β_g -CLOSED SETS IN BITOPOLOGICAL SPACES

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ABSTRACT

In this paper, the notion of closed sets called (p, q) - β -generalized closed set, (p, q) - β_g - closed sets are introduced and some of its properties are discussed in bitopological spaces.

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1. INTRODUCTION AND PRELIMINARIES

In 1986, Andrijevic. D, introduced the concept of β -open sets in topological spaces. In 1963, J. C. Kelly[1] introduced the notion of bitopological spaces. In 1991, Sundaram[4] introduced the concept of generalized closed sets in bitopological spaces. The notion has been studied extensively in recent years by many topologists.

Definition 1.1:[2] A subset A of a topological space (X, τ) is called β -closed if $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$.

Definition 1.2:[3] The β -frontier of A denoted by $\text{Fr}_\beta(A)$, is defined by $\text{Fr}_\beta(A) = \text{Cl}_\beta(A) \setminus \text{Int}_\beta(A)$.

Definition 1.3:[3] A subset A of a topological space (X, τ) is called β gclosed if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \beta O(X, \tau)$.

Definition 1.4:[2] A subset A of a topological space (X, τ) is called $g\beta$ -closed if $\beta \text{cl}(A) \subseteq G$ whenever $A \subseteq G$ and G is open in X .

Definition 1.5 :[3] A subset A of a bitopological space (X, τ_p, τ_q) is said to be (p, q) -generalized closed if $\tau_q - \text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_p$ for $p, q = 1, 2$ and $p \neq q$.

2. (P, Q) - B_G -CLOSED SETS

Definition 2.1: A subset A of a bitopological space (X, τ_p, τ_q) is called (p, q) - β_g -closed if $\tau_q - \text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \beta O(X, \tau_q)$ for $p, q = 1, 2$ and $p \neq q$.

Definition 2.2: A subset A of a bitopological space (X, τ_p, τ_q) is called (p, q) - $g\beta$ -closed if $\tau_q - \text{Cl}_\beta(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_p$ for $p, q = 1, 2$ and $p \neq q$.

Definition 2.3: A subset A of a bitopological space (X, τ_p, τ_q) is called (p, q) - β_g -closed if $\tau_q - \text{Cl}_\beta(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \beta O(X, \tau_q)$ for $p, q = 1, 2$ and $p \neq q$.

Example 2.3 : Let $X = \{a, b, c\}$ with the topologies $\tau_p = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_q = \{X, \emptyset, \{b\}, \{a, b\}, \{a, c\}\}$. Thus $\beta O(X, \tau_p) = X, \emptyset, \{a\}, \{b\}, \{b, c\}$ and $\{a, b\}$. The set $\{a\}$ is (p, q) - β_g -closed but it is not (p, q) - β_g -closed. Also the set $\{b\}$ is (p, q) - $g\beta$ -closed but it is not (p, q) - β_g -closed.

Theorem 2.4: The union of two (p, q) - β_g -closed sets is (p, q) - β_g -closed.

Proof: Let $A \cup B \subseteq U$, then $A \subseteq U$ and $B \subseteq U$ where $U \in \beta O(X, \tau_p)$. As A and B are (p, q) - β_g -closed, $\tau_q - Cl(A) \subseteq U$ and $\tau_q - Cl(B) \subseteq U$. Hence $\tau_q - Cl(A \cup B) = \tau_q - Cl(A) - \tau_q - Cl(B) \subseteq U$

Remark 2.5: The intersection of two (p, q) - β_g -closed sets need not be (p, q) - β_g -closed as shown by the following example.

Example 2.6 : Let $X = \{a, b, c\}$ with the topologies $\tau_p = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\tau_q = \{X, \phi, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. If $A = \{a, b\}$ and $B = \{a, c\}$ are two (p, q) - β_g -closed sets but $A \cap B$ is not (p, q) - β_g -closed.

Theorem 2.7: If a subset A of (X, τ_p, τ_q) is (p, q) - β_g -closed, then $\tau_q Cl(A) \setminus A$ contains no non empty τ_p -closed subset of (X, τ_p, τ_q) .

Proof: Let F be a τ_p -closed subset contained in $\tau_q - Cl(A) \setminus A$. Clearly $A \subseteq X \setminus F$ where A is (p, q) - β_g -closed and $X \setminus F$ is a τ_p -open subset of X . Thus $\tau_q - Cl(A) \subseteq X \setminus F$ or $F \subseteq X \setminus (\tau_q - Cl(A))$. Then $F \subseteq (X \setminus (\tau_q - Cl(A))) \cap (\tau_q - Cl(A) \setminus A) \subseteq (X \setminus (\tau_q - Cl(A))) \cap (\tau_q - Cl(A)) = \phi$.

Remark 2.8: The converse of the above theorem need not be true in general as shown by the following example.

Example 2.9 : Let $X = \{a, b, c\}$ with the topologies $\tau_p = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\tau_q = \{X, \phi, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. If $A = \{a, c\}$ then $\tau_q - Cl(A) \setminus A = \{b\}$ does not contain non empty closed set. But A is not (p, q) - β_g -closed set in (X, τ_p, τ_q) .

Definition 2.10 : The topologies τ_p and τ_q on a non empty set X are said to have the property ε if $A \in \beta C(X, \tau_p)$ and $B \in \beta C(X, \tau_q)$ imply $A \cap B \in \beta C(X, \tau_q)$

Theorem 2.11: Let τ_p and τ_q have the property ε . Then a subset A is (p, q) - β_g -closed if and only if $\tau_q - Cl(A) \setminus A$ contains no non-empty $F \in \beta C(X, \tau_p)$

Proof: Suppose that A is (p, q) - β_g -closed. Let $F \in \beta C(X, \tau_p)$ such that $F \subseteq \tau_q - Cl(A) \setminus A$. Then $A \subseteq X \setminus F$. Since A is (p, q) - β_g -closed, we have $\tau_q - Cl(A) \subseteq X \setminus F$. Consequently $F \subseteq X \setminus \tau_q - Cl(A)$. Hence $F \subseteq (\tau_q - Cl(A) \subseteq A \setminus A) \cap (\tau_q - Cl(A)) = \phi$. Therefore $F = \phi$.

Now, suppose that $\tau_q - Cl(A) \subseteq A \setminus A$ contains no non empty $F \in \beta C(X, \tau_q)$. Let $A \subseteq G$ and $G \in \beta O(X, \tau_q)$. If $\tau_q - Cl(A) \subseteq G$, then $\tau_q - Cl(A) \cap (X \setminus G) \neq \phi$, $\tau_q - Cl(A) \cap (X \setminus G) \in \beta C(X, \tau_p)$ and $\tau_q - Cl(A) \cap (X \setminus G) \subseteq \tau_q - Cl(A) \setminus A$. Therefore A is (p, q) - β_g -closed.

Theorem 2.12: If A is (p, q) - β_g -closed and $A \subseteq B \subseteq \tau_q - Cl(A)$, then B is (p, q) - β_g -closed.

Proof: Let $B \subseteq U$ and $U \in \beta O(X, \tau_p)$. Then $A \subseteq U$ and A is (p, q) - β_g -closed. Hence $\tau_q - Cl(B) \subseteq \tau_q - Cl(A) \subseteq U$ and B is (p, q) - β_g -closed.

Theorem 2.13: If A is an (p, q) - β_g -closed set in (X, τ_p, τ_q) and $A \in \beta O(X, \tau_p)$ then A is τ_q -closed.

Proof: Since $A \in \beta O(X, \tau_p)$ and A is (p, q) - β_g -closed, $\tau_q - Cl(A) \subseteq A$ and hence A is τ_q -closed.

Theorem 2.14: If A is τ_q -closed, then A is (p, q) - β_g -closed.

Proof: Straight forward

Theorem 2.15: For each $x \in X$, either $\{x\} \in \beta C(X, \tau_p)$. Then $X \setminus \{x\}$ is not in $\beta O(X, \tau_p)$ and the only subset that in $\beta O(X, \tau_p)$

and contains $X \setminus \{a\}$ is X . Therefore $\tau_q - \text{Cl}(X \setminus \{x\}) \subseteq X$ and so $X \setminus \{x\}$ is $(p, q) - \beta_g$ - closed.

Theorem 2.16: If A is $(p, q) - \beta_g$ - closed and $A \subseteq U \in \beta O(X, \tau_p)$. Then $\tau_q - \text{Fr}_\beta(U) \subset \tau_q - \text{Int}(X \setminus A)$.

Proof: Let A be $(p, q) - \beta_g$ - closed and $A \subseteq U \in \beta O(X, \tau_p)$. Then $\tau_q - \text{Cl}(A) \subseteq U$. Suppose that $x \in \tau_q - \text{Fr}_\beta(U)$. Since $U \in \beta O(X, \tau_p)$, $\tau_q - \text{Fr}_\beta(U) = \tau_q - \text{Cl}_\beta(U) \setminus U$. Therefore x does not belongs to U . This shows that $x \in \tau_q - \text{Int}(X \setminus A)$ and hence $\tau_q - \text{Fr}_\beta(U) \subset \tau_q - \text{Int}(X \setminus A)$.

Definition 2.17: A subset A in (X, τ_p, τ_q) is said to be $(p, q) - \beta_g$ - open if $X \setminus A$ is $(p, q) - \beta_g$ - closed.

Theorem 2.18: The intersection of two $(p, q) - \beta_g$ - open sets is $(p, q) - \beta_g$ - open.

Proof: Straight forward.

Theorem 2.19: A subset A is $(p, q) - \beta_g$ - open if and only if $F \subseteq \tau_q - \text{Int}(A)$ whenever $F \in \beta C(X, \tau_q)$ and $F \subseteq A$.

Proof: Suppose that $F \subseteq \tau_q - \text{Int}(A)$ whenever $F \in \beta C(X, \tau_q)$ and $F \subseteq A$. Let $X \setminus A \subseteq G$, where $G \in \beta O(X, \tau_p)$. Hence $X \setminus G \subseteq A$. By assumption $X \setminus G \subseteq \tau_q - \text{Int}(A)$. This implies that $X \setminus \tau_q - \text{Int}(A) \subseteq G$ and hence $\tau_q - \text{Cl}(X \setminus A) \subseteq G$. Hence $X \setminus A$ is $(p, q) - \beta_g$ - closed. i.e., $(p, q) - \beta_g$ - open. Now let A be an $(p, q) - \beta_g$ - open set. Then $X \setminus A$ is $(p, q) - \beta_g$ - closed.

Also let $F \in \beta C(X, \tau_q)$ such that $F \subseteq A$. Then $X \setminus F \in \beta O(X, \tau_p)$. Therefore whenever $X \setminus A \subseteq X \setminus F$, $\tau_q - \text{Cl}(X \setminus A) \subseteq X \setminus F$. Hence $F \subseteq \tau_q - \text{Int}(A)$.

Theorem 2.20: If $A \in \tau_q$, then A is $(p, q) - \beta_g$ - open.

Proof: Straight forward.

Theorem 2.21: Let τ_p and τ_q have the property ε . Then, a subset A is $(p, q) - \beta_g$ - open if and only if $G = X$ whenever $G \in \beta O(X, \tau_p)$ and $\tau_q - \text{Int}(A) \cup (X \setminus A) \subseteq G$.

Proof: Let A be $(p, q) - \beta_g$ - open, $G \in \beta O(X, \tau_p)$ and $\tau_q - \text{Int}(A) \cup (X \setminus A) \subseteq G$. Then $X \setminus G \subseteq (X \setminus (\tau_q - \text{Int}(A))) \cap (X \setminus (X \setminus A)) = (X \setminus (\tau_q - \text{Cl}(A))) \cap (X \setminus A) = \tau_q - \text{Cl}(A) \cap (X \setminus A) = \tau_q - \text{Cl}(A) \cap (X \setminus A)$. Since $X \setminus A$ is $(p, q) - \beta_g$ - closed, $X \setminus G \in \beta O(X, \tau_p)$, by Theorem 2.7, it follows that $X \setminus G = \emptyset$. Therefore $X = G$. Conversely, suppose that $F \in \beta C(X, \tau_p)$ and $F \subseteq G$. Then $\tau_q - \text{Int}(A) \cup (X \setminus A) \subseteq \tau_q - \text{Int}(A) \cup (X \setminus F)$. Therefore A is $(p, q) - \beta_g$ - open.

Theorem 2.22: If $\tau_q - \text{Int}(A) \subseteq B \subseteq A$ and A is $(p, q) - \beta_g$ - open, then B is $(p, q) - \beta_g$ - open.

Proof: Suppose that $\tau_q - \text{Int}(A) \subseteq B \subseteq A$ and A is $(p, q) - \beta_g$ - open. Then $X \setminus A \subseteq X \setminus B \subseteq \tau_q - \text{Cl}(X \setminus A)$ and $X \setminus A$ $(p, q) - \beta_g$ - closed. By using Theorem 2.8, B is $(p, q) - \beta_g$ - open.

Theorem 2.23: Let τ_p and τ_q have the property ε . Then, a subset A is $(p, q) - \beta_g$ - closed if and only if $\tau_q - \text{Cl}(A) \setminus A$ is $(p, q) - \beta_g$ - open.

Proof: Suppose that A is $(p, q) - \beta_g$ - closed. Let $F \subseteq \tau_q - \text{Cl}(A) \setminus A$, where $F \in \beta C(X, \tau_p)$. By using Theorem 2.7, $F = \emptyset$. Therefore $F \subseteq \text{Int}(\tau_q - \text{Cl}(A) \setminus A)$ and by theorem 2.15 $\tau_q - \text{Cl}(A) \setminus A$ is $(p, q) - \beta_g$ open. Now, let $A \subseteq G$ where $G \in \beta C(X, \tau_p)$. Then $\tau_q - \text{Cl}(A) \cap (X \setminus G) \subseteq \tau_q - \text{Cl}(A) \cap (X \setminus A) = \tau_q - \text{Cl}(A) \setminus A$. Since $\tau_q - \text{Cl}(A) \cap (X \setminus G) \in \beta O(X, \tau_p)$ and $\tau_q - \text{Cl}(A) \setminus A$ is $(p, q) - \beta_g$ - open, by Theorem 2.15, we have $\tau_q - \text{Cl}(A) \cap (X \setminus G) \subseteq \tau_q - \text{Int}(\tau_q - \text{Cl}(A) \setminus A) = \emptyset$ Hence A is $(p, q) - \beta_g$ - closed.

Theorem 2.24: A subset A is $(p, q) - \beta_g$ - closed if and only if $\tau_q - \text{Cl}_\beta(\{x\}) \cap A \neq \emptyset$ for every $x \in \tau_q - \text{Cl}(A)$.

Proof: Suppose that $\tau_q - \text{Cl}_\beta(\{x\}) \cap A \neq \emptyset$ for every $x \in \tau_q - \text{Cl}(A)$. Then $X \setminus \tau_q - \text{Cl}_\beta(\{x\}) \in \tau_q - \text{Cl}(A)$ such that $A \subseteq X \setminus \tau_q - \text{Cl}_\beta(\{x\})$. Furthermore $x \in \tau_q - \text{Cl}(A) \setminus (X \setminus \tau_q - \text{Cl}_\beta(\{x\}))$ and hence $x \in \tau_q - \text{Cl}(A) \cap (X \setminus \tau_q - \text{Cl}_\beta(\{x\}))$. This shows that A is not $(p, q) - \beta_g$ - closed. Now, suppose that A is not $(p, q) - \beta_g$ - closed. There exists $U \in \beta O(X, \tau_p)$ such that $A \subseteq U$ and $\tau_q - \text{Cl}(A) \setminus U \neq \emptyset$. There exists $x \in \tau_q - \text{Cl}(A)$ such that x does not belong to U . Therefore $\tau_q - \text{Cl}_\beta(\{x\}) \cap A = \emptyset$ for some $x \in \tau_q - \text{Cl}(A)$.

3. PROPERTIES OF $(P, Q) - B_G$ - CLOSED SETS

Definition 3.1: A bitopological space (X, τ_p, τ_q) is said to be (p, q) - normal if for disjoint τ_p - closed sets F_1 and F_2 there exists $U_1, U_2 \in \tau_q$ such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$

Theorem 3.2: Let (X, τ_p, τ_q) be a bitopological space. Then the following properties are equivalent:

1. (X, τ_p, τ_q) is (p, q) - normal.
2. For any disjoint τ_p - closed sets F_1, F_2 , there exists $(p, q) - \beta_g$ - open sets V_1, V_2 such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $V_1 \cap V_2 = \emptyset$.
3. For any τ_p - closed sets F , and τ_p - open set U containing F , there exists $(p, q) - \beta_g$ - open set V such that $F \subset U \subset \tau_q - \text{Cl}(V) \subset U$.
4. For any τ_p - closed sets F , and τ_p - open set U containing F , there exists an τ_q - open set G such that $F \subset G \subset \tau_q - \text{Cl}(G) \subset U$.
5. For any disjoint τ_p - closed sets F_1, F_2 , there exists $(p, q) - \beta_g$ - open sets V such that $F_1 \subset V$ and $\tau_q - \text{Cl}(V) \cap F_2 = \emptyset$.
6. For any disjoint τ_p - closed sets F_1, F_2 , there exists an τ_q - open set G such that $F_1 \subset G$ and $\tau_q - \text{Cl}(G) \cap F_2 = \emptyset$.

Proof:

(1) $\Rightarrow$ (2): Clear from Theorem 2.16.

(2) $\Rightarrow$ (3): Let F be a τ_p - closed set $U \in \tau_p$ such that $F \subset U$. Then $F, X \setminus U$ are disjoint τ_p - closed sets by (2) there exists $(p, q) - \beta_g$ - open sets V_1, V_2 such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $V_1 \cap V_2 = \emptyset$. Since V_2 $(p, q) - \beta_g$ open, by using Theorem 2.15, $X \setminus U \subset \tau_q - \text{Int}(V_2)$ and $\tau_q - \text{Int}(V_2)$ is τ_q open. Therefore we obtain $F \subset V_1 \subset \tau_q - \text{Int}(V_1) \subset X \setminus (\tau_q - \text{Int}(V_2)) \subset U$. Put $V = V_1$, then we obtain $F \subset V \subset \tau_q - \text{Cl}(V) \subset U$.

(3) $\Rightarrow$ (4): Let F be a τ_p - closed set $U \in \tau_p$ such that $F \subset U$. Then by using (3), there exists an $(p, q) - \beta_g$ - open set V such that $F \subset V$ $(p, q) - \beta_g \subset U$; By using Theorem 2.15 $F \subset \tau_q - \text{Int}(V)$. Put $G = \tau_q - \text{Int}(V)$. Then $F \subset G \subset \tau_q - \text{Cl}(V) \subset \tau_q - \text{Cl}(V) \subset U$.

(4) $\Rightarrow$ (5): Let F_1, F_2 be any disjoint τ_p - closed sets, Since $X \setminus F_2$ is a τ_p - open set such that $F_1 \subset X \setminus F_2$, by (4) there exists an τ_p - open set V such that $F_1 \subset V \subset \tau_q - \text{Cl}(V) \subset X \setminus F_2$. By using Theorem 2.16, V is $(p, q) - \beta_g$ - open. Furthermore, we have $F_1 \subset V$ and $\tau_q - \text{Cl}(V) \cap F_2 = \emptyset$.

(5) $\Rightarrow$ (6): Let F_1, F_2 , be any disjoint τ_p - closed sets. Then there exists an $(p, q) - \beta_g$ - open set V such that $F_1 \subset V$ and $\tau_q - \text{Cl}(V) \cap F_2 = \emptyset$. By using 2.15 $F_1 \subset \tau_q - \text{Int}(V)$. Then we have $G \in \tau_q$, $F_1 \subset G$ and $\tau_q - \text{Cl}(G) \cap F_2 = \emptyset$.

(6) $\Rightarrow$ (1): Let F_1, F_2 be any disjoint τ_p - closed sets. Then by (6) there exists $G \in \tau_q$ such that $F_1 \subset G$ and $\tau_q - \text{Cl}(G) \cap F_2 = \emptyset$. Then U_1, U_2 , be any disjoint τ_p - open sets. $F_1 \subset U_1$ and $F_2 \subset U_2$. This shows that (X, τ_p, τ_q) is (p, q) - normal.

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