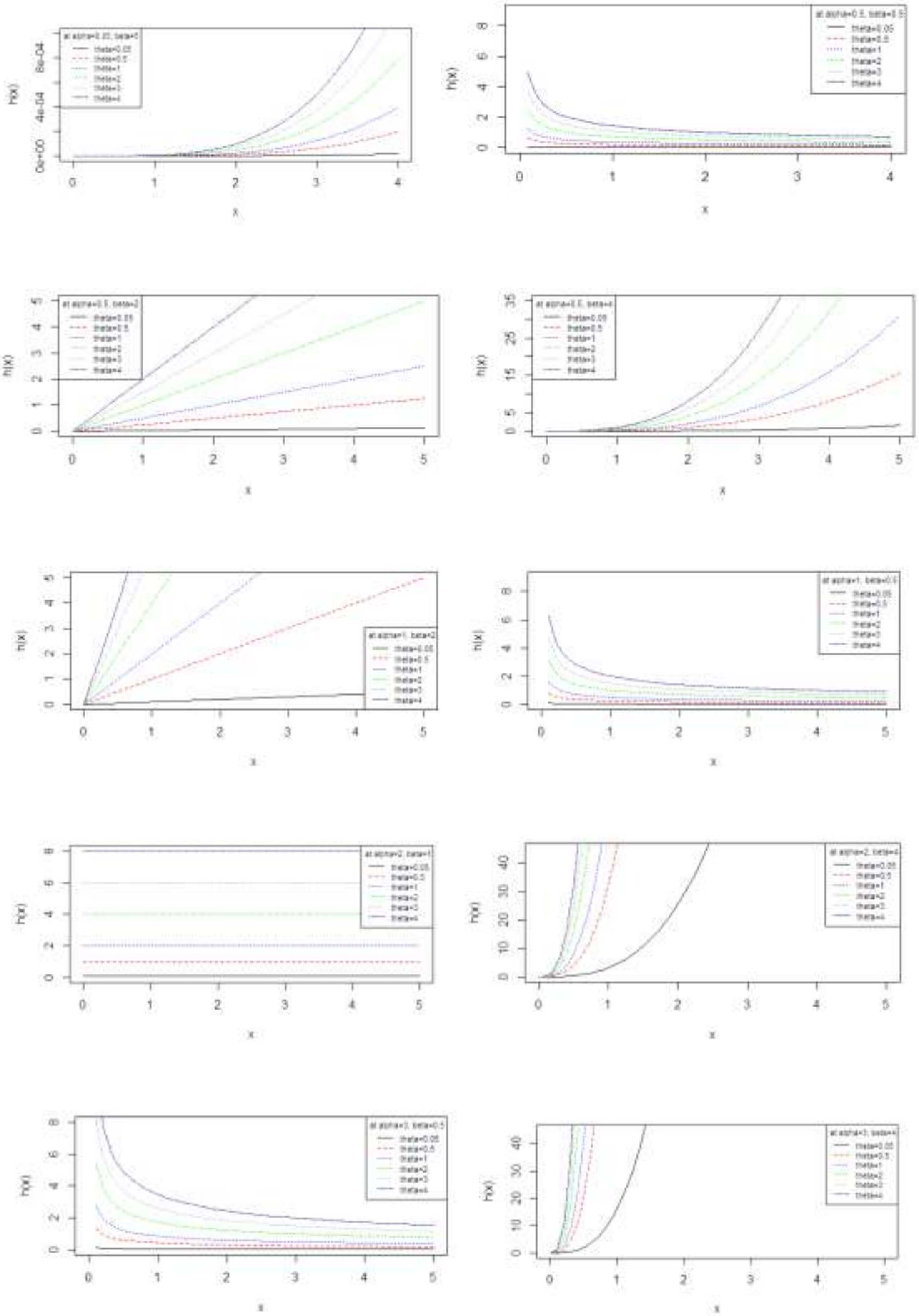

Fig.3. Graphs of $h(x)$ of GWD for varying of values of parameters θ, α and β (continuation)

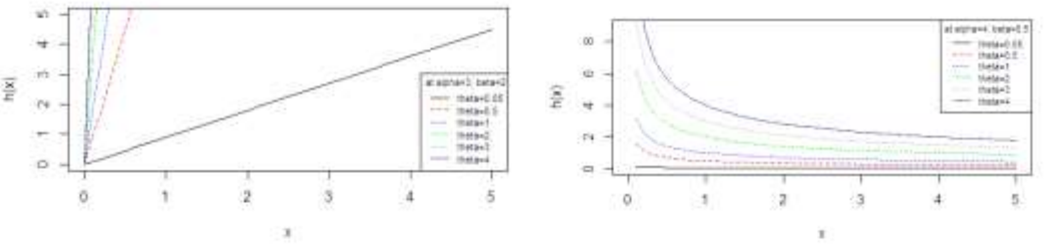

Fig.3. Graphs of $h(x)$ of GWD for varying of values of parameters θ, α and β (continuation)

5. Stochastic Ordering

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $\left(X \le_{st} Y\right)$ if $F_X\left(x\right) \ge F_Y\left(x\right)$ for all x

(ii) hazard rate order $\left(X \le_{hr} Y\right)$ if $h_X\left(x\right) \ge h_Y\left(x\right)$ for all x

(iii) mean residual life order $\left(X \le_{mrl} Y\right)$ if $m_X\left(x\right) \le m_Y\left(x\right)$ for all x

(iv) likelihood ratio order $\left(X \le_{lr} Y\right)$ if $\dfrac{f_X\left(x\right)}{f_Y\left(x\right)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$\begin{array}{c} X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \\ \Downarrow \\ X \le_{st} Y \end{array}$$

GWD is ordered with respect to the strongest 'likelihood ratio ordering' as shown in the following theorem:

Theorem: Let X ~ GWD$\left(\theta_1, \alpha_1, \beta_1\right)$ and Y ~ GWD$\left(\theta_2, \alpha_2, \beta_2\right)$. If one of the following conditions satisfied

(i) $\theta_1 = \theta_2, \alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$ (ii) $\theta_1 = \theta_2, \beta_1 = \beta_2$ and $\alpha_1 > \alpha_2$

(iii) $\theta_1 > \theta_2, \alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$

then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X\left(x;\theta_1,\alpha_1,\beta_1\right)}{f_Y\left(x;\theta_2,\alpha_2,\beta_2\right)} = \left(\frac{\beta_1\,\alpha_1^{\beta_1}\theta_1}{\beta_2\,\alpha_2^{\beta_2}\theta_2}\right) x^{\beta_1-\beta_2}\, e^{-\left\{\theta_1\alpha_1^{\beta_1}x^{\beta_1}-\theta_2\alpha_2^{\beta_2}x^{\beta_2}\right\}} \quad ; \; x > 0$$

Now

$$\ln\frac{f_X\left(x;\theta_1,\alpha_1,\beta_1\right)}{f_Y\left(x;\theta_2,\alpha_2,\beta_2\right)} = \ln\left(\frac{\beta_1\,\alpha_1^{\beta_1}\theta_1}{\beta_2\,\alpha_2^{\beta_2}\theta_2}\right) + \left(\beta_1-\beta_2\right)\ln x - \left(\theta_1\alpha_1^{\beta_1}x^{\beta_1} - \theta_2\alpha_2^{\beta_2}x^{\beta_2}\right).$$

.

This gives $$\frac{d}{dx}\ln\frac{f_X(x;\theta_1,\alpha_1,\beta_1)}{f_Y(x;\theta_2,\alpha_2,\beta_2)}=\frac{\beta_1-\beta_2}{x}-\left(\theta_1\beta_1\alpha_1^{\beta_1}x^{\beta_1-1}-\theta_2\beta_2\alpha_2^{\beta_2}x^{\beta_2-1}\right).$$

Thus under the given conditions in the theorem, $\frac{d}{dx}\ln\frac{f_X(x;\theta_1,\alpha_1,\beta_1)}{f_Y(x;\theta_2,\alpha_2,\beta_2)}<0$. This means that $X\le_{lr}Y$ and hence $X\le_{hr}Y$, $X\le_{mrl}Y$ and $X\le_{st}Y$.

6. Maximum Likelihood Estimation of Parameters

Let $\left(x_1,x_2,x_3,\dots,x_n\right)$ be a random sample of size n from GWD$\left(\theta,\alpha,\beta\right)$. Then natural log likelihood function is given by

$$\ln L=\sum_{i=1}^{n}\ln f\left(x_i;\theta,\alpha,\beta\right)=n\left(\beta\ln\alpha+\ln\beta+\ln\theta\right)+\left(\beta-1\right)\sum_{i=1}^{n}\ln x_i-\theta\alpha^{\beta}\sum_{i=1}^{n}x_i^{\beta}$$

The maximum likelihood estimate (MLE) $\left(\hat{\theta},\hat{\alpha},\hat{\beta}\right)$ of $\left(\theta,\alpha,\beta\right)$ of GWD (2.1) are the solutions of the following log likelihood equations

$$\frac{\partial\ln L}{\partial\theta}=\frac{n}{\theta}-\alpha^{\beta}\sum_{i=1}^{n}x_i^{\beta}=0$$

$$\frac{\partial\ln L}{\partial\alpha}=\frac{n\beta}{\alpha}-\theta\,\beta\,\alpha^{\beta-1}\sum_{i=1}^{n}x_i^{\beta}=0$$

$$\frac{\partial\ln L}{\partial\beta}=n\ln\alpha+\frac{n}{\beta}+\sum_{i=1}^{n}\ln x_i-\theta\alpha^{\beta}\ln\alpha\sum_{i=1}^{n}x_i^{\beta}-\theta\alpha^{\beta}\sum_{i=1}^{n}x_i^{\beta}\ln x_i=0$$

These three log likelihood equations do not seem to be solved directly because these cannot be expressed in closed form. However, Fisher's scoring method can be applied to solve these equations iteratively. For, we have

$$\frac{\partial^2\ln L}{\partial\theta^2}=-\frac{n}{\theta^2}$$

$$\frac{\partial^2\ln L}{\partial\alpha^2}=-\frac{n\beta}{\alpha^2}-\theta\,\beta\left(\beta-1\right)\alpha^{\beta-2}\sum_{i=1}^{n}x_i^{\beta}$$

$$\frac{\partial^2\ln L}{\partial\beta^2}=-\left[\frac{n}{\beta^2}+\theta\alpha^{\beta}\sum_{i=1}^{n}x_i^{\beta}\left(\ln x_i\right)^2+\theta\alpha^{\beta}\left(\ln\alpha\right)^2\sum_{i=1}^{n}x_i^{\beta}+2\theta\alpha^{\beta}\left(\ln\alpha\right)\sum_{i=1}^{n}x_i^{\beta}\ln x_i\right]$$

$$\frac{\partial^2\ln L}{\partial\theta\,\partial\alpha}=-\beta\,\alpha^{\beta-1}\sum_{i=1}^{n}x_i^{\beta}=\frac{\partial^2\ln L}{\partial\alpha\,\partial\theta}$$

$$\frac{\partial^2\ln L}{\partial\theta\,\partial\beta}=-\left[\alpha^{\beta}\ln\alpha\sum_{i=1}^{n}x_i^{\beta}+\alpha^{\beta}\sum_{i=1}^{n}x_i^{\beta}\ln x_i\right]=\frac{\partial^2\ln L}{\partial\beta\,\partial\theta}$$

$$\frac{\partial^2\ln L}{\partial\alpha\,\partial\beta}=\frac{n}{\alpha}-\theta\,\alpha^{\beta-1}\sum_{i=1}^{n}x_i^{\beta}-\theta\,\beta\,\alpha^{\beta-1}\ln\alpha\sum_{i=1}^{n}x_i^{\beta}-\theta\,\beta\,\alpha^{\beta-1}\sum_{i=1}^{n}x_i^{\beta}\ln x_i=\frac{\partial^2\ln L}{\partial\beta\,\partial\alpha}$$

The MLE $\left(\hat{\theta},\hat{\alpha},\hat{\beta}\right)$ of $\left(\theta,\alpha,\beta\right)$ of GWD (2.1) are the solution of the following equations

$$\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \theta^2} & \frac{\partial^2 \ln L}{\partial \theta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \theta \partial \beta} \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \theta} & \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \ln L}{\partial \beta \partial \theta} & \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0 \\ \hat{\beta}=\beta_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \\ \hat{\beta}-\beta_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \ln L}{\partial \theta} \\ \frac{\partial \ln L}{\partial \alpha} \\ \frac{\partial \ln L}{\partial \beta} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0 \\ \hat{\beta}=\beta_0}}$$

where θ_0, α_0 and β_0 are the initial values of θ, α and β. These equations are solved iteratively till sufficiently close estimates of $\hat{\theta}, \hat{\alpha}$ *and* $\hat{\beta}$ are obtained. In this paper, R-software has been used to estimate the parameters θ, α and β for the considered datasets.

7. Goodness of Fit

In this section, the application and goodness of fit of GWD has been discussed with a real lifetime dataset from engineering and the fit has been compared with one parameter exponential and Lindley distributions, two-parameter Gamma, Weibull, Lognormal, GED and WLD, and three-parameter GGD and GLD. The following dataset has been considered for the goodness of fit of the considered distributions.
The dataset in table 1 is given by Birnbaum and Saunders (1969) on the fatigue life of 6061 – T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The dataset consists of 100 observations with maximum stress per cycle 31,000 psi. The data ($\times 10^{-3}$) are presented below (after subtracting 65**).**

Table 1: The data is regarding fatigue life of 6061 – T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second, available in Birnbaum and Saunders (1969)

5	25	31	32	34	35	38	39	39	40
42	43	43	43	44	44	47	47	48	49
49	49	51	54	55	55	55	56	56	56
58	59	59	59	59	59	63	63	64	64
65	65	65	66	66	66	66	66	67	67
67	68	69	69	69	69	71	71	72	73
73	73	74	74	76	76	77	77	77	77
77	77	79	79	80	81	83	83	84	86
86	87	90	91	92	92	92	92	93	94
97	98	98	99	101	103	105	109	136	147

The pdf and the cdf of the considered distributions are presented in table 2.

Table 2: **pdf and cdf of GLD, WLD, GED, Lognormal and Lindley distributions**

Distributions	pdf/cdf	
GLD	Pdf	$f\left(x;\theta,\alpha,\beta\right)=\frac{\theta^{\alpha+1}}{\left(\beta+\theta\right)}\frac{x^{\alpha-1}}{\Gamma\left(\alpha+1\right)}\left(\alpha+\beta x\right)e^{-\theta x}$
	Cdf	$F\left(x;\theta,\alpha,\beta\right)=1-\frac{\alpha\left(\beta+\theta\right)\Gamma\left(\alpha,\theta x\right)+\beta\left(\theta x\right)^{\alpha}e^{-\theta x}}{\left(\beta+\theta\right)\Gamma\left(\alpha+1\right)}$
WLD	Pdf	$f\left(x;\theta,\alpha\right)=\frac{\theta^{\alpha+1}}{\left(\theta+\alpha\right)}\frac{x^{\alpha-1}}{\Gamma\left(\alpha\right)}\left(1+x\right)e^{-\theta x}$
	Cdf	$F\left(x;\theta,\alpha\right)=1-\frac{\left(\theta+\alpha\right)\Gamma\left(\alpha,\theta x\right)+\left(\theta x\right)^{\alpha}e^{-\theta x}}{\left(\theta+\alpha\right)\Gamma\left(\alpha\right)}$
GED	pdf	$f\left(x;\theta,\alpha\right)=\theta\alpha\left(1-e^{-\theta x}\right)^{\alpha-1}e^{-\theta x}$
	cdf	$F\left(x;\theta,\alpha\right)=\left(1-e^{-\theta x}\right)^{\alpha}$
Lognormal	pdf	$f\left(x;\theta,\alpha\right)=\frac{1}{\sqrt{2\pi}\alpha x}e^{-\frac{1}{2}\left(\frac{\log x-\theta}{\alpha}\right)^{2}}$
	cdf	$F\left(x;\theta,\alpha\right)=\phi\left(\frac{\log x-\theta}{\alpha}\right)$
Lindley	pdf	$f\left(x;\theta\right)=\frac{\theta^{2}}{\theta+1}\left(1+x\right)e^{-\theta x}$
	cdf	$F\left(x;\theta\right)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}$

Table 3: Summary of the ML estimates of parameters

Model	ML Estimates		
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
GWD	2.6864	0.0097	3.2122
GGD	0.01880	4.8293	1.3071
GLD	0.1152	7.0755	0.4492
WLD	0.1128	6.7274	------
GED	0.4140	9.6988	-------
Weibull	0.0021	-------	1.4573
Gamma	0.1125	17.4395	------
Lognormal	4.1579	0.41112	------
Lindley	0.0288	--------	-------
Exponential	0.0146	-------	--------

In order to compare considered distributions maximum likelihood estimates (MLEs) of parameters have been presented in table 3 and values of $-2\ln L$, AIC (Akaike information criterion), K-S Statistic (Kolmogorov-Smirnov Statistic) and p-value for the considered dataset have been computed and presented in table 4. The formulae for AIC and K-S Statistics are as follows: $AIC = -2\log L + 2k$, and $K\text{-}S = \underset{x}{\text{Sup}}\left|F_n(x) - F_0(x)\right|$, where k being the number of parameters involved in the respective distributions, n is the sample size and $F_n(x)$ is the empirical distribution function. The best distribution corresponds to the lower values of $-2\ln L$, AIC and K-S statistic and higher p- value.

Table 4: Summary of Goodness of fit by K-S Statistic

Model	$-2\log L$	AIC	K-S	p-value
GWD	908.52	914.52	0.071	0.695
GGD	912.44	918.44	0.087	0.429
GLD	914.95	920.44	0.098	0.281
WLD	915.56	919.56	0.099	0.273
GED	927.78	931.78	0.854	0.000
Weibull	982.66	986.66	0.986	0.000
Gamma	915.76	919.76	0.895	0.000
Lognormal	937.59	941.59	0.345	0.000
Lindley	983.11	985.11	0.252	0.000
Exponential	1044.87	1046.87	0.366	0.000

It is obvious from the goodness of fit in table 4 that GWD gives much closer fit than GGD,GLD,WLD, GED, Weibull, Gamma, Lognormal, Lindley and exponential and hence it can be considered an important distribution for modeling lifetime data over these distributions. The variance-covariance matrix of the parameters θ, α and β of GWD for the considered dataset has been presented in table 5.

Table 5: Variance-covariance matrix of the parameters θ, α and β of GWD for dataset

$$\begin{array}{cc} & \begin{array}{ccc} \hat{\theta} & \hat{\alpha} & \hat{\beta} \end{array} \\ \begin{array}{c} \hat{\theta} \\ \hat{\alpha} \\ \hat{\beta} \end{array} & \begin{bmatrix} 0.1690 & -0.0404 & 0.0221 \\ -0.0404 & 0.0097 & -0.0060 \\ 0.0221 & -0.0060 & 0.0597 \end{bmatrix} \end{array}$$

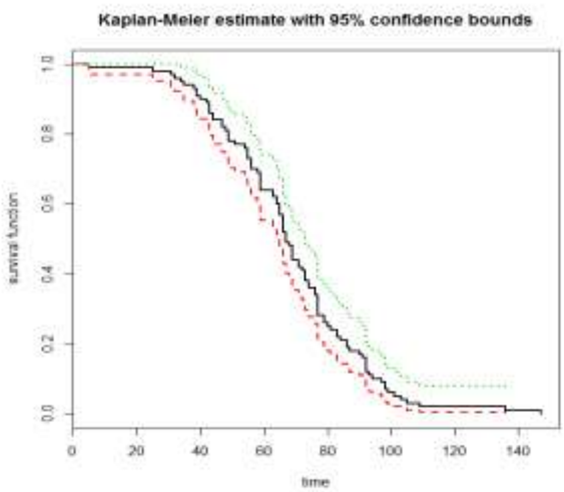

Fig4: Kaplan Meier plot on considered data set

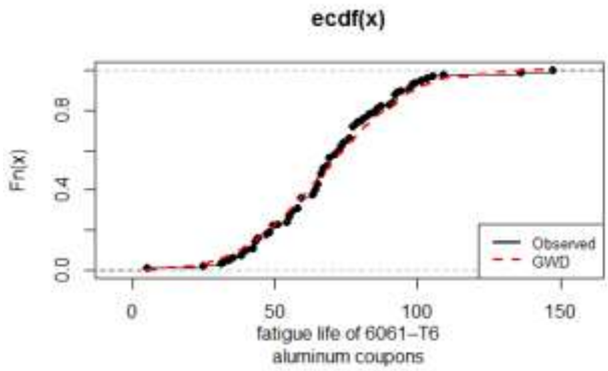

Fig.5: Fitted cdf plot on considered dataset

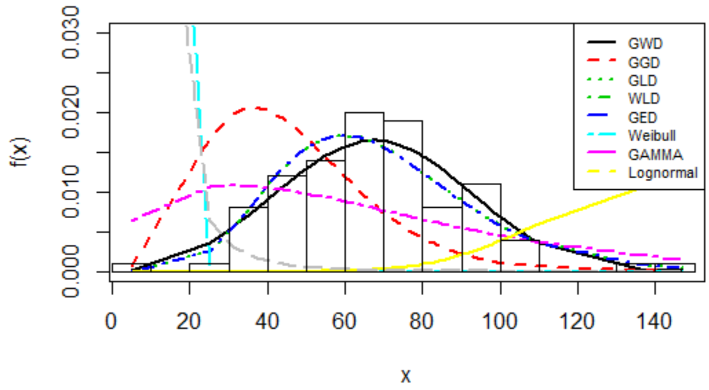

Fig.6: fitted probability plot of distributions on considered data set.

8. Concluding Remarks

A three-parameter generalization of Weibull distribution (GWD) introduced by Weibull (1951), which includes the two-parameter Weibull introduced by Weibull (1951) and exponential distribution as special cases, has been proposed and investigated. Its hazard rate function and stochastic ordering has been discussed and their behaviors have been studied graphically. Maximum likelihood estimation has been discussed for parameter estimates of GWD. The goodness of fit of the proposed distribution has been discussed with a real lifetime dataset. The proposed distribution is useful for the dataset whose hazard rate is monotonically decreasing, constant, monotonically increasing or bathtub shape. Further, it is worth mentioning that, although we have used non-censored data but it can also be used for censored data including left censoring, right censoring and interval censoring

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