COMPUTATIONAL SOLUTION OF TWO-POINT BOUNDARY VALUE PROBLEMS WITH DERIVATIVE BOUNDARY CONDITIONS USING CUBIC-SPLINE PROCESS

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ABSTRACT

An attempt has been made here to study a two –point BVP with the mixed BCs. A cubic spline function is used to arrive at the solution of the second-order linear two-point BVPs with considerable slope conditons. While implementing the method we are applied three point recursion algorithm termed as “Thomas algorithm “, after incorporating the cubic-splines involves some parameters. These parameters may be suitably chosen to obtain an non-oscillation solution of the governing equation. The same method has been implemented on some linear problems and realised , the numerically computed results are in line with the analytical solutions available in the literature.

KEYWORDS: Boundary Value Problems (BVPs), Boundary Conditions (BCs) Cubic Spline (CS), Thomas Algorithm, Differential Equation (DE).

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INTRODUCTION

Two point BVPs are common in numerous fields of Engineering and more so in study of Fluids, Convection dominated flow problems with high Reynolds number, singularly perturbed problems, Hydrodynamics etc. Such kind of problems cannot be solved analytically always due to the methods available in the literature. Recourse must be adopted to get the nature of such system by applying parallel computing, numerical methods. Particularly in the numerical methods we come across FEM, FDM and FVM. In these methods we have to develop a grid or mesh in order to get the solution at each interior point at the selected domain where problems is defined. After discretization is incorporated we come across a difference equation. A suitable process must be selected with designed algorithm to reach numerical solution. Numerically computed solution must be analyzed with the analytical solution for constancy, stability and convergence. The selected problems are available in [1, 3, 12] where numerical solutions are discussed and developed the consistency, stability and convergent analysis mechanism are discussed. In [6, 9] we can see the various methods to solve differential equations computationally. In the literature study [4, 5, 7, 12] various numerical methods are developed and discussed. We have attempted to solve a two-point BVP using CS approximation method and three-term recursive algorithm [14, 15]. Since closed form solutions for most of these problems are not available so one has to resort to get the solutions of such problems in approximation technology. Due to the advent of digital computer it has become possible to consider and solve such a huge task. The most commonly employed approximate methods for solving such problems are Finite Difference Methods, Numerical Quadrature method and Finite Volume Methods. In this present research paper a two point BVP with Neumann BCs are considered. Taylor’s series can be used as an approximation to convert a second order DE into a first order

DE. Such first order differential equation again transformed into a finite difference equation. Numerical integration method subsequently Thomas algorithm employed to get the approximate solution in the defined region. It is observed that numerically computed solutions are in good agreement on par with the available closed form solutions.

Two point BVPs occur very frequently in branches of Engineering. Since closed form solutions are not possible for all we try to find solutions of such problems using approximations. The most frequently adoptable approximate methods for solving such problems are FDM, Numerical quadrature method and FVM.

In this present research problem a two point BVP with differential BCs are considered. Taylor's series approximation is used to reduce a second order DE into first order differential equation subsequently the first order DE transformed into a finite difference equation. Numerical integration method and subsequently Thomas algorithm employed to get the approximate solution in the defined region at each mesh point for the various perturbation parameter values. It is observed that numerically computed solutions are in line with available closed form solutions.

EXPLANATION

Two-point BVPs with singular perturbation is given by

$$\varepsilon y''(x) + \frac{c}{x} y'(x) + b(x)y(x) = d(x), \quad 0 < x < 1 \quad (1)$$

Where ε is a perturbed parameter very close to zero such that $0 < \varepsilon \ll 1$ and $c \geq 1$ with BCs

$$y'(0) = \alpha, \quad y(1) = \beta.$$

We try to modify (1) at $x = 0$ (singular point) to get

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x), \quad (2)$$

$$y'(0) = \alpha, \quad y(1) = \beta \quad (3)$$

$$\text{Where } p(x) = \begin{cases} 0, & (x = 0), \\ \frac{c}{\varepsilon x}, & (x \neq 0), \end{cases} \quad q(x) = \begin{cases} \frac{b(0)}{\varepsilon(c+1)}, & (x = 0), \\ b(x, \varepsilon), & (x \neq 0) \end{cases} \quad \text{and } r(x) = \begin{cases} \frac{d(0)}{\varepsilon(1+c)}, & (x = 0), \\ d(x, \varepsilon), & (x \neq 0) \end{cases} \quad (4)$$

Here assume the grid points of the mesh as x_i with $x_0 = 0, x_n = 1$ with $h = x_i - x_{i-1} > 0$ and construct the spline function as given below.

Construction of Spline Function

For the given set of data points (x_i, y_i) the data defined $(i=0, 1, 2, 3, 4, \dots, n)$ and let $S_i(x)$ be the CS defined in the interval $[x_i, x_{i+1}]$. As we aware of the conditions of it are

- $S_i(x)$ is approximately cubic in every sub-interval $[x_i, x_{i+1}]$, $i=0, 1, 2, 3, \dots, n-1$
- $S_i(x) = y_i$
- $S_i(x)$, $S_i'(x)$ and $S_i''(x)$ are continuous in $[x_0, x_n]$

$$\bullet \quad S_i''(x_0), S_i''(x_n) = 0$$

For deriving the associated equations of the cubic spline, we perceive the second derivatives of spline to be of first

$$S_i''(x) = \frac{1}{h_i} [(x_i - x)M_{i-1} + (x - x_{i-1})M_i] \quad (5)$$

degree. So we consider the condition in $S_i(x)$

Where h is the step width defined by

$h = x_i - x_{i-1}$, $S_i''(x_i) = M_i$ for all i . It is evident that the spline or smooth curves that connect two are more specific points.

Integrating (5) two times with respect to the independent variable we have

$$S_i(x) = \frac{1}{h_i} \left[(x_i - x)^3 M_{i-1} + \frac{(x - x_{i-1})^3}{6} M_i \right] + c_i(x_i - x) + d_i(x - x_{i-1}) \quad (6)$$

Where c_i & d_i are constants to be calculated.

Apply the conditions $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ so we have

$$c_i = \frac{1}{h_i} \left[y_{i-1} - \frac{h_i^2}{6} M_{i-1} \right], d_i = \frac{1}{h_i} \left[y_i - \frac{h_i^2}{6} M_i \right] \quad (7)$$

Substitute c_i & d_i in equation (6) we have

$$S_i(x) = \frac{1}{h_i} \left[\frac{(x_i - x)^3}{6} M_{i-1} + \frac{(x - x_{i-1})^3}{6} M_i + \left(y_{i-1} - \frac{h_i^2}{6} M_{i-1} \right) (x_i - x) + \left(y_i - \frac{h_i^2}{6} M_i \right) (x - x_{i-1}) \right] \quad (8)$$

In (8) the spline second derivatives M_i are not known still. To calculate them we apply the other condition of continuity of $S_i'(x)$. Differentiate Equation (8) one can have

$$S_i'(x) = \frac{1}{h_i} \left[-3 \left(\frac{(x_i - x)^2}{6} \right) M_{i-1} + \frac{3(x - x_{i-1})^2}{6} M_i - \left(y_{i-1} - \frac{h_i^2}{6} M_{i-1} \right) + y_i - \frac{h_i^2}{6} M_i \right] \quad (9)$$

Since $S_i(x)$ is differentiable function so it's right hand and left hand derivatives exists and they are evidently equal and for equal intervals $h_{i+1} = h_i = h$

$$S'(x) = M_{i-1} \left[\frac{h}{6} - \frac{(x_i - x)^2}{2h} \right] + M_i \left[\frac{(x - x_{i-1})^2}{2h} - \frac{h}{6} \right] + \frac{(y_i - y_{i-1})}{h} \quad (10)$$

From (10), we get

$$S'(x_i^+) = -\frac{h}{3}M_i - \frac{h}{6}M_{i+1} + \frac{y_{i+1} - y_i}{h}, \quad i = 0, 1, 2, \dots, n-1 \quad (11)$$

$$S'(x_i^-) = \frac{h}{3}M_i + \frac{h}{6}M_{i-1} + \frac{y_i - y_{i-1}}{h}, \quad i = 0, 1, 2, \dots, n \quad (12)$$

The continuity condition $S'(x_i^+) = S'(x_i^-)$ gives

$$\frac{h}{6}M_{i-1} + \frac{2h}{3}M_i + \frac{h}{6}M_{i+1} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h}, \quad i = 1, 2, \dots, n-1 \quad (13)$$

By using it in any sub-interval $[x_i, x_{i+1}]$ with $h_{i+1} = x_{i+1} - x_i$

For equal intervals $h_{i+1} = h_i = h$ we can write by utilizing eqn. (8) the following

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2} (y_{i+1} - 2y_i + y_{i-1}), \quad i = 1, 2, \dots, n-1 \quad (14)$$

The simultaneous system of equations (14) has special significance. If M_0 and M_n are known, then equation (14) can be transformed to

$$h_{n-1}M_{n-2} + 2(h_{n-1} + h_n)M_{n-1} = 6 \left[\frac{y_n - y_{n-1}}{h_n} - \frac{y_{n-1} - y_{n-2}}{h_{n-1}} \right] - h_n M_n \quad (15)$$

Equation (14) is an $(n-1)$ system of equations along with two conditions in (4) for natural spline, the complete system can be solved for the values M_i , $(i=1, 2, \dots, n-1)$. The system of equations (15) is tri-diagonal which can be solved by Gaussian elimination method or by applying efficient and accurate method known as Thomas algorithm (simple version of Gauss elimination) when the M_i are known. So that equation (8) gives the required cubic spline in each sub-interval $[x_{i-1}, x_i]$. Also the numerical derivatives can be calculated from using (9) with the virtue of finite slopes in each sub-interval.

$$S(x) = M_{i-1} \frac{(x_i - x)^3}{6h} + M_i \frac{(x - x_{i-1})^3}{6h} + \left(y_{i-1} - \frac{h^2}{6}M_{i-1} \right) \left(\frac{x_i - x}{h} \right) + \left(y_i - \frac{h^2}{6}M_i \right) \frac{(x - x_{i-1})}{h} \quad (16)$$

Where $M_i = S''(x_i)$ and $y_i = y(x_i)$. If the function $y(x)$ at all points $y(x) \in C^4[0,1]$, then the CS function $S(x)$ approximates $y(x)$ at all points in the interval $[0,1]$ up to fourth order. The unknowns M_i are connected by continuity condition on $S'(x)$. Use Eq. (2) at equidistant grid points $x_i = x_0 + ih$, $i = 0, 1, 2, \dots, n$ to determine Cubic Spline interpolating polynomial. Using a known approach [2], we have

$$A_i y_{i-1} - B_i y_i + C_i y_{i+1} = D_i, \quad i = 1, 2, \dots, n-1 \quad (17)$$

Where

$$A_i = b_i \left[1 - \frac{h}{2} p_{i-1} + \frac{h^2}{6} q_{i-1} \right], \quad B_i = a_i \left[1 + \frac{h}{2} p_{i+1} \right] + b_i \left[1 - \frac{h}{2} p_{i-1} \right] - \frac{2h^2}{3} q_i d_i,$$

$$C_i = a_i \left[1 + \frac{h}{2} p_{i+1} + \frac{h^2}{6} q_{i+1} \right], \quad D_i = \frac{h^2}{6} (b_i r_{i-1} + 4e_i r_i + a_i r_{i+1}),$$

$$a_i = 1 - \frac{h}{3} p_{i-1} + \frac{h}{3} p_i - \frac{h^2}{12} p_i p_{i-1}, \quad b_i = 1 - \frac{h}{3} p_i + \frac{h}{3} p_{i+1} - \frac{h^2}{12} p_i p_{i+1},$$

$$e_i = 1 - \frac{h^2}{12} p_{i-1} p_{i+1} + \frac{7h}{24} (p_{i+1} - p_{i-1}).$$

The BCs (3) and (4) in discrete form is given by

$$S_0 y_0 = \alpha S_1 y_1 + S_2, \quad (18)$$

$$y_n = \beta \quad (19)$$

$$S_0 = \frac{h}{3a_1} \left(q_0 - \frac{p_0}{h} - \frac{p_1}{2h} - \frac{p_0 p_1}{4} + \frac{h}{4} p_1 q_0 - \frac{3a_1}{h^2} \right) \quad (20)$$

$$S_1 = \frac{h}{3a_1} \left(-\frac{p_0}{h} - \frac{q_1}{2} - \frac{p_1}{2h} - \frac{p_0 p_1}{4} - \frac{3a_1}{h^2} \right), \quad (21)$$

$$S_2 = \frac{h}{3a_1} \left(r_0 + \frac{r_1}{2} + \frac{h}{4} p_1 r_0 \right), \quad (22)$$

Eq. (17) together with Eqs.(18) and (19) form 'n+1' system of equations with n+1 unknowns which would be used to solve the unknowns. This System has been solved using the simple algorithm defined by Thomas. The algorithm pivots the unknown values by back substitution method. The starting recurrence relation we use is

$$y_i = W_i y_{i+1} + T_i, \quad i = 0, 1, 2, \dots, n-1 \quad (23)$$

Make Use of Eqn (23) in (17) to get

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}} \quad (24)$$

$$\text{and } T_i = \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \quad (25)$$

The value of W_0 and T_0 and required to find W_i and T_i

From (18) , we have

$$W_0 = \frac{S_1}{S_0}, \quad (26)$$

$$T_0 = \frac{S_2}{S_0} \quad (27)$$

We compute W_i and T_i from Eqn (24) and (25) using forward process with the initial conditions .

Example Problems: To test the efficianccy of the selected method we have given some exampels into consideration. The patterns revealed the tendency of the solution.

Numerical Examples

Example-1

We take a second order DE with Neumann boudary conditons

$$-y''(x) - \frac{2}{x}y'(x) + (1-x^2)y(x) = x^4 - 2x^2 + 7,$$

$$y'(0) = 0, \quad y(1) = 0$$

The exact solution is $y(x) = 1 - x^2$ and numerical approximations are given below.

Table 1: Numerical Results for Example 1

Argument(x)	Step width(h=0.05)	Step width h=0.025	Exact solution
0	0.99998	1.000002	1.000000
0.025	0.999370	0.999377	0.999375
0.050	0.997490	0.997502	0.995700
0.1	0.989995	0.990002	0.990000
0.2	0.959998	0.960002	0.960000
0.3	0.909997	0.910001	0.910000
0.4	0.839988	0.840001	0.840000
0.5	0.749999	0.750001	0.750000
0.8	0.359998	0.359884	0.360000
0.9	0.190002	0.190000	0.190000
1	0.000000	0.000000	0.000000

Example-2

Consider the following BVP

$$\frac{d^2 y}{dx^2} + \frac{2}{x} \frac{dy}{dx} - 4y = -2$$

with the prescribed Neumann BCs

$$\frac{dy}{dx} = 0 \text{ at } x = 0, y = 5.5 \text{ at } x = 1. \text{ The analytical solution for example-2 is given by}$$

$$y(x) = \frac{1}{2} + \frac{5 \sinh 2x}{x \sinh 2}$$

The values have been compared and tabulated in Table 2

Table 2: Numerical Results for Example 2

Argument value (x)	Spline method (h=0.05)	Spline method(h=0.025)	Analytical solution
0	3.256907	3.256904	3.257206
0.1	3.275328	3.275299	3.275624
0.2	3.331029	3.331006	3.333132
0.3	3.425356	3.425289	3.425641
0.4	3.560588	3.560566	3.560864
0.5	3.740011	3.740009	3.740271
0.6	3.977901	3.977851	3.968246
0.7	4.240188	4.240169	4.250393
0.8	4.592999	4.592878	4.593706
0.9	5.000623	5.000586	5.006766
1	5.599990	5.599658	5.500000

Similary we can generate numerical approximation values by lowering the step width accordingly we can get the most plausible values similar to the above tables.

Figure 1

Figure 2

Figure 3

Figure 4

CONCLUSIONS

We have defined a two point BVP with prescribed BCs, described and demonstrated the methodology of the applied CS method for solving BVPs with single perturbation. We have analysed, developed and designed a discrete three term recursive relation by using this method. Such three term recursive difference equation is solved using Thomas algorithm. This method is simple, precise, elegant and user friendly to be implemented on a computer. However here we have implemented this method with some selected problems available in the literature due to comparative study. More particularly examples of homogeneous and non-homogeneous singular BVPs are considered and implemented. Numerically obtained results are tabulated and compared with the exact solutions or analytical solution. After comparative study we come to a conclusion that the numerical results are in perfect correlation with exact solutions available in the literature. The algorithm has a strong convergent analogy by observing the tabulated values throughout the region. Even for small step size also the tendency of the convergence is retained throughout the domain. We establish here that the method used here is in correlation with exact solution to a great extent. The convergence, consistency and stability of the solution is uniform throughout the defined region.

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