(M, MAP)/(PH, PH)/1 queue with Nonpremptive Priority, Working Interruption and Protection

[1] A. Krishnamoorthy [1,3],[2] Divya V [2,4]

[1]Department of Mathematics
CMS College,Kottayam-686001,India.
[2]Department of Mathematics
N.S.S. College, Cherthala-688556

Abstract

In this paper we consider a (M,MAP)/(PH,PH)/1 queue with nonpremptive priority, working interruption and protection. Two types of priority classes of customers where type I customers arrive according to a Poisson process and type II customers arrive according to Markovian Arrival Process are considered. Service time of both type I and type II customers follow mutually independent phase type distributions. The number of type I customers in the system is restricted to a maximum of L. Also type I customers are assumed to have a non-premptive priority over type II customers. Customer services are subject to interruption by a self induced mechanism. The interruptions occur according to Poisson process. Instead of stopping service completely, the service continues at slower rate during interruption. Also we assume that an interruption occuring while customer is already under interruption will not affect the customer.The server continues to serve at this lower rate until interruption is fixed. The duration of interruption is assumed to be exponentially distributed. A protection mechanism to diminish the effect of interruptions on type I customers service is arranged.The protection for the service of type I customers is provided at the epoch of realization of the clock which starts ticking up the moment a type I customer is taken for service. Type II customers are not provided protection against interruption during their service. Also we assume that type I customers get service at a faster rate starting from the epoch of providing service protection. We analyse the distribution of service time duration of both type I and type II customers and the distribution of a p-cycle. Also we provide LSTs of busy cycle, busy period of type I customers generated during the service time of a type II customer and LSTs of waiting time distributions of type I and type II customers. Also we compute the expected number of interruptions during a type I and a type II service. We perform numerical computations to evaluate important system characteristics and also optimal system cost using a cost function .

Keywords: (M,MAP)/(PH,PH)/1 queue, nonpremptive priority, working interruption, protection

1 Introduction

Queues with interruption play an important role in day to day life. We encounter different kinds of interruptions in various activities like internet browsing, banking, medical check ups, in supermarkets etc. The works so far reported in the literature discuss about interruptions such as server induced, customer induced, enviornment dependent service interruptions, server vacations,

---

[1] Author for correspondence: E-mail:achyuthacusat@gmail.com [3]Emeritus Fellow(EMERITUS-2017-18 GEN 10822(SA-II)), University Grants commission, India

[2] E-mail:divyavelayudhannair@gmail.com. [4] Research is supported by the University Grants Commission, Govt. Of India, under Faculty Development Programme(Grant No.F.No.FIP/12th Plan/KLKE008 TF 04) in Department of Mathematics, Cochin University of Science and Technology, Cochin-22

vacation interruptions and arrival of a priority customer. The first reported work on queues with service interruption is by White and Christie in 1958 in which they considered a two-priority single server system with the low priority customer in service pre-empted on arrival of a high priority customer. Even in the case of single class customer system, the customer in service has to wait whenever a system breakdown occurs. The interrupted service starts from the very beginning (repeat) or from where it got interrupted (resumption) on completion of interruption. These two cases are separately considered in Keilson [2], Gaver [4] and by several other researchers. Fiems et al. [3] introduced probability measures for repeat/resumption on completion of interruption without assigning any rule. Krishnamoorthy et al. [6] are the first to give a specific rule for resumption/repetition of service. We refer the review paper by Krishnamoorthy et al. [5]for details on queueing models with system induced service interruption (priority queues not included).

Varghese et al. [12] introduced a new type of interruption called customer induced interruption in which a customer interrupts own service. They considered an infinite capacity queueing system with a single server in which customers arrive according to a Poisson process with the service time following an exponential distribution. The interruptions occur according to a Poisson process and the duration of each interruption follows an exponential distribution. The self-interrupted customers enter into a finite buffer of size K. Any interrupted customer, finding the buffer full, is considered lost. Those interrupted customers who complete their interruptions move into another buffer of same size and are given a nonpreemptive priority over new customers. They evaluated several performance measures. Numerical illustrations of the system behavior are also provided and also discussed an optimization problem through an illustrative example. Krishnamoorthy et al. [7] extended this to a multi-server queueing system. They investigated the behavior of the queueing system, several performance measures are evaluated and numerical illustrations of the system behavior are provided. Also an optimization problem to maximize the revenue with respect to number of servers is employed and optimal buffer size for the self-interrupted customers are discussed through two illustrative examples. Dudin et al. [13] extended these to MMAP/PH(PH)/c queue with negative arrivals. Varghese and Krishnamoorthy [8] considered a single-server retrial queue with infinite capacity of the primary buffer and finite capacity of the orbit to which customers arrive according to a Poisson process, and the service time follows phase-type distribution. The customer-induced interruption occurs according to a Poisson process. The self-interrupted customers enter into orbit. Any interrupted customer, finding the orbit full, is considered lost. The interrupted customers retries for service after the interruption is completed. Several performance measures were evaluated and some numerical illustrations of the system behavior were provided.

In this paper we consider a single server queueing model with two priority classes of customers where the type I customers are assumed to have a non-premptive priority over type II customers.We consider customer induced interruption during own service. Instead of stopping service completely, the service continues at slower rate during interruption. The protection for the service of type I customers is provided at the epoch of realization of the clock which starts at the epoch at which the type I customer is taken for service. The rest of the paper is arranged as follows. The mathematical formulation is given in section 2. Section 3 provides steady state analysis of the model. Waiting time analysis of type I and type II customers are discussed in sections 4. Expected number of interruptions during type I and type II services are discussed in sections 5 and 6 respectively. Some other performance measures are discussed in section 7. A related cost function is discussed in section 8. Some numerical results are discussed in section 9. Proofs of two theorems stated in section 4, are given in appendix.

Notations and abbreviations used in the sequel:

- $\mathbf{e}(a)$ = Column vector of $1'$s of order a
- $\mathbf{e}$ =Column vector of $1'$s of appropriate order.

- $CTMC$: Continuous time Markov chain.
- I_a = identity matrix of order a.
- $\mathbf{e}_a(b)$ = column vector of order b with 1 in the ath position and the remaining entries zero.
- MAP: Markovian Arrival Process
- LST: Laplace-Steiltges Transform
- $LIQBD$: Level independent Quasi-Birth and-Death
- WI: Working Interruption
- Parameters: λ-arrival rate of type I customers,γ- arrival rate of interruptions, η- parameter of exponential duration of interruption, δ- parameter of exponential protection clock.

2 Mathematical formulation

We consider a single server queue with two priority classes of customers type I and type II with the former arriving according to a Poisson process of rate λ and the latter according to Markovian Arrival Process with representation(D_0, D_1). Service time of both types follow distinct phase type distributions with representations PH(α,T) of order m_1 and PH(β, S) of order m_2 respectively. The number of type I customers in the system is restricted to a maximum of L. Also type I customers are assumed to have a non-premptive priority over type II customers. Customer services are subject to interruption by a self induced mechanism. While in interruption arrival of another interruption doesnot affect the customer.The interruptions occur according to Poisson process with rate γ. Instead of stopping the service of that customer completely,it continues at slower rate during interruption. That is, the service time of type I and type II ,during an interruption follow phase type distributions with representation PH$(\alpha, \theta T)$ and PH$(\beta, \theta' S)$, $0 < \theta, \theta' < 1$ respectively.Thus $\mu = [\alpha(-T)^{-1}\mathbf{e}]^{-1}$ is the normal service rate and $\theta\mu$ is the interrupted service rate of type I customers and $\mu' = [\beta(-S)^{-1}\mathbf{e}]^{-1}$ and $\theta'\mu'$ are respectively the corresponding rates of normal and interrupted services of type II customers. The server continues to serve at this lower rate until a random clock expires. The duration of interruption is assumed to be exponentially distributed with parameter η. A protection mechanism to diminish the effect of interruptions on type I customers service is arranged. An exponential random clock with mean $\frac{1}{\delta}$ is started simultaneously with each type I service. The protection for the service of type I customers is provided at the epoch of realization of this clock. Type II customers are not provided protection against interruption during their service. Also we assume that the service time of type I customers on activation of protection clock, follows phase type distribution with representation PH$(\alpha, \phi T)$, $\phi > 1$ and finite.

Let $Q^* = D_0 + D_1$ be the generator matrix of the type II arrival process and π^* be its stationary probability vector. Hence π is the unique (positive) probability vector satisfying $\pi^* Q^* = 0$, $\pi^* \mathbf{e} = 1$. The constant $\beta^* = \pi^* D_1 \mathbf{e}$, referred to as $fundemental\ rate$, gives the expected number of type II arrivals per unit of time in the stationary version of the MAP. It is assumed that the two arrival processes are mutually independent and are also independent of the service time distributions.

2.1 The QBD process

The model described in section 1 can be studied as a LIQBD process. First we introduce the followiing notations:

At time t:

$N_1(t)$: number of type II customers in the system,

$N_2(t)$: number of type I customers in the system

$$J(t) = \begin{cases} 0, \textit{if the type I customer in service is unprotected/type II customer is in service} \\ 1, \textit{if the type I customer in service is protected} \end{cases}$$

$$K(t) = \begin{cases} 0, & \textit{if the server provides service to type I customer in WI} \\ 1, & \textit{if the server provides service to type II customer in WI} \\ 2, & \textit{if the server provides normal service to type I customer} \\ 3, & \textit{if the server provides normal service to type II customer} \end{cases}$$

$S(t)$: the phase of service when the server is busy

$M(t)$: the phase of arrival of the type II customer.

It is easy to verify that $\{(N_1(t), N_2(t), J(t), K(t), S(t), M(t)): t \geq 0\}$ is a LIQBD with state space

$$l(0) = \{(0,k)/1 \leq k \leq n\} \cup \{(0, i_2, 0, j_2, k_1, k_2)/1 \leq i_2 \leq L, j_2 = 0 \text{ } or \text{ } 2, 1 \leq k_1 \leq m_1, 1 \leq k_2 \leq n\} \cup \{(0, i_2, 1, 2, k_1, k_2)/1 \leq i_2 \leq L, 1 \leq k_1 \leq m_1, 1 \leq k_2 \leq n\}$$

For $i_1 \geq 1$,

$$\{(i_1, 0, 0, j_2, k_1, k_2)/j_2 = 1 \text{ } or \text{ } 3, 1 \leq k_1 \leq m_1, 1 \leq k_2 \leq n\} \cup \{(i_1, i_2, 0, j_2, k_1, k_2): 1 \leq i_2 \leq L, j_2 = 0 \text{ } or \text{ } 2, 1 \leq k_1 \leq m_1, 1 \leq k_2 \leq n\} \cup \{(i_1, i_2, 0, j_2, k_1, k_2): 1 \leq i_2 \leq L, j_2 = 1 \text{ } or \text{ } 3, 1 \leq k_1 \leq m_2, 1 \leq k_2 \leq n\} \cup \{(i_1, i_2, 1, 2, k_1, k_2)/1 \leq i-2 \leq L, 1 \leq k_1 \leq m_1, 1 \leq k_2 \leq n\}$$

The infinitesimal generator of this CTMC is

$$Q_1 = \begin{bmatrix} B_0 & C_0 & & & \\ B_1 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}.$$

where B_0 contains transitions within the level 0; C_0 represents transitions from level 0 to level 1; B_1 represents transitions from level 1 to level 0; A_0 represents transitions from level g to level $g+1$ for $g \geq 1$, A_1 represents transitions within the level g for $g \geq 1$ and A_2 represents transitions from level g to $g-1$ for $g \geq 2$. The boundary blocks B_0, C_0, B_1 are of orders $n(1+3m_1L) \times n(1+3m_1L)$, $n(1+3m_1L) \times (2m_2n + (3m_1+2m_2)nL)$, $(2m_2n + (3m_1+2m_2)nL) \times n(1+3m_1L)$ respectively. A_0, A_1, A_2 are square matrices of order $2m_2n + (3m_1+2m_2)nL$. Define the entries of $B_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}, C_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}, B_{1_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$ as transition submatrices which contains transitions of the form $(0, h_1, i_1, j_1, k_1, l_1) \to (0, h_2, i_2, j_2, k_2, l_2), (0, h_1, i_1, j_1, k_1, l_1) \to (1, h_2, i_2, j_2, k_2, l_2)$ *and* $(1, h_1, i_1, j_1, k_1, l_1) \to (0, h_2, i_2, j_2, k_2, l_2)$ respectively. Define the entries of $A_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}, A_{1_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}, A_{2_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$ as transition submatrices which contains transitions of the form $(g, h_1, i_1, j_1, k_1, l_1) \to (g+1, h_2, i_2, j_2, k_2, l_2)$, where $g \geq 1, (g, h_1, i_1, j_1, k_1, l_1) \to (g, h_2, i_2, j_2, k_2, l_2)$, where $g \geq 1, (g, h_1, i_1, j_1, k_1, l_1) \to (g-1, h_2, i_2, j_2, k_2, l_2)$, where $g \geq 2$ respectively. Since none or one event alone could take place in a short interval of time with positive probability, in general, a transition such as $(g_1, h_1, i_1, j_1, k_1, l_1) \to (g_2, h_2, i_2, j_2, k_2, l_2)$ has positive rate only for exactly one of $g_1, h_1, i_1, j_1, k_1, l_1$ different from $g_2, h_2, i_2, j_2, k_2, l_2$.

$$B_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)} =$$

$$\begin{cases}
\lambda(\alpha \otimes I_n) & h_1=0, h_2=1; i_2=0; j_2=2, 1\le k_2\le m_1, \\ & 1\le l_1,l_2\le n \\
\lambda I_{m_1 n} & 1\le h_1\le L-1, h_2=h_1+1; i_1=i_2=0; j_1=j_2, \\ & j_1=0 \ or\ 2; 1\le k_1,k_2\le m_1; 1\le l_1,l_2\le n \\
\lambda I_{m_1 n} & 1\le h_1\le L-1, h_2=h_1+1; i_1=i_2=1; j_1=j_2=2; \\ & 1\le k_1,k_2\le m_1; 1\le l_1,l_2\le n \\
\theta T^0\otimes I_n & h_1=1, h_2=0; i_1=0; j_1=0; 1\le k_1\le m_1; \\ & 1\le l_1,l_2\le n \\
T^0\otimes I_n & h_1=1, h_2=0; i_1=0; j_1=2; 1\le k_1\le m_1; \\ & 1\le l_1,l_2\le n \\
\phi T^0\otimes I_n & h_1=1, h_2=0; i_1=1; j_1=2; 1\le k_1\le m_1; \\ & 1\le l_1,l_2\le n \\
\theta T^0\alpha\otimes I_n & 2\le h_1\le L, h_2=h_1-1; i_1=i_2=0; j_1=0, j_2=2; \\ & 1\le k_1,k_2\le m_1; 1\le l_1,l_2\le n \\
T^0\alpha\otimes I_n & 2\le h_1\le L, h_2=h_1-1; i_1=i_2=0; j_1=j_2=2; \\ & 1\le k_1,k_2\le m_1; 1\le l_1,l_2\le n \\
\phi T^0\alpha\otimes I_n & 2\le h_1\le L, h_2=h_1-1; i_1=1, i_2=0; j_1=j_2=2; \\ & 1\le k_1,k_2\le m_1; 1\le l_1,l_2\le n \\
\eta I_{m_1 n} & 1\le h_1\le L, h_1=h_2, i_1=i_2=0; j_1=0, j_2=2; \\ & 1\le k_1,k_2\le m_1; 1\le l_1,l_2\le n \\
\gamma I_{m_1 n} & 1\le h_1\le L, h_1=h_2; i_1=i_2=0; j_1=2, j_2=0; \\ & 1\le k_1,k_2\le m_1; 1\le l_1,l_2\le n \\
\delta I_{m_1 n} & 1\le h_1\le L, h_1=h_2; i_1=0, i_2=1; j_1=0\ or\ 2, j_2=2; \\ & 1\le k_1,k_2\le m_1; 1\le l_1,l_2\le n \\
D_0-\lambda I_n & h_1=h_2=0; 1\le l_1,l_2\le n \\
\theta T\oplus D_0-(\lambda+\eta+\delta)I_{m_1 n} & 1\le h_1\le L-1, h_1=h_2; i_1=i_2=0; \\ & j_1=j_2=0; 1\le k_1,k_2\le m_1; 1\le l_1,l_2\le n \\
T\oplus D_0-(\lambda+\gamma+\delta)I_{m_1 n} & 1\le h_1\le L-1, h_1=h_2; i_1=i_2=0; j_1=j_2=2; \\ & 1\le k_1,k_2\le m_1; 1\le l_1,l_2\le n \\
\phi T\oplus D_0-\lambda I_{m_1 n} & 1\le h_1\le L-1, h_1=h_2; i_1=i_2=1; j_1=j_2=2; \\ & 1\le k_1,k_2\le m_1; 1\le l_1,l_2\le n
\end{cases}$$

$$C_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}=\begin{cases}
\beta\otimes D_1 & h_1=h_2=0; i_2=0; j_2=3; 1\le k_2\le m_2; 1\le l_1,l_2\le n \\
I_{m_1}\otimes D_1 & 1\le h_1\le L, h_1=h_2; i_1=i_2=0; j_1=j_2=0; 1\le k_1,k_2,\le m_1; \\ & 1\le l_1,l_2\le n \\
I_{m_1}\otimes D_1 & 1\le h_1\le L, h_1=h_2; i_1=i_2=0; j_1=j_2=2; 1\le k_1,k_2\le m_1; \\ & 1\le l_1,l_2\le n \\
I_{m_1}\otimes D_1 & 1\le h_1\le L, h_1=h_2; i_1=i_2=1; j_1=j_2=2; 1\le k_1,k_2\le m_1; \\ & 1\le l_1,l_2\le n
\end{cases}$$

$$B_{1_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}=\begin{cases}
\theta' S^0\otimes I_n & h_1=h_2=0; i_1=0; j_1=1; 1\le k_1\le m_2, 1\le l_1,l_2\le n \\
S^0\otimes I_n & h_1=h_2=0; i_1=0; j_1=3; 1\le k_1\le m_2, 1\le l_1,l_2\le n \\
\theta' S^0\alpha\otimes I_n & h_1=h_2, 1\le h_1\le L; i_1=i_2=0; j_1=1, j_2=2;; 1\le k_1\le m_2, \\ & 1\le k_2\le m_1; 1\le l_1,l_2\le n \\
S^0\alpha\otimes I_n & h_1=h_2, 1\le h_1\le L; i_1=i_2=0; j_1=3, j_2=2;; 1\le k_1\le m_2, \\ & 1\le k_2\le m_1; 1\le l_1,l_2\le n
\end{cases}$$

$$A_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)} = \begin{cases} I_{m_2} \otimes D_1 & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ I_{m_2} \otimes D_1 & i_1 = i_2 = 0; j_1 = j_2 = 3; 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ I_{m_1} \otimes D_1 & 1 \le h_1 \le L; h_1 = h_2; i_1 = i_2 = 0; j_1 = j_2 = 0 \text{ or } 2; 1 \le k_1, k_2 \le m_1; \\ & 1 \le l_1, l_2 \le n \\ I_{m_1} \otimes D_1 & 1 \le h_1 \le L; h_1 = h_2; i_1 = i_2 = 1; j_1 = j_2 = 2; 1 \le k_1, k_2 \le m_1; \\ & 1 \le l_1, l_2 \le n \\ I_{m_2} \otimes D_1 & 1 \le h_1 \le L; h_1 = h_2; i_1 = i_2 = 0; j_1 = j_2 = 1 \text{ or } 3; 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \end{cases}$$

$$A_{2_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)} = \begin{cases} \theta' S^0 \beta \otimes I_n & h_1 = h_2 = 0; i_1 = i_2 = 0; j_1 = 1, j_2 = 3; 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ S^0 \beta \otimes I_n & h_1 = h_2 = 0; i_1 = i_2 = 0; j_1 = j_2 = 3; 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ \theta' S^0 \alpha \otimes I_n & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 1, j_2 = 2; 1 \le k_1 \le m_2, \\ & 1 \le k_2 \le m_1; 1 \le l_1, l_2 \le n \\ S^0 \alpha \otimes I_n & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 3, j_2 = 2; 1 \le k_1 \le m_2, \\ & 1 \le k_2 \le m_1, 1 \le l_1, l_2 \le n \end{cases}$$

$$A_{1_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)} =$$

$$\begin{cases} \lambda I_{m_1 n} & 1 \le h_1 \le L-1, h_2 = h_1 + 1; i_1 = i_2 = 0; j_1 = j_2 = 0 \text{ or } 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \lambda I_{m_2 n} & 0 \le h_1 \le L-1, h_2 = h_1 + 1; i_1 = i_2 = 0; j_1 = j_2 = 1 \text{ or } 3; \\ & 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \lambda I_{m_1 n} & 1 \le h_1 \le L-1, h_2 = h_1 + 1; i_1 = i_2 = 1; j_1 = j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \theta T^0 \beta \otimes I_n & h_1 = 1, h_2 = 0; i_1 = i_2 = 0; j_1 = 0, j_2 = 3; 1 \le k_1 \le m_1, \\ & 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ T^0 \beta \otimes I_n & h_1 = 1, h_2 = 0; i_1 = i_2 = 0; j_1 = 2, j_2 = 3; 1 \le k_1 \le m_1, \\ & 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \phi T^0 \beta \otimes I_n & h_1 = 1, h_2 = 0; i_1 = 1, i_2 = 0; j_1 = 2, j_2 = 3; 1 \le k_1 \le m_1, \\ & 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \theta T^0 \alpha \otimes I_n & 2 \le h_1 \le L, h_2 = h_1 - 1; i_1 = i_2 = 0; j_1 = 0, j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ T^0 \alpha \otimes I_n & 2 \le h_1 \le L, h_2 = h_1 - 1; i_1 = i_2 = 0; j_1 = j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \phi T^0 \alpha \otimes I_n & 2 \le h_1 \le L, h_2 = h_1 - 1; i_1 = i_2 = 1; j_1 = j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \eta I_{m_1 n} & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 0, j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \eta I_{m_2 n} & 0 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 1, j_2 = 3; \\ & 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \gamma I_{m_1 n} & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 2, j_2 = 0; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \gamma I_{m_2 n} & 0 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 3, j_2 = 1; \\ & 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \delta I_{m_1 n} & 1 \le h_1 \le L, h_1 = h_2; i_1 = 0, i_2 = 1; j_1 = 0 \text{ or } 2, j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \theta' S \oplus D_0 - (\lambda + \eta) I_{m_2 n} & h_1 = h_2 = 0; i_1 = i_2 = 0; j_1 = j_2 = 1, 1 \le k_1, k_2 \le m_2, 1 \le l_1, l_2 \le n \\ S \oplus D_0 - (\lambda + \gamma) I_{m_2 n} & h_1 = h_2 = 0; i_1 = i_2 = 0; j_1 = j_2 = 3, 1 \le k_1, k_2 \le m_2, 1 \le l_1, l_2 \le n \\ \theta T \oplus D_0 - (\lambda + \eta + \delta) I_{m_1 n} & 1 \le h_1 \le L-1, h_1 = h_2; i_1 = i_2 = 0, j_1 = j_2 = 0, 1 \le k_1, k_2 \le m_1, \end{cases}$$

3 Steady State Analysis

Let $\pi = (\pi_{\mathbf{0}}, \pi_{\mathbf{1}}, \dots, \pi_{\mathbf{L}})$ denote the steady state probability vector of the generator

$$A = A_0 + A_1 + A_2 = \begin{bmatrix} F_0 & F_1 & & & & \\ F_2 & F_3 & \lambda I & & & \\ & F_4 & F_3 & \lambda I & & \\ & & \ddots & \ddots & \ddots & \\ & & F_4 & F_3 & \lambda I & \\ & & & F_4 & F_5 & \end{bmatrix}. \text{ ie,}$$

$$\pi A = 0, \pi e = 1. \tag{1}$$

In the above,

$$F_0(k,l) = \begin{cases} \theta' S \oplus D_0 - (\lambda+\eta) I_{m_2 n} + I_{m_2} \otimes D_1 & k=1, l=1 \\ \eta I_{m_2 n} + \theta' S^0 \beta \otimes I_n & k=1, l=2 \\ \gamma I m_2 n & k=2, l=1 \\ S \oplus D_0 - (\lambda+\gamma) I_{m_2 n} + S^0 \beta \otimes I_n + I_{m_2} \otimes D_1 & k=2, l=2 \end{cases}, F_1(k,l) = \begin{cases} \lambda I_{m_2 n} & k=1, l=2 \\ \lambda I_{m_2 n} & k=2, l=4 \\ 0 & otherwise, \end{cases}$$

$$F_2(k,l) = \begin{cases} \theta T^0 \beta \otimes I_n & k=1, l=2 \\ T^0 \beta \otimes I_n & k=3, l=2 \\ \phi T^0 \beta \otimes I_n & k=5, l=2 \\ 0 & otherwise, \end{cases}, F_3(k,l) = \begin{cases} \theta T \oplus D_0 - (\lambda+\eta+\delta) I_{m_1 n} + I_{m_1} \otimes D_1 & k=1, l=1 \\ \eta I_{m_1 n} & k=1, l=3 \\ \delta I_{m_1 n} & k=1, l=5 \\ \theta' S \oplus D_0 - (\lambda+\eta) I_{m_2 n} + I_{m_2} \otimes D_1 & k=2, l=2 \\ \theta' S^0 \alpha \otimes I_n & k=2, l=3 \\ \eta I_{m_2 n} & k=2, l=4 \\ \gamma I_{m_1 n} & k=3, l=1 \\ T \oplus D_0 - (\lambda+\gamma+\delta) I_{m_1 n} + I_{m_1} \otimes D_1 & k=3, l=3 \\ \delta I_{m_1 n} & k=3, l=5 \\ \gamma I_{m_2 n} & k=4, l=2 \\ S^0 \alpha \otimes I_n & k=4, l=3 \\ S \oplus D_0 - (\lambda+\gamma) I_{m_2 n} + I_{m_2} \otimes D_1 & k=4, l=4 \\ \phi T \oplus D_0 - \lambda I_{m_1 n} + I_{m_1} \otimes D_1 & k=5, l=5 \\ 0 & otherwise \end{cases}$$

$$F_4(k,l) = \begin{cases} \theta T^0 \alpha \otimes I_n & k=1, l=3 \\ T^0 \alpha \otimes I_n & k=3, l=3 \\ \phi T^0 \alpha \otimes I_n & k=5, l=3 \\ 0 & otherwise \end{cases}, F_5(k,l) =$$

$$\begin{cases} \theta T \oplus D_0 - (\eta+\delta) I_{m_1 n} + I_{m_1} \otimes D_1 & k=1, l=1 \\ \eta I_{m_1 n} & k=1, l=3 \\ \delta I_{m_1 n} & k=1, l=5 \\ \theta' S \oplus D_0 - \eta I_{m_2 n} + I_{m_2} \otimes D_1 & k=2, l=2 \\ \theta' S^0 \alpha \otimes I_n & k=2, l=3 \\ \eta I_{m_2 n} & k=2, l=4 \\ \gamma I_{m_1 n} & k=3, l=1 \\ T \oplus D_0 - (\gamma+\delta) I_{m_1 n} + I_{m_1} \otimes D_1 & k=3, l=3 \\ \delta I_{m_1 n} & k=3, l=5 \\ \gamma I_{m_2 n} & k=4, l=2 \\ S^0 \alpha \otimes I_n & k=4, l=3 \\ S \oplus D_0 - \gamma I_{m_2 n} + I_{m_2} \otimes D_1 & k=4, l=4 \\ \phi T \oplus D_0 + I_{m_1} \otimes D_1 & k=5, l=5 \\ 0 & otherwise \end{cases}$$

with dimensions of F_0, F_1, F_2 be $2m_2n \times 2m_2n$, $2m_2n \times (3m_1 + 2m_2)n$,$(3m_1 + 2m_2)n \times 2m_2n$ respectively. F_3, F_4 *and* F_5 are square matrices of order $(3m_1 + 2m_2)n$.

The $LIQBD$ description of the model indicates that the queueing system is stable (see Neuts [9]) if and only if the left drift exceeds that of right drift. That is,

$$\pi A_0 e < \pi A_2 e. \tag{2}$$

The vector π cannot be obtained directly in terms of the parametres of the model. The inequality (2) is simplified in(5) below. From (1)we get

$$\pi_i = \pi_{i-1} \mathcal{U}_{i-1}, 1 \le i \le L \tag{3}$$

where

$$\mathcal{U}_0 = -F_1 (F_3 + \mathcal{U}_1 F_4)^{-1}$$

$$\mathcal{U}_i = \begin{cases} -\lambda (F_3 + U_{i+1} F_4)^{-1} & for\ 1 \le i \le L-2 \\ -\lambda F_5^{-1} & for\ i = L-1. \end{cases}$$

From the normalizing condition $\pi \mathbf{e} = 1$ we have

$$\pi_{\mathbf{0}} \left(\sum_{j=0}^{L-1} \prod_{i=0}^{j} \mathcal{U}_i + I \right) e = 1. \tag{4}$$

The inequality (2) gives the stability condition as

$$\pi_{\mathbf{0}} \left[(I_{(2m_2)} \otimes D_1) e + \sum_{i=0}^{L-1} \prod_{j=0}^{i} \mathcal{U}_j (I_{3m_1+2m_2} \otimes D_1) e \right] <$$
$$\pi_{\mathbf{0}} \left[[e_1(2)(\theta' S^0 \beta \otimes I) + e_2(2) S^0 \beta \otimes I] e(m_2 n) + \sum_{i=0}^{L-1} \prod_{j=0}^{i} \mathcal{U}_j [e_2(5) \theta' S^0 \alpha \otimes I) + e_4(5)(S^0 \alpha \otimes I)] e(m_2 n) \right]. \tag{5}$$

Let $\mathbf{x}$ be the steady state probability vector of Q.We partition this vector as $\mathbf{x} = (\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2} \ldots)$, where $\mathbf{x_0}$ is of dimension $n(1 + 3m_1 L)$ and $\mathbf{x_1}, \mathbf{x_2}, \ldots$ are each of dimension $n(2m_2 + (3m_1 + 2m_2)L)$. Under the stability condition, we have $\mathbf{x_i} = \mathbf{x}_1 R^{i-1}, i \ge 2$, where the matrix R is the minimal nonnegative solution to the matrix quadratic equation

$$R^2 A_2 + R A_1 + A_0 = 0$$

and the vectors $\mathbf{x_0}$ and $\mathbf{x_1}$are obtained by solving the equations

$$\mathbf{x_0} B_0 + \mathbf{x_1} B_1 = 0 \tag{6}$$

$$\mathbf{x_0} C_0 + \mathbf{x_1} (A_1 + R A_2) = 0 \tag{7}$$

subject to the normalizing condition

$$\mathbf{x_0} \mathbf{e} + \mathbf{x_1} (I - R)^{-1} \mathbf{e} = 1 \tag{8}$$

3.1 Analysis of service time of a type I customer

The duration of service of a type I customer is a phase type distribution with representation (α', S_1) where the underlying MC has state space $\{(i,j,k): i = 0, j = 0\ or\ 2, 1 \le k \le m_1\} \cup \{(i,2,k): i = 1, 1 \le k \le m_1)\} \cup \{*\}$ where i denotes the status of the protection clock, j, the status of the server, k, the service phase and *, the absorbing state indicating service completion. The infinitesimal generator is

$$\mathcal{S}_1 = \begin{bmatrix} S_1 & S_1^0 \\ \mathbf{0} & 0 \end{bmatrix}, where, S_1 = \begin{bmatrix} \theta T - (\eta+\delta)I_{m_1} & \eta I_{m_1} & \delta I_{m_1} \\ \gamma I_{m_1} & T - (\gamma+\delta)I_{m_1} & \delta I_{m_1} \\ \mathbf{0} & \mathbf{0} & \phi T \end{bmatrix}\ and\ S_1^0 = \begin{bmatrix} \theta T^0 \\ T^0 \\ \phi T^0 \end{bmatrix}$$

The initial probability vector is $\alpha' = [\mathbf{0} \quad \alpha \quad \mathbf{0}]$, , where $\mathbf{0}$ is a zero matrix of order $1 \times m_1$.

Thus the service time distribution of a type I customer is $Ph(\alpha', S_1)$ of order $3m_1n$.

3.2 Analysis of service time of a type II customer

The duration of service of a type II customer turn out to be a phase type distribution (β', S_2) where the underlying MC has state space $\{(i,j): i = 1\ or\ 3, 1 \le j \le m_2\} \cup \{*\}$ where i denotes the status of the server, j, the service phase and *, the absorbing state indicating service completion. The infinitesimal generator is

$$\mathcal{S}_2 = \begin{bmatrix} S_2 & S_2^0 \\ \mathbf{0} & 0 \end{bmatrix}, where, S_2 = \begin{bmatrix} \theta' S - \eta I_{m_2} & \eta I_{m_2} \\ \gamma I_{m_2} & S - \gamma I_{m_2} \end{bmatrix}\ and\ S_2^0 = \begin{bmatrix} \theta' S^0 \\ S^0 \end{bmatrix}$$

The initial probability vector is $\beta' = [\mathbf{0} \quad \alpha]$, where $\mathbf{0}$ is a zero matrix of order $1 \times m_2$. Thus we have the service time distribution of a type II customer is $Ph(\beta', S_2)$ of order $2m_2n$.

4 Waiting time analysis

4.1 Type I Customer

To find the waiting time of a type I customer who joins for service at time t, we have to consider different possibilities depending on the status of server at that time. Let $W(t)$ be the waiting time of a type I customer who arrives at time t and $W^*(s)$ be the corresponding LST.

Case I

Suppose that E_1 denote the event the system is in the state $(0, v), 1 \le v \le n$ when the tagged type I customer arrives. Let $W^*(s/E_1)$ denote the corresponding LST.Then

$$W^*(s/E_1) = 1$$

Case II

E_2 be the event that the system is in the state $(n_1, a, 0, 0, u, v), n_1 \ge 0, 1 \le a \le L-1, 1 \le u \le m_1, 1 \le v \le n$, when the tagged customer arrives the system. In this case the waiting time is the sum of the residual service time of the type I customer in service when the tagged customer arrives and service time of $a-1$ remaining type I customers. Let $W^*(s/E_2)$ represent the corresponding conditonal LST. Then

$$W^*(s/E_2) = (e_u(3m_1)(sI - S_1)^{-1}S_1^0)(\alpha'(sI - S_1)^{-1}S_1^0)^{a-1}.$$

Case III

E_3 denotes the event: the system is in the state $(n_1, a, 0, 2, u, v), n_1 \ge 0, 1 \le a \le L-1, 1 \le u \le m_1, 1 \le v \le n$, when the tagged customer arrives the system. In this case the waiting time is the sum of the residual service time of the type I customer in service when the tagged customer arrives and service times of $a-1$ remaining type I customers. With $W^*(s/E_3)$ as the corresponding conditonal LST, we have

$$W^*(s/E_3) = (e_{m_1+u}(3m_1)(sI - S_1)^{-1}S_1^0)(\alpha'(sI - S_1)^{-1}S_1^0)^{a-1}.$$

Case IV

E_4 denotes the event: the system is in the state $(n_1, a, 1,2, u, v), n_1 \geq 0, 1 \leq a \leq L-1, 1 \leq u \leq m_1, 1 \leq v \leq n,$ when the tagged customer arrives the system. In this case the waiting time is the sum of the residual service time of the type I customer in service when the tagged customer arrives and service times of $a-1$ remaining type I customers. Let $W^*(s/E_4)$ represent the corresponding conditonal LST. Then

$$W^*(s/E_4) = (e_{2m_1+u}(3m_1)(sI - S_1)^{-1}S_1^0)(\alpha'(sI - S_1)^{-1}S_1^0)^{a-1}.$$

Case V

E_5 denotes the event: the system is in the state $(n_1, a, 0,1, u, v), n_1 \geq 1, 0 \leq a \leq L-1, 1 \leq u \leq m_2, 1 \leq v \leq n,$ when the tagged customer arrives the system. In this case the waiting time is the sum of the residual service time of the type II customer in service when the tagged customer arrives and service times of a remaining type I customers. Let $W^*(s/E_5)$ represent the corresponding conditonal LST. Then

$$W^*(s/E_5) = (e_u(2m_2)(sI - S_2)^{-1}S_2^0)(\alpha'(sI - S_1)^{-1}S_1^0)^{a}.$$

Case VI

E_6 denotes the event: the system is in the state $(n_1, a, 0,3, u, v), n_1 \geq 1, 0 \leq a \leq L-1, 1 \leq u \leq m_2, 1 \leq v \leq n,$ when the tagged customer arrives the system. In this case the waiting time is the sum of the residual service time of the type II customer in service when the tagged customer arrives and service times of a remaining type I customers. Let $W^*(s/E_6)$ represent the corresponding conditonal LST. Then

$$W^*(s/E_6) = (e_{m_2+u}(2m_2)(sI - S_2)^{-1}S_2^0)(\alpha'(sI - S_1)^{-1}S_1^0)^{a}.$$

Thus the LST of the waiting time

$$W^*(s) = \frac{1}{d}\Big[\sum_{v=1}^{n} x_{0,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/E_2)x_{n_1,a,0,0,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/E_3) x_{n_1,a,0,2,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/E_4)x_{n_1,a,1,2,u,v} + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m_2}\sum_{v=1}^{n} W^*(s/E_5)x_{n_1,a,0,1,u,v} + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m_2}\sum_{v=1}^{n} W^*(s/E_6)x_{n_1,a,0,3,u,v}\Big] \quad (9)$$

$$where, d = \sum_{v=1}^{n} x_{0,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} x_{n_1,a,0,0,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} x_{n_1,a,0,2,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} x_{n_1,a,1,2,u,v} + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m_2}\sum_{v=1}^{n} x_{n_1,a,0,1,u,v} + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m_2}\sum_{v=1}^{n} x_{n_1,a,0,3,u,v}$$

4.2 Type II customer

To find the LST of the waiting time distribution of a type II customer, we have to compute certain distributions. We proceed to such computations.

Definition 1 *Consider the duration of time with p type I customers in the system at a service commencement epoch of type I customers until the number of type I customers become zero for the first time, we call this a p-cycle, denoted by* B_p.

4.2.1 Distribution of a p-cycle

This is a phase type distribution with representation (γ_p, T_1) where the underlying Markov chain has state space $\{(i, j, k, l): 1 \leq i \leq L, j = 0, k = 0 \text{ or } 2, 1 \leq l \leq m_1\} \cup \{(i, j, k, l): 1 \leq i \leq L, j = 1, k = 2, 1 \leq l \leq m_1\} \cup \{*\}$ and i, j, k, l and * respectively denote the number of type I customers in the system, the status of the protection clock, the status of the server, the service phase and the absorbing state indicating that the number of type I customers become zero. The infinitesimal generator T_1 of $B_p(t)$ has the form

$$T_1 = \begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_1^0 \\ 0 & 0 \end{bmatrix}, where \mathcal{T}_1 = \begin{bmatrix} E_1 & \lambda I_{m_1} & & & \\ E_2 & E_1 & \lambda I_{m_1} & & \\ & \ddots & \ddots & \ddots & \\ & & E_2 & E_1 & \lambda I_{m_1} \\ & & & E_2 & E_3 \end{bmatrix},$$

$$\mathcal{T}_1^0 = \begin{bmatrix} E^0 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

where

$$E_1 = \begin{bmatrix} \theta T - (\lambda + \eta + \delta) I_{m_1} & \eta I_{m_1} & \delta I_{m_1} \\ \gamma I & T - (\lambda + \gamma + \delta) I_{m_1} & \delta I_{m_1} \\ 0 & 0 & \phi T - \lambda I_{m_1} \end{bmatrix} \begin{bmatrix} 0 & \theta T^0 \alpha & 0 \\ 0 & T^0 \alpha & 0 \\ 0 & \phi T^0 \alpha & 0 \end{bmatrix}.$$

$$E_3 = \begin{bmatrix} \theta T - (\eta + \delta) I_{m_1} & \eta I_{m_1} & \delta I_{m_1} \\ \gamma I_{m_1} & T - (\gamma + \delta) I_{m_1} & \delta I_{m_1} \\ 0 & 0 & \phi T \end{bmatrix}$$

$$and\ E^0 = \begin{bmatrix} \theta T^0 \\ T^0 \\ \phi T^0 \end{bmatrix}.$$

The initial probabilty vector is

$$\gamma_p = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \gamma' & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, 1 \leq p \leq L$$

where $\mathbf{0}$ s a zero matrix of order $1 \times 3m_1$, with $\gamma' = \begin{bmatrix} \mathbf{0} & \alpha & \mathbf{0} \end{bmatrix}, 1 \leq p \leq L$ is in the pth position and $\mathbf{0}$ is a zero matrix of order $1 \times m_1$.

We can compute the LST of the length of the busy period as $\gamma_p (sI - \mathcal{T}_1)^{-1} \mathcal{T}_1^0$

4.2.2 LST of the busy cycle generated by type I customers arriving during the service time of a type II customer

Theorem 1

The LST of the busy cycle generated by type I customers arriving during the service time of a type II customer is given by

$$\hat{B}_{c_L}(s) = \beta'[(s+\lambda)I - S_2]^{-1} S_2^0 + \sum_{p=1}^{L-1} \gamma_p (sI - \mathcal{T}_1)^{-1} \mathcal{T}_1^0 \lambda^p \beta' [(s+\lambda)I - S_2]^{-(p+1)} S_2^0 + \gamma_L (sI - \mathcal{T}_1)^{-1} \mathcal{T}_1^0$$
$$\beta'[\lambda^{-1}((s+\lambda)I - S_2)]^{-L} [I - \lambda[(s+\lambda)I - S_2]^{-1}]^{-1} [(s+\lambda)I - S_2]^{-1} S_2^0 \quad (10)$$

Proof.

The proof is given in the appendix.

4.2.3 LST of the busy period of type I customers generated during the service time of a type II customer

Theorem 2

The LST of the busy period generated by type I customers arriving during the service time of a type II customer is given by

$$\hat{B}_L(s) = \beta'[\lambda I - S_2]^{-1}S_2^0 + \sum_{p=1}^{L-1} \gamma_p(sI - T_1)^{-1}T_1^0\lambda^p\beta'[\lambda I - S_2]^{-(p+1)}S_2^0 + \gamma_L(sI - T_1)^{-1}T_1^0$$

$$\beta'[\lambda^{-1}(\lambda I - S_2)]^{-L}[I - \lambda[\lambda I - S_2]^{-1}]^{-1}[\lambda I - S_2]^{-1}S_2^0 \quad (11)$$

Proof.

The proof is given in the appendix.

Now, to find the waiting time of a type II customer who joins for service at time t, we have to consider different possibilities depending on the status of server at that time. Let $W(t)$ be the waiting time of a type II customer who arrives at time t and $W^*(s)$ be the corresponding LST.

Case I

Suppose that F_1 denotes the event the system is in the state $(0, v), 1 \le v \le n$ when the tagged customer arrives. Let $W^*(s/F_1)$ denote the corresponding LST.Then

$$W^*(s/F_1) = 1$$

Case II

F_2 be the event that the system is in one of the states $(b, a, 0,0, u, v), b \ge 0, 1 \le a \le L, 1 \le u \le m_1, 1 \le v \le n$ when the tagged customer arrives. In this case , the waiting time is the length of the busy cycle generated by a type I customers starting from his arrival epoch plus lengths of busy cycles of type I customers generated during service times of each of the b type II customers. Let $W^*(s/F_2)$ denote the corresponding LST.Then

$$W^*(s/F_2) = e_{(a-1)3m_1+u}(3Lm_1)(sI - T_1)^{-1}T_1^0(\hat{B}_{c_L}(s))^b$$

Case III

F_3 denotes the event the system is in one of the states $(b, a, 0,2, u, v), b \ge 0, 1 \le a \le L, 1 \le u \le m_1, 1 \le v \le n$ when the tagged customer arrives. In this case, the waiting time is the length of the busy cycle generated by a type I customers starting from his arrival epoch plus lengths of busy cycles of type I customers generated during service times of each of the b type II customers. Let $W^*(s/F_3)$ denote the corresponding LST.Then

$$W^*(s/F_3) = e_{(a-1)3m_1+m_1+u}(3Lm_1)(sI - T_1)^{-1}T_1^0(\hat{B}_{c_L}(s))^b$$

Case IV

F_4 denotes the event the system is in one of the states $(b, a, 1,2, u, v), b \ge 0, 1 \le a \le L, 1 \le u \le m_1, 1 \le v \le n$ when the tagged customer arrives. In this case, the waiting time is the length of the busy cycle generated by a type I customers starting from his arrival epoch plus lengths of busy cycles of type I customers generated during service times of each of the b type II customers. Let $W^*(s/F_4)$ denote the corresponding LST.Then

$$W^*(s/F_4) = e_{(a-1)3m_1+2m_1+u}(3Lm_1)(sI - T_1)^{-1}T_1^0(\hat{B}_{c_L}(s))^b$$

Case V

F_5 denotes the event the system is in one of the states $(b, a, 0,1, u, v), b \ge 1, 0 \le a \le L, 1 \le u \le m_2, 1 \le v \le n$ when the tagged customer arrives. In this case, the waiting time is the length of residual service time of the type II customer in service plus length of the busy period generated by type I customers arriving during the service time of the type II customer in service plus lengths of busy cycles of type I customers generated during service time of each of the $b-1$ type II customers. Let $W^*(s/F_5)$ denote the corresponding LST.Then

$$W^*(s/F_5) = e_u(2m_2)(sI - S_2)^{-1}S_2^0\hat{B}_L(s)(\hat{B}_{c_L}(s))^{b-1}$$

Case VI

F_6 denotes the event the system is in one of the states $(b, a, 0,3, u, v)$ when the tagged customer arrives. In this case the waiting time is the length of residual service time of the type II customer in service plus the length of the busy period generated by type I customers arriving during the service time of the type II customer in service plus lengths of busy cycles of type I customers generated during service time of each of the $b-1$ type II customers. Let $W^*(s/F_6)$ denote the corresponding LST.Then

$$W^*(s/F_6) = e_{m_2+u}(2m_2)(sI - S_2)^{-1}S_2^0\hat{B}_L(s)(\hat{B}_{c_L}(s))^{b-1}$$

Thus the LST of the waiting time

$$W^*(s) = \sum_{v=1}^{n} x_{0,v} + \sum_{b=0}^{\infty}\sum_{a=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/F_2)x_{b,a,0,0,u,v} + \sum_{b=0}^{\infty}\sum_{a=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/F_3) x_{b,a,0,2,u,v} + \sum_{b=0}^{\infty}\sum_{a=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/F_4)x_{b,a,1,2,u,v} + \sum_{b=1}^{\infty}\sum_{a=0}^{L}\sum_{u=1}^{m_2}\sum_{v=1}^{n} W^*(s/F_5)x_{b,a,0,1,u,v} + \sum_{b=1}^{\infty}\sum_{a=0}^{L}\sum_{u=1}^{m_2}\sum_{v=1}^{n} W^*(s/F_6)x_{b,a,0,3,u,v} \quad (12)$$

5 Expected number of interruptions during a single type I service

5.1 Distribution of duration of time till interruptions occur during a single type I service

Consider the Markov process, $\chi_1 = (N(t), J(t), K(t))$, where $N(t)$ denotes the number of interruptions upto time t, $J(t)$, status of the server (providing normal or interrupted service) and $K(t)$, the service phase at time t. The state space of the process is given by $\{(0,2,k)/1 \le k \le m_1\} \cup \{(i,j,k)/i \ge 1, j = 0 \text{ or } 2, 1 \le k \le m_1\} \cup \{*_1\} \cup \{*_2\}$ where $*_1$ denotes the absorbing state indicating the service completion and $*_2$ denotes the absorbing state indicating the realization of protection. The infinitesimal generator of the process is given by

$$\mathcal{U} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \delta e(m_1) & T^0 & T-(\gamma+\delta)I_{m_1} & \gamma I_{m_1} & 0 & 0 & 0 & \cdots \\ \delta e(m_1) & \theta T^0 & 0 & \theta T-(\eta+\delta)I_{m_1} & \eta I_{m_1} & 0 & 0 & \cdots \\ \delta e(m_1) & T^0 & 0 & 0 & T-(\gamma+\delta)I_{m_1} & \gamma I_{m_1} & 0 & \cdots \\ \delta e(m_1) & \theta T^0 & 0 & 0 & 0 & \theta T-(\eta+\delta)I_{m_1} & \eta I_{m_1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

5.2 Distribution of number of interruptions during a single type I service

Let y_k be the probabaility that the number of interruptions during a single type I service is k. Then y_k is the probabilty that the absorption occurs from the level k for the process χ_1. Hence y_k are given by

$$y_0 = -\alpha(T-(\gamma+\delta)I))^{-1}(T^0+\delta e)$$

and for $k = 1,2,3, \ldots$

$$y_k = \alpha(T-(\gamma+\delta)I)^{-1}\gamma I((\theta T-(\eta+\delta)I)^{-1}\eta I(T-(\gamma+\delta)I)^{-1}\gamma I)^{k-1}(\theta T-(\eta+\delta)I)^{-1} ((\theta T^0+\delta e) - \eta I(T-(\gamma+\delta)I)^{-1}(T^0+\delta e)) \quad (13)$$

Therefore, the expected number of interruptions during any particular type I customer service,

$$E(i) = \sum_{k=0}^{\infty} k y_k = \alpha(T-(\gamma+\delta)I)^{-1}\gamma I((I-(\theta T-(\eta+\delta)I)^{-1}\eta I(T-(\gamma+\delta)I)^{-1}\gamma I))^{-2} (\theta T-(\eta+\delta)I)^{-1}((\theta T^0+\delta e) - \eta I(T-(\gamma+\delta)I)^{-1}(T^0+\delta e)). \quad (14)$$

6 Expected number of interruptions during a single type II service

6.1 Distribution of duration of time till interruptions occur during a single type II service

Consider the Markov process, $\chi_2 = (N(t), J(t), K(t))$, where $N(t)$ denotes the number of interruptions, $J(t)$, status of the server (providing normal or interrupted service) and $K(t)$, the service phase at time t. The state space of the process of the process is given by $\{(0,3,k)/1 \leq k \leq m_2\} \cup \{(i,j,k)/i \geq 1, j = 1\ or\ 3, 1 \leq k \leq m_2\} \cup \{*\}$ where $*$ denotes the absorbing state indicating the service completion. The infinitesimal generator of the process is given by

$$\mathcal{U} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ S^0 & S-\gamma I_{m_2} & \gamma I_{m_2} & 0 & 0 & 0 & \cdots \\ \theta' S^0 & 0 & \theta' S - \eta I_{m_2} & \eta I_{m_2} & 0 & 0 & \cdots \\ S^0 & 0 & 0 & S-\gamma I_{m_2} & \gamma I_{m_2} & 0 & \cdots \\ \theta' S^0 & 0 & 0 & 0 & \theta' S - \eta I_{m_2} & \eta I_{m_2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

6.2 Distribution of number of interruptions during a single type II service

Let z_k be the probabaility that the number of interruptions during a single type II service is k. Then z_k is the probabilty that the absorption occurs from the level k for the process χ_2. Hence z_k are given by

$$z_0 = -\alpha(S-\gamma I))^{-1}S^0$$

and for $k = 1,2,3,\ldots$

$$z_k = \alpha(S-\gamma I)^{-1}\gamma I((\theta' S - \eta I)^{-1}\eta I(S-\gamma I)^{-1}\gamma I)^{k-1}(\theta' S - \eta I)^{-1}(\theta' S^0 - \eta I((S-\gamma I)^{-1}S^0) \quad (15)$$

Therefore, the expected number of interruptions during any particular type II customer service,

$$E(i) = \sum_{k=0}^{\infty} k z_k = \alpha(S-\gamma I)^{-1}\gamma I(I - (\theta' S - \eta I)^{-1}\eta I(S-\gamma I)^{-1}\gamma I)^{-2} (\theta' S - \eta I)^{-1}(\theta' S^0 - \eta I(S-\gamma I)^{-1}S^0). \quad (16)$$

7 Other Performance Measures

- The probability that the server is idle:

$$p_{idle} = \sum_{v=1}^{n} x_{0,v}.$$

- Mean number of type I customers in the system:

$$E_{nsh} = \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} n_2 x_{n_1,n_2,0,0,u,v} + \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_2}\sum_{v=1}^{n} n_2 x_{n_1,n_2,0,1,u,v} + \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} n_2 x_{n_1,n_2,0,2,u,v} + \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_2}\sum_{v=1}^{n} n_2 x_{n_1,n_2,0,3,u,v} + \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} n_2 x_{n_1,n_2,1,2,u,v}$$

- Mean number of type II customers in the system:

$$E_{nsl} = \sum_{n_1=0}^{\infty} n_1 x_{n_1}\mathbf{e}$$

- The fraction of time during which the system is protected:

$$T_p = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{L} \sum_{u=1}^{m_1} \sum_{v=1}^{n} x_{n_1,n_2,1,2,u,v}$$

- The fraction of time the server is providing service to type I customers during WI:

$$T_{ih} = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{L} \sum_{u=1}^{m_1} \sum_{v=1}^{n} x_{n_1,n_2,0,0,u,v}$$

- The fraction of time the server is providing service to type II customers during WI:

$$T_{il} = \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{L} \sum_{u=1}^{m_2} \sum_{v=1}^{n} x_{n_1,n_2,0,1,u,v}$$

- The fraction of time the server is providing service to type I customers in normal mode:

$$T_{nh} = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{L} \sum_{u=1}^{m_1} \sum_{v=1}^{n} x_{n_1,n_2,0,2,u,v}$$

- The fraction of time the server provides service to type II customers in normal mode:

$$T_{nl} = \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{L} \sum_{u=1}^{m_2} \sum_{v=1}^{n} x_{n_1,n_2,0,3,u,v}$$

8 Analysis of a cost function

We construct a cost function based on the above performance measures.

Let

C_h: Holding cost for retaining a type I customer

C_l: Holding cost for retaining a type II customer

C_p: Unit time cost of providing service with protection

C_{ih}: Unit time cost of providing service when the server is providing service to type I customer in WI

C_{il}: Unit time cost of providing service when the server is providing service to type II customer in WI

C_{nh}: Unit time cost of providing service when the server is providing service to type I customer in normal mode

C_{nl}: Unit time cost of providing service when the server is providing service to type II customer in normal mode

Then the expected cost per unit time,

$$C = E_{nsh} \times C_h + E_{nsl} \times C_l + T_p \times \phi C_p + T_{ih} \times \theta C_{ih} + T_{il} \times \theta' C_{il} + T_{nh} \times C_{nh} + T_{nl} \times C_{nl}$$

9 Numerical Results

For the arrival process of type II customers, we consider the following two sets of matrices for D_0 and D_1:

1. MAP with negetive correlation (MNA)

$$D_0 = \begin{bmatrix} -0.8101 & 0.8101 & 0 \\ 0 & -1.3497 & 0 \\ 0 & 0 & -40.5065 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.0810 & 0 & 1.2687 \\ 38.0761 & 0 & 2.4304 \end{bmatrix}$$

2. MAP with positive correlation (MPA)

$$D_0 = \begin{bmatrix} -0.8101 & 0.8101 & 0 \\ 0 & -1.3497 & 0 \\ 0 & 0 & -40.5065 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1.2687 & 0 & 0.0810 \\ 2.4304 & 0 & 38.0761 \end{bmatrix}$$

These two MAP processes are normalized so as to have an arrival rate of 1. The arrival process labeled MNA has correlated arrivals with correlation between two successive interarrival times given by -0.4211 and the arrival process corresponding to the one labelled MPA has a positive correlation with value 0.4211.

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.3493	1.2748	1.2194	1.1774	1.1450	1.1193	1.0985	1.0815	1.0672	1.0552
E_{nsl}	49.9733	19.8907	13.2051	10.3241	8.7368	7.7382	7.0548	6.5593	6.1843	5.8910
T_p	0.0334	0.0324	0.0318	0.0313	0.0308	0.0305	0.0302	0.0300	0.0298	0.0296
T_{ih}	0.1298	0.1104	0.0955	0.0838	0.0746	0.0672	0.0611	0.0559	0.0516	0.0479
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.3924	0.3988	0.4032	0.4063	0.4086	0.4103	0.4116	0.4126	0.4134	0.4141
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	33.7635	31.2805	30.9839	**30.9648**	**31.0063**	**31.0595**	**31.1111**	**31.1575**	**31.1982**	**31.2335**

Table 1: Effect of θ: Fix $L = 3, \theta' = 0,6, \lambda = 2, \eta = 0.5, \delta = 1, \gamma = 0.6$ and $\phi = 4$

Tables 1 to 6 contain the effect of different parameters on various performance measures and on the cost function when the arrival process of type II customer is MNA and tables 7 to 12 contain the effect of different parameters on various performance measures and on the cost function when the arrival process of type II customer is MPA.

Table 1 indicates the effect of the parameter θ on various performance measures and the cost function.As θ increases, type I customers get faster service during WI and hence E_{nsh} decreases. Then more number of type II customers also get service and hence E_{nsl} also decreases. T_p and T_{ih} also decreases since the expected number of type I customers during WI decreases. As θ increases, T_{il} and T_{nl} remains fixed due to the diminished effect of θ on type II customers and T_{nh} increases due to the fact that the system stays in WI serving type I customers for lesser time and hence it stays more in normal mode serving type I customers. As θ increases, the system cost first decreases, reach an optimal value(30.9648) corresponding to $\theta = 0.4$ and then increases.

ϕ	1	1.5	2	2.5	3	3.5	4	4.5	5
E_{nsh}	1.3572	1.1902	1.1112	1.0658	1.0366	1.0162	1.0013	0.9898	0.9808
E_{nsl}	1.1787×10^4	12.1182	7.6530	6.1872	5.4634	5.0334	4.7491	4.5473	4.3968
T_p	0.1581	0.1087	0.0826	0.0665	0.0557	0.0479	0.0420	0.0374	0.0337
T_{ih}	0.0482	0.0497	0.0504	0.0507	0.0509	0.0511	0.0512	0.0513	0.0513
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.3591	0.3702	0.3751	0.3778	0.3795	0.3806	0.3814	0.3820	0.3824
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	1.2112×10^3	34.4147	**34.2737**	**34.2923**	**34.3216**	**34.3469**	**34.3673**	**34.3837**	**34.3969**

Table 2: Effect of ϕ: Fix $L = 3, \theta = 0.7, \theta' = 0,6, \lambda = 2, \eta = 0.5, \delta = 1.5$ and $\gamma = 0.6$

Table 2 indicates the effect of the parameter ϕ on various performance measures and the cost function. As ϕ increases, the type I customers in protected mode get faster service and hence

E_{nsh} decreases. As a result, E_{nsl} also decreases. As expected T_p also decreases. As ϕ increases, T_{ih} and T_{nh} increase since T_p decreases. T_{il} and T_{nl} remains unchanged since ϕ has only a small effect on low priority customers. As ϕ increases, the system cost first decreases, reach an optimal value(34.2737) corresponding to $\phi = 2$ and then increases.

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.3590	1.3225	1.2883	1.2562	1.2260	1.1975	1.1706	1.1452	1.1212	1.0985
E_{nsl}	1071.6	57.1361	29.5220	19.9883	15.1618	12.2491	10.3021	8.9100	7.8661	7.0548
T_p	0.0035	0.0069	0.0102	0.0133	0.0164	0.0193	0.0222	0.0250	0.0276	0.0302
T_{ih}	0.0865	0.0831	0.0798	0.0767	0.0737	0.0709	0.0683	0.0657	0.0633	0.0611
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.4750	0.4674	0.4599	0.4526	0.4454	0.4384	0.4315	0.4247	0.4181	0.4116
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	129.7496	29.2871	27.4764	**27.4443**	**27.8543**	**28.4282**	**29.0719**	**29.7453**	**30.4286**	**31.1111**

Table 3: Effect of δ: Fix $L = 3, \theta = 0.7, \theta' = 0{,}6, \lambda = 2, \eta = 0.5, \gamma = 0.6$ and $\phi = 4$

Table 3 indicates the effect of the parameter δ on various performance measures and the cost function. As δ increases, protection clock realizes quickly and hence T_p increases, so T_{ih} and T_{nh} decreases. But T_{il} and T_{nl} remains unchanged since δ has only a small effect on low priority customers. In this case also, as δ increases, the system cost first decreases, reach an optimal value(27.4443) corresponding to $\delta = 0.4$ and then increases.

η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.1161	1.1112	1.1067	1.1025	1.0985	1.0948	1.0913	1.0880	1.0848	1.0819
E_{nsl}	7.7025	7.5160	7.3475	7.1944	7.0548	6.9270	6.8096	6.7013	6.6012	6.5083
T_p	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0303	0.0303	0.0303
T_{ih}	0.0663	0.0649	0.0636	0.0623	0.0611	0.0599	0.0587	0.0576	0.0566	0.0555
T_{il}	0.0994	0.0958	0.0924	0.0893	0.0863	0.0836	0.0810	0.0785	0.0762	0.0740
T_{nh}	0.4058	0.4074	0.4089	0.4103	0.4116	0.4129	0.4141	0.4153	0.4165	0.4175
T_{nl}	0.3403	0.3425	0.3445	0.3464	0.3482	0.3499	0.3514	0.3529	0.3543	0.3556
C	31.6461	31.5012	31.3642	31.2344	31.1111	30.9939	30.8823	30.7759	30.6743	30.5772

Table 4: Effect of η: Fix $L = 3, \theta = 0.7, \theta' = 0{,}6, \lambda = 2, \eta = 0.5, \gamma = 0.6$ and $\phi = 4$

Table 4 indicates the effect of the parameter η on various performance measures and the cost function. As η increases, the server turns to normal mode quickly. Hence T_{nh} and T_{nl} increase and E_{nsh}, E_{nsl}, T_{ih} and T_{il} decrease. η has only a very small effect on T_p. The cost function decreases as η increases.

θ'	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
E_{nsh}	1.3367	1.1562	1.0535	0.9894	0.9467	0.9166	0.8945	0.8779	0.8649
E_{nsl}	56.6142	10.2087	6.2053	4.7515	4.0095	3.5628	3.2658	3.0547	2.8973
T_p	0.0464	0.0490	0.0504	0.0512	0.0517	0.0521	0.0523	0.0525	0.0527
T_{ih}	0.0369	0.0389	0.0400	0.0406	0.0410	0.0413	0.0415	0.0417	0.0418
T_{il}	0.2089	0.1576	0.1260	0.1049	0.0898	0.0785	0.0697	0.0627	0.0569
T_{nh}	0.3182	0.3357	0.3452	0.3508	0.3544	0.3568	0.3586	0.3599	0.3608
T_{nl}	0.3791	0.3685	0.3622	0.3580	0.3551	0.3529	0.3512	0.3499	0.3488
C	38.4471	**35.5315**	**36.0777**	**36.5074**	**36.8103**	**37.0278**	**37.1887**	**37.3113**	**37.4072**

Table 5: Effect of θ': Fix $L = 3, \theta = 0.7, \lambda = 2, \eta = 0.8, \delta = 2, \gamma = 0.6$ and $\phi = 4$

Table 5 indicates the effect of the parameter θ' on various performance measures and the cost function. As expected, T_{il} decreases and hence E_{nsl} and E_{nsh} decrease, T_{ih} , T_{nh} and T_p increase since type I customers have high priority. As a result, T_{nl} decreases. As θ' increases, the system cost first decreases, reach an optimal value(35.5315) corresponding to $\theta' = 0.2$ and then increases.

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	0.9997	1.0204	1.0407	1.0604	1.0797	1.0985	1.1169	1.1349	1.1525	1.1697
E_{nsl}	4.4562	4.8646	5.3190	5.8279	6.4019	7.0548	7.8046	8.6747	9.6973	10.9167
T_p	0.0301	0.0301	0.0302	0.0302	0.0302	0.0302	0.0302	0.0303	0.0303	0.0303
T_{ih}	0.0113	0.0220	0.0324	0.0423	0.0519	0.0611	0.0699	0.0784	0.0866	0.0945
T_{il}	0.0160	0.0313	0.0459	0.0599	0.0734	0.0863	0.0988	0.1107	0.1222	0.1333
T_{nh}	0.4586	0.4485	0.4378	0.4293	0.4203	0.4116	0.4033	0.3953	0.3876	0.3801
T_{nl}	0.3904	0.3812	0.3725	0.3640	0.3560	0.3482	0.3407	0.3336	0.3267	0.3200
C	27.0694	27.9294	28.7606	29.5661	30.3486	31.1111	31.8570	32.5901	33.3148	34.0369

Table 6: Effect of γ: Fix $L = 3, \theta = 0.7, \lambda = 2, \eta = 0.8, \delta = 2, \gamma = 0.6$ and $\phi = 4$

Table 6 indicates the effect of the parameter γ on various performance measures and the cost function. As γ increases, more interruptions occur during service and hence both E_{nsh} and E_{nsl} increases. T_p also increases in a slow rate. As γ increases T_{ih} and T_{il} increase and T_{nh} and T_{nl} decrease since the system stays more time in interruption mode. As γ increases, the cost function increases. Note the sharpness in decrease of the value of E_{nsl} is quite pronounced. However the trend is not seen in table 4 which gives the effect of η.

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.3471	1.2716	1.2167	1.1761	1.1451	1.1208	1.1013	1.0853	1.0721	1.0609
E_{nsl}	343.0679	141.3074	96.1158	76.4713	65.5556	58.6331	53.8616	50.3784	47.7263	45.6412
T_p	0.0334	0.0324	0.0318	0.0313	0.0308	0.0305	0.0302	0.0300	0.0298	0.0296
T_{ih}	0.1298	0.1104	0.0955	0.0838	0.0746	0.0672	0.0611	0.0559	0.0516	0.0479
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.3924	0.3988	0.4032	0.4063	0.4086	0.4103	0.4116	0.4126	0.4134	0.4141
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	63.0719	43.4206	39.2737	37.5789	36.6882	36.1497	35.7932	35.5413	35.3548	35.2114

Table 7: Effect of θ: Fix $L = 3, \theta' = 0{,}6, \lambda = 2, \eta = 0.5, \delta = 1, \gamma = 0.6$ and $\phi = 4$

Table 7 indicates the effect of the parameter θ on various performance measures and the cost function. In this case also E_{nsh} and E_{nsl} decreases as θ increases. But the values of E_{nsl} is much high when the arrival process of type II customer is MPA. All other values are same as in the case of MNA. But the cost function decreases as θ increases.

ϕ	1	1.5	2	2.5	3	3.5	4	4.5	5
E_{nsh}	1.3571	1.1900	1.1141	1.0711	1.0436	1.0244	1.0104	0.9996	0.9911
E_{nsl}	4.4374×10^4	90.9874	58.6062	47.8674	42.5211	39.3245	37.1995	35.6852	34.5516
T_p	0.1581	0.1087	0.0826	0.0665	0.0557	0.0479	0.0420	0.0374	0.0337
T_{ih}	0.0482	0.0497	0.0504	0.0507	0.0509	0.0511	0.0512	0.0513	0.0513
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.3591	0.3702	0.3751	0.3778	0.3795	0.3806	0.3814	0.3820	0.3824
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	4.4699×10^3	42.3015	39.3705	38.4629	38.0309	37.7801	37.6169	37.5023	37.4175

Table 8: Effect of ϕ: Fix $L = 3, \theta = 0.7, \theta' = 0{,}6, \lambda = 2, \eta = 0.5, \delta = 1.5\ and\ \gamma = 0.6$

Table 8 indicates the effect of the parameter ϕ on various performance measures and the cost function. Both E_{nsh} and E_{nsl} decrease as ϕ increases. The cost function and E_{nsl} decreases sharply as ϕ increases from 1 to 1.5. However, with further increase in ϕ value does not produce that decrease in values of cost function and E_{nsl}.

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.3589	1.3214	1.2867	1.2545	1.2244	1.1965	1.1703	1.1458	1.1229	1.1013
E_{nsl}	7.5197×10^3	411.6330	214.5271	146.4434	111.9502	91.1156	77.1729	67.1914	59.6955	53.8616
T_p	0.0035	0.0069	0.0102	0.0133	0.0164	0.0193	0.0222	0.0250	0.0276	0.0302
T_{ih}	0.0865	0.0831	0.0798	0.0767	0.0737	0.0709	0.0683	0.0657	0.0633	0.0611
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.4750	0.4674	0.4599	0.4526	0.4454	0.4384	0.4315	0.4247	0.4181	0.4116
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	774.5584	64.7362	45.9761	40.0889	37.5324	36.3143	35.7588	**35.5737**	**35.6123**	**35.7932**

Table 9: Effect of δ: Fix $L = 3, \theta = 0.7, \theta' = 0{,}6, \lambda = 2, \eta = 0.5, \gamma = 0.6\ and\ \phi = 4$

Table 9 indicates the effect of the parameter δ on various performance measures and the cost function. Both E_{nsh} and E_{nsl} decrease as δ increases.In this case, as δ increases, the system cost first decreases, reaches an optimal value(35.5737) corresponding to $\delta = 0.8$ and then increases. Both E_{nsl} and the cost show sharp decrease in their values when δ moves from 0.1 to 0.2. Thereafter the decrease is not that pronounced.

η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.1184	1.1136	1.1093	1.1051	1.1013	1.0976	1.0942	1.0910	1.0880	1.0851
E_{nsl}	58.6679	57.2868	56.0367	54.8999	53.8616	52.9096	52.0337	51.2250	50.4761	49.7807
T_p	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0303	0.0303	0.0303
T_{ih}	0.0663	0.0649	0.0636	0.0623	0.0611	0.0599	0.0587	0.0576	0.0566	0.0555
T_{il}	0.0994	0.0958	0.0924	0.0893	0.0863	0.0836	0.0810	0.0785	0.0762	0.0740
T_{nh}	0.4058	0.4074	0.4089	0.4103	0.4116	0.4129	0.4141	0.4153	0.4165	0.4175
T_{nl}	0.3403	0.3425	0.3445	0.3464	0.3482	0.3499	0.3514	0.3529	0.3543	0.3556
C	36.7438	36.4795	36.2344	36.0062	35.7932	35.5936	35.4062	35.2298	35.0634	34.9061

Table 10: Effect of η Fix $L = 3, \theta = 0.7, \theta' = 0{,}6, \lambda = 2, \eta = 0.5, \gamma = 0.6\ and\ \phi = 4$

Table 10 indicates the effect of the parameter η on various performance measures and the cost function. Both E_{nsh} and E_{nsl} decrease as η increases.The cost fuction decreases as η increases.

θ'	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.3380	1.1630	1.0642	1.0027	0.9616	0.9325	0.9111	0.8947	0.8819	0.8716
E_{nsl}	417.6867	79.1550	49.1289	37.9315	32.0828	28.4903	26.0604	24.3078	22.9843	21.9498
T_p	0.0464	0.0490	0.0504	0.0512	0.0517	0.0521	0.0523	0.0525	0.0527	0.0528
T_{ih}	0.0369	0.0389	0.0400	0.0406	0.0410	0.0413	0.0415	0.0417	0.0418	0.0419
T_{il}	0.2089	0.1576	0.1260	0.1049	0.0898	0.0785	0.0697	0.0627	0.0569	0.0521
T_{nh}	0.3182	0.3357	0.3452	0.3508	0.3544	0.3568	0.3586	0.3599	0.3608	0.3616
T_{nl}	0.3791	0.3685	0.3622	0.3580	0.3551	0.3529	0.3512	0.3499	0.3488	0.3479
C	74.5558	42.4295	40.3754	39.8320	39.6251	39.5285	39.4764	39.4450	39.4244	39.4098

Table 11: Effect of θ': Fix $L = 3, \theta = 0.7, \lambda = 2, \eta = 0.8, \delta = 2, \gamma = 0.6\ and\ \phi = 4$

Table 11 indicates the effect of the parameter θ' on various performance measures and the cost function. Both E_{nsh} and E_{nsl} decrease as θ' increases.The cost fuction decreases as θ' increases, as it is to be expected. However, there is a sharp decrease in value of E_{nsl} when θ' moves from 0.1 to 0.2. For higher values of θ', the initial sharpness in decrease is not seen.

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.0050	1.0251	1.0448	1.0640	1.0829	1.1013	1.1193	1.1369	1.1542	1.1711
E_{nsl}	34.0325	37.1338	40.5924	44.4740	48.8618	53.8616	59.6115	66.2942	74.1569	83.5428
T_p	0.0301	0.0301	0.0302	0.0302	0.0302	0.0302	0.0302	0.0303	0.0303	0.0303
T_{ih}	0.0113	0.0220	0.0324	0.0423	0.0519	0.0611	0.0699	0.0784	0.0866	0.0945
T_{il}	0.0160	0.0313	0.0459	0.0599	0.0734	0.0863	0.0988	0.1107	0.1222	0.1333
T_{nh}	0.4586	0.4485	0.4378	0.4293	0.4203	0.4116	0.4033	0.3953	0.3876	0.3801
T_{nl}	0.3904	0.3812	0.3725	0.3640	0.3560	0.3482	0.3407	0.3336	0.3267	0.3200
C	30.0298	31.1586	32.2900	33.4325	34.5961	35.7932	37.0389	38.3530	39.7616	41.3003

Table 12: Effect of γ: Fix $L = 3, \theta = 0.7, \lambda = 2, \eta = 0.8, \delta = 2, \gamma = 0.6$ and $\phi = 4$

Table 12 indicates the effect of the parameter γ on various performance measures and the cost function. Both E_{nsh} and E_{nsl} increase as γ increases. As expected, the cost increases as γ increases.

Conclusion

In this paper, we considered a (M,MAP)/(PH,PH)/1 queue with non premptive priority, exponentially distributed working interruptions and protection. We analysed the distribution of service time of type I and type II customers and the distribution of a p-cycle. Also we provided LSTs of busy cycle, busy period of type I customers generated during the service time of a type II customer. For the waiting time distributions of type I and type II customers, we provided an analysis using LST and the matrix analytic method. We also performed some numerical experiments to evaluate some performance measures and also found optimal values using a cost function. Extension of the model discussed to multi-server is proposed to be taken up in a future study.

References

[1] Chakravarthy,S.R.(2001), *The Batch Markovian Arrivl Process: A Review and future work*,Advances in probability Theory and Stochastic Processes,Eds., A Krishnamoorthy et al., 21-49, Notable Publications, Inc.,New Jersey

[2] Keilson,J.(1962),*Queues Subject to Service Interruption*, Annals of Mathematical Statistics, Vol.33,1314-1322.

[3] Fiems, D., Maertens, T and Bruneel, H. (2008),*Queueing systems with different types of interruptions*, European Journal of Operations Research, vol. 188, 3, 838-845.

[4] Gaver D.P.(1962),*A Waiting Line with Interrupted Service, including Priorities*,Journal of the Royal Statistical Society, Vol.24, No.1,73-90.

[5] Krishnamoorthy A., Pramod, P. K. and Chakravarthy, S.R.(2012), *Queues with interruption: A survey*, TOP-Spanish journal of Statistics and Operations Research, DOI 10.1007/s11750-012-0256-6.

[6] Krishnamoorthy A., Pramod, P. K. and Deepak, T.G.(2009), *On a queue with interruptions and repeat/resumption of service*, Non Linear Analysis, Theory,Methods and Applications, Vol.71, Issue 12, e1673-e1683.

[7] Krishnamoorthy, A. and Varghese Jacob.(2012),*Analysis of Customer Induced Interruption in a*

multiserver system, Neural, Parallel and Scientific Computations, 20, 153-172.

[8] Krishnamoorthy, A. and Varghese Jacob (2015),*Analysis of customer induced interruption and retrial of interrupted customers*,American Journal of Mathematical and Management Sciences, Vol.34, Issue 4,343-366.

[9] Neuts,M.F.(1981),*Matrix Geometric Solutions in Stochastic Models: An Algorithmic Approach*,The Johns Hopkins University Press,Baltimore,MD

[10] Takagi H.(1991), *Queueing Analysis, Volume I: Vacation and Priority Systems, Part 1*, North Holland, Amsterdam

[11] Takagi H.(1993), *Queueing Analysis, Volume II: Finite systems*, North Holland, Amsterdam

[12] Varghese Jacob, Chakravarthy, S.R. and Krishnamoorthy A.(2012)*On a Customer Induced Interruption in a service system*, Journal of Stochastic analysis and Applications, 30,1-13.

[13] Dudin A.N., Varghese jacob and Krishnamoorthy A.(2013),*A multiserver queueing system with service interruption, partial protection and repetition of service*, Annals of Operations Research, Vol.233, Issue 1,101-121.

[14] Krishnamoorthy, A.,Gopakumar, B. and Viswanath,C. N.(2009),*A queueing model with interruption Resumption/Restart and Reneging*, Bulletin of Kerala Mathematical Association, Special Issue- Guest Editor S.R.S. Varadhan; 29-45.

Appendix

Proof of Theorem 1

Proof. Let B_{c_L} denote the length of the busy cycle generated by type I customers arriving during the service time of a type II customer , $\hat{B}_{c_L}(s)$ the LST of the length of the busy cycle and l the number of type I customers that arrive during service time of type II customer.

Then $B_{c_L} = X + B_L^1 + \cdots B_L^l$ where X denote the service time of the type II customer in service, B_L^j the busy period generated by jth type I customers that arrive during X, where $1 \leq j \leq l$.

$$\begin{aligned}
\hat{B}_{c_L}(s) &= E(e^{-sB_{c_L}}) \\
&= \int_{x=0}^{\infty} E(e^{-sB_{c_L}}/X = x)P(x \leq X < x + dx) \\
&= \int_{x=0}^{\infty} \sum_{p=0}^{\infty} E(e^{-sB_{c_L}}/X = x, l = p)P(l = p/X = x)P(x \leq X < x + dx) \\
&= \int_{x=0}^{\infty} \sum_{p=0}^{\infty} E(e^{-sB_{c_L}}/X = x, l = p)\frac{e^{-\lambda x}(\lambda x)^p}{p!}\beta' e^{S_2 x} S_2^0 dx \\
&= \int_{x=0}^{\infty} e^{-(s+\lambda)x}\beta' e^{S_2 x} S_2^0 dx + \int_{x=0}^{\infty} \sum_{p=1}^{L-1} e^{-sx}\gamma_p(sI - T_1)^{-1}T_1^0 \frac{e^{-\lambda x}(\lambda x)^p}{p!} \qquad (17) \\
&\quad \beta' e^{S_2 x} S_2^0 dx + \int_{x=0}^{\infty} \sum_{p=L}^{\infty} e^{-sx}\gamma_L(sI - T_1)^{-1}T_1^0 \frac{e^{-\lambda x}(\lambda x)^p}{p!}\beta' e^{S_2 x} S_2^0 dx \\
&= \beta'[(s + \lambda)I - S_2]^{-1}S_2^0 + \sum_{p=1}^{L-1} \gamma_p(sI - T_1)^{-1}T_1^0 \frac{\lambda^p \beta'}{p!}\int_{x=0}^{\infty} x^p e^{-[(s+\lambda)I-S_2]x} S_2^0 dx \\
&\quad + \sum_{p=L}^{\infty} \gamma_L(sI - T_1)^{-1}T_1^0 \frac{\lambda^p}{p!}\beta' \int_{x=0}^{\infty} x^p e^{[-(s+\lambda)I-S_2]x} S_2^0 dx
\end{aligned}$$

We have,

$$\int_{x=0}^{\infty} x^p e^{-[(s+\lambda)I-S_2]x} dx = \frac{p!}{[(s+\lambda)I-S_2]^{p+1}} \qquad (18)$$

Substituting (18) in (17) , its third term

$$\begin{aligned}
&= \sum_{p=L}^{\infty} \gamma_L(sI - T_1)^{-1}T_1^0 \lambda^p \beta'[(s + \lambda)I - S_2]^{-(p+1)}S_2^0 \\
&= \gamma_L(sI - T_1)^{-1}T_1^0 \beta' \sum_{p=L}^{\infty} [\lambda^{-1}[(s + \lambda)I - S_2]]^{-p}[(s + \lambda)I - S_2]^{-1}S_2^0 \\
&= \gamma_L(sI - T_1)^{-1}T_1^0 \beta'[\lambda^{-1}[(s + \lambda)I - S_2]]^{-L} \sum_{q=0}^{\infty} [\lambda^{-1}[(s + \lambda)I - S_2]]^{-q} \\
&\quad [(s + \lambda)I - S_2]^{-1}S_2^0 \qquad (19) \\
&= \gamma_L(sI - T_1)^{-1}T_1^o \beta'[\lambda^{-1}[(s + \lambda)I - S_2]]^{-L}[I - \lambda[(s + \lambda)I - S_2]^{-1}]^{-1} \\
&\quad [(s + \lambda)I - S_2]^{-1}S_2^0
\end{aligned}$$

Substituting (19) in (17) gives

$$\hat{B}_{c_L}(s) = \beta'[(s + \lambda)I - S_2]^{-1}S_2^0 + \sum_{p=1}^{L-1} \gamma_p(sI - T_1)^{-1}T_1^0 \lambda^p \beta'[(s + \lambda)I - S_2]^{-(p+1)}S_2^0 + \gamma_L(sI - T_1)^{-1}$$
$$T_1^0 \beta'[\lambda^{-1}[(s + \lambda)I - S_2]]^{-L}[I - \lambda[(s + \lambda)I - S_2]^{-1}]^{-1}[(s + \lambda)I - S_2]^{-1}S_2^0 \qquad (20)$$

Proof of theorem 2

Proof. Let B_L denote the length of the busy period generated by type I customers arriving during the service time of a type II customer , $\hat{B}_L(s)$ the LST of the length of the busy period and l the number of type I customers that arrive during service time of type II customer.

Then $B_L = B_L^1 + \cdots B_L^l$,where B_L^j denote the busy period generated by jth type I customers that arrive during X, where $1 \leq j \leq l$. Proceeding as in the above proof, we get the required result.