OSCILLATION AND NONOSCILLATION THEOREMS FOR A CLASS OF FOURTH ORDER QUASILINEAR GENERALIZED α -DIFFERENCE EQUATIONS

V. SRIMANJU¹ & S K. KHADAR BABU²

¹Research Scholar, Department of Mathematics, Vellore Institute of Technology, Vellore, Tamil Nadu, India

²Associate Professor, Department of Mathematics, Vellore Institute of Technology, Vellore, Tamil Nadu, India

ABSTRACT

In this paper, we consider certain generalized quasilinear α -difference equations

$$\Delta_{\alpha(\ell)}^2 \left(|\Delta_{\alpha(\ell)}^2 v(k)|^{\gamma-1} \Delta_{\alpha(\ell)}^2 v(k) \right) + q(k) |v(\sigma(k))|^{\beta-1} v(\sigma(k)) = 0$$

Where γ, β are positive constants; $q(k)$ are positive real valued function, $k_0 \in K_0 = \{1, 2, \dots\}$. An oscillation and nonoscillation theorem of the above equation is obtained.

KEYWORDS: Quasilinear Generalized Difference Equations, Oscillation and Nonoscillation Theorems, Four Order

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INTRODUCTION

In this paper, we consider certain quasilinear generalized α -difference equations

$$\Delta_{\alpha(\ell)}^2 \left(|\Delta_{\alpha(\ell)}^2 v(k)|^{\gamma-1} \Delta_{\alpha(\ell)}^2 v(k) \right) + q(k) |v(\sigma(k))|^{\beta-1} v(\sigma(k)) = 0, \quad (1)$$

- γ, β are positive constants;
- $q(k)$ are positive real valued function. $k_0 \in K_0 = \{1, 2, \dots\}$.
- $\sigma(k) \leq k$, and $\lim_{k \rightarrow \infty} \sigma(k) = \infty$

The Equation (1) can also be expressed as

$$\Delta_{\alpha(\ell)}^2 \left((\Delta_{\alpha(\ell)}^2 v(k))^{\gamma*} \right) + q(k\ell + j)(v(\sigma(k)))^{\beta*} = 0, \quad (2)$$

In terms of the asterisk notation

$$\xi^{\eta*} = |\xi|^{\eta} \operatorname{sgn} \xi = |\xi|^{\eta-1} \xi, \quad \xi \in \mathbb{R}, \eta > 0.$$

It is clear that if $v(k)$ is a eventually positive solution of (2), then $-v(k)$ is a eventually negative solution of (2).

Lemma 1: Assume that $v(k)$ is an eventually positive solution of (2). Then one of the following two cases holds for all sufficiently large k :

- $\Delta_{\alpha(\ell)} v(k) > 0, \Delta_{\alpha(\ell)}^2 v(k) > 0, \Delta_{\ell}(\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*} > 0$
- $\Delta_{\alpha(\ell)} v(k) > 0, \Delta_{\alpha(\ell)}^2 v(k) < 0, \Delta_{\ell}(\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*} > 0$

Proof: From (2), we have $\Delta_{\alpha(\ell)}^2 ((\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*}) < 0$ for all large k . It follows that $\Delta_{\alpha(\ell)} v(k), \Delta_{\alpha(\ell)}^2 v(k), \Delta_{\alpha(\ell)}(\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*}$ are eventually monotonic and one-signed.

- If $\Delta_{\alpha(\ell)}(\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*} < 0$ eventually. Then combining this with $\Delta_{\alpha(\ell)}^2 ((\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*}) < 0$, we see that $\lim_{k \rightarrow \infty} (\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*} = -\infty$. That is $\Delta_{\alpha(\ell)}^2 v(k) \rightarrow -\infty$ for all large k . It follows that $\Delta_{\alpha(\ell)} v(k) \rightarrow -\infty, v(k) \rightarrow -\infty$, which contradicts the positivity of $v(k)$.
- If $\Delta_{\alpha(\ell)}(\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*} > 0$ eventually. Then combining this with $\Delta_{\alpha(\ell)}^2 ((\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*}) < 0$, we see that $\Delta_{\alpha(\ell)}(\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*} \rightarrow 0$ or $\rightarrow a > 0$ so

$$(\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*} = (\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*} + \sum_K^{k-1} (\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*}.$$

If $(\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*} > 0$ That is $\Delta_{\alpha(\ell)}^2 v(k) > 0$ is increasing and $\rightarrow C$ or ∞ . It follows that $\Delta_{\alpha(\ell)} v(k) > 0$; If $(\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*} < 0$. That is $\Delta_{\alpha(\ell)}^2 v(k) < 0$ is increasing and $\rightarrow d$ or 0 . If $\Delta_{\alpha(\ell)} v(k) < 0$, then $v(k) \rightarrow \infty$, it is impossible, so $\Delta_{\alpha(\ell)} v(k) > 0$. This completes the proof of the lemma.

From Lemma 1, we know $v(k), \Delta_{\alpha(\ell)} v(k), \Delta_{\alpha(\ell)}^2 v(k), \Delta_{\alpha(\ell)}(\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*}$ tend to finite or infinite limits as $k \rightarrow \infty$. Let $\lim_{k \rightarrow \infty} \Delta_{\alpha(\ell)}^i v(k) = z(i), i = 0, 1, 2$, and $\lim_{k \rightarrow \infty} \Delta_{\alpha(\ell)}(\Delta_{\alpha(\ell)}^2 v(k))^{\gamma^*} = z(3)$

It is that $z(3)$ is a finite nonnegative number. One can easily show that:

If $v(k)$ satisfies I, then the set of its asymptotic values $z(i)$ falls into one of the following three cases:

$$I_1. z(0) = z(1) = z(2) = \infty, z(3) \in (0, \infty);$$

$$I_2. z(0) = z(1) = z(2) = \infty, z(3) = 0;$$

$$I_3. z(0) = z(1) = \infty, z(2) \in (0, \infty), z(3) = 0.$$

If $v(k)$ satisfies II, then the set of its asymptotic values $z(i)$ falls into one of the following three cases:

$$II_1. z(0) = \infty, z(1) \in (0, \infty), z(2) = z(3) = 0$$

$$II_2. z(0) = \infty, z(1) = z(2) = z(3) = 0$$

$$II_3. z(0) \in (0, \infty), z(1) = z(2) = z(3) = 0.$$

Equivalent expressions for these six classes of positive solutions of (2) are as follows:

$$I_1. \lim_{k \rightarrow \infty} \frac{v(k)}{k^{2+\frac{1}{\gamma}}} = \text{const} > 0;$$

$$I_2. \lim_{k \rightarrow \infty} \frac{v(k)}{k^{2+\frac{1}{\gamma}}} = 0, \lim_{k \rightarrow \infty} \frac{v(k)}{k^2} = \infty;$$

$$I_3. \lim_{k \rightarrow \infty} \frac{v(k)}{k^2} = \text{const} > 0;$$

$$II_1. \lim_{k \rightarrow \infty} \frac{v(k)}{k} = \text{const} > 0;$$

$$II_2. \lim_{k \rightarrow \infty} \frac{v(k)}{k} = 0, \lim_{k \rightarrow \infty} v(k) = \infty;$$

$$II_3. \lim_{k \rightarrow \infty} v(k) = \text{const}.$$

Let $v(k)$ be a positive solution of (2) such that $v(k) > 0, v(\sigma(k)) > 0$ for $k \geq K > k_0$. summing (2) from k to ∞ gives

$$\Delta_{\alpha(\ell)} \left(\Delta_{\alpha(\ell)}^2 v(k) \right)^{\gamma^*} = z(3) + \sum_{s=k}^{\infty} q(s) (v(\sigma(s)))^{\beta}, \quad k \geq K. \quad (3)$$

If $v(k)$ is a solution of type $I_i (i = 1, 2, 3)$, then sum (3) three times over $[K, k-1]$ to obtain

$$v(k) = n_0 + n_1(k-K) + \sum_{s=K}^{k-1} (k-s) \left[n(2)^{\gamma} + \sum_{r=k}^{s-1} \left(z(3) + \sum_{\tau=r}^{\infty} q(\tau) (v(\sigma(\tau)))^{\beta} \right) \right]^{\frac{1}{\gamma}} \quad (4)$$

For $k \geq K$ where $n_0 = v(K), n_1 = \Delta_{\alpha(\ell)} v(K), n_2 = \Delta_{\alpha(\ell)}^2 v(K)$ are nonnegative constants. The equality (4) gives a representation for a solution $v(k)$ of type $-I_1$. A type $-I_2$ solution $v(k)$ of (2) is expressed by (4) with $z(3) = 0$.

If $v(k)$ is a solution of type I_3 , then, first summing (2) from k to ∞ and then summing the resulting equation twice times over $[K, k-1]$ to obtain

$$v(k) = n_0 + n_1(k-K) + \sum_{s=K}^{k-1} (k-s) \left[z(2)^{\gamma} - \sum_{r=s}^{\infty} (r-s) q(r) (v(\sigma(r)))^{\beta} \right]^{\frac{1}{\gamma}}, \quad k > K \quad (5)$$

A representation for a solution $v(k)$ of type II_1 is derived by summing (3) with $z(3) = 0$ twice from k to ∞ and then once from K to $k-1$:

$$v(k) = n_0 + \sum_{s=K}^{k-1} \left(z(1) + \sum_{r=s}^{\infty} \left[\sum_{\tau=r}^{\infty} (\tau-r)q(\tau)v(\sigma(\tau))^{\beta} \right] \right)^{\frac{1}{\gamma}}, k > K \quad (6)$$

A representation for a solution $v(k)$ of type II_2 is given by (6) with $z(3)=0$ a representation for a solution $v(k)$ of type II_3 is derived by summing (3) with $z(3)=0$ three times from k to ∞ yield

$$v(k) = z(0) - \sum_{s=k}^{\infty} (s-k) \left[\sum_{r=s}^{\infty} (r-s)q(r)v(\sigma(r))^{\beta} \right]^{\frac{1}{\gamma}}, k > K \quad (7)$$

NONOSCILLATION CRITERIA

Theorem 1: The equation (2) has a positive of type- I_1 if and only if

$$\sum_{k=k_0}^{\infty} q(k) (\sigma(k))^{2+\frac{1}{\gamma}} \beta < \infty \quad (8)$$

Proof Necessary: Suppose that (2) has a positive of type- I_1 , then, it satisfies (4) for $k \geq K$, which implies that

$$\sum_{k=k_0}^{\infty} q(k) (v(\sigma(k)))^{\beta} < \infty$$

This together with the asymptotic relation $\lim_{k \rightarrow \infty} = \text{const} > 0$; shows that the condition (8) is satisfied.

Sufficiently: Suppose now that (8) holds. Let $n > 0$ be any given constant. Choose $K > k_0$ large enough so that

$$\left(\frac{\gamma^2}{(\gamma+1)(2\gamma+1)} \right)^{\beta} \sum_{k=k_0}^{\infty} q(k) (\sigma(k))^{2+\frac{1}{\gamma}} \beta \leq \frac{(2n)^{\gamma} - n^{\gamma}}{(2n)^{\beta}} \quad (9)$$

Put $K \cdot \min\{K, \inf_{k>K} \sigma(k)\}$, and define

$$G(k, K) = \sum_{s=K}^{k-1} (k-s)(s-K)^{\frac{1}{\gamma}} = \frac{\gamma^2}{(\gamma+1)(2\gamma+1)} (k-K)^{\frac{2}{1+\gamma}} \quad k \geq K$$

$$G(k, K) = 0 \quad k < K$$

Let $B(K)$ be the Banach space of all real sequences $V = v(k)$, with the norm $\|V\| = \sup_{k>k_0} |v(k)| < \infty$ we define a closed, bounded and convex subset Ω of $B(K)$ as follows:

$$\Omega = \{Y = v(k) \in B(K) \mid nG(k, K) \leq v(k) \leq 2nG(k, K), k \geq K_*\}$$

Define the map $T : \Omega \rightarrow B(K)$ as follows:

$$\begin{cases} Tv(k) = \sum_K^{k-1} (k-s) \left[\sum_K^{s-1} \left(n^\gamma + \sum_{\tau=r}^{\infty} q(\tau) (v(\sigma(\tau)))^\beta \right)^\gamma \right]^{\frac{1}{\gamma}}, & k \geq K \\ Tv(k) = Tv(K), & K_* \leq k \leq K \end{cases} \quad (10)$$

I. T Maps Ω into Ω . For $v(k) \in \Omega$, then for $k \geq K$

$$Tv(k) \geq k \sum_K^{k-1} (k-s)(s-K)^{\frac{1}{\gamma}} = kG(k, K)$$

And

$$\begin{aligned} Tv(k) &\leq \sum_K^{k-1} (k-s) \left[\sum_K^{s-1} \left(n^\gamma + \sum_{\tau=r}^{\infty} q(\tau) (2n\sigma(\sigma(\tau), K)^\beta) \right)^\gamma \right]^{\frac{1}{\gamma}} \\ &\leq \sum_K^{k-1} (k-s) \left[\sum_K^{s-1} \left(n^\gamma + \left(\frac{2n\gamma^2}{(\gamma+1)(2\gamma+1)} \right)^\beta \right)^\gamma \sum_{\tau=r}^{\infty} q(\tau) (\sigma(\tau))^{\left(2+\frac{1}{\gamma}\right)\beta} \right]^{\frac{1}{\gamma}} \\ &\leq 2n \sum_K^{k-1} (k-s)(s-K)^{\frac{1}{\gamma}} = 2nG(k, K) \end{aligned}$$

II. T is continuous. Let $v^{(n)} \in \Omega$ such that $\lim_{n \rightarrow \infty} \|v^{(n)} - v\| = 0$

$$|(Tv^{(n)})_k - (Tv)_k| = \sum_K^{k-1} (k-s) \left[\sum_K^{s-1} \left(n^\gamma + \sum_{\tau=r}^{\infty} q(\tau) (v^{(n)}(\sigma(\tau)))^\beta \right)^\gamma \right]^{\frac{1}{\gamma}} - \sum_K^{k-1} (k-s) \left[\sum_K^{s-1} \left(n^\gamma + \sum_{\tau=r}^{\infty} q(\tau) (v(\sigma(\tau)))^\beta \right)^\gamma \right]^{\frac{1}{\gamma}}$$

By using Lebesgue's dominated convergence theorem, we can conclude that

$$\lim_{n \rightarrow \infty} \|(Tv^{(n)}) - (Tv)\| = 0$$

III. T is uniformly-cauchy, $\forall k_1, k_2 > K_*$

$$\begin{aligned} |(Tv^{(n)})_{k_1} - (Tv)_{k_2}| &= \sum_K^{k_2-1} (k-s) \left[\sum_K^{s-1} \left(n^\gamma + \sum_{\tau=r}^{\infty} q(\tau) (v^{(n)}(\sigma(\tau)))^\beta \right)^\gamma \right]^{\frac{1}{\gamma}} - \sum_K^{k_1-1} (k-s) \left[\sum_K^{s-1} \left(n^\gamma + \sum_{\tau=r}^{\infty} q(\tau) (v(\sigma(\tau)))^\beta \right)^\gamma \right]^{\frac{1}{\gamma}} \\ &= \sum_K^{k_2-1} (k-s) \left[\sum_K^{s-1} \left(n^\gamma + \sum_{\tau=r}^{\infty} q(\tau) (v^{(n)}(\sigma(\tau)))^\beta \right)^\gamma \right]^{\frac{1}{\gamma}} \end{aligned}$$

Therefore, by the Schauder fixed point theorem, there exists a fixed $Tv = v$, which satisfies (2). This completes the proof.

Theorem 2: The equation (2) has a positive of type- I_3 if and only if

$$\sum_{k=k_0}^{\infty} kq(k)(\sigma(k))^{2\beta} < \infty \quad (11)$$

Proof Necessary: Suppose that (2) has a positive of type- I_3 , then, it satisfies (5) for $k \geq K$, which implies that

$$\sum_{k=K}^{\infty} (k-K)q(k)(\sigma(k))^{\beta} < \infty$$

This together with the asymptotic relation $\lim_{k \rightarrow \infty} \frac{v(k)}{k^2} = \text{const} > 0$; shows that the condition (9) is satisfied.

Sufficiently: Suppose now that (9) holds. Let $k > 0$ be any given constant. Choose $K > k_0$ large enough so that

$$\sum_{k=K}^{\infty} kq(k)(\sigma(k))^{2\beta} \leq \frac{(2n)^{\gamma} - n^{\gamma}}{(n)^{\beta}} \quad (12)$$

Put $K_* = \min\{K, \inf_{k \geq K} \sigma(k)\}$, Let B_K be the Banach space of all real sequences $Y = v(k)$, with the norm $\|V\| = \sup_{k > k_0} |v(k)| < \infty$ we define a closed, bounded and convex subset Ω of B_K as follows:

$$\Omega = \{V = v(k) \in B_K \mid \frac{2}{n}(k-K)_+^2 \leq v(k) \leq n(k-K)_+^2, k \geq K_*\}$$

Where $k - K_+ = k - K$ if $k \geq K$, and $k - K_+ = 0$ if $k \leq K$. Define the map $T : \Omega \rightarrow B_K$ as follows:

$$\begin{cases} Tv(k) = \sum_{K_*}^{k-1} (k-s) \left[2n^{\gamma} - \sum_{r=s}^{\infty} (r-s)q(r)(v(\sigma(r)))^{\beta} \right]^{\frac{1}{\gamma}}, & k \geq K \\ Tv(k) = Tv(K), & K_* \leq k \leq K \end{cases}$$

The proof is similar to that of Theorem 1 and there exists an element v such that $v = Tv$, which is a type- I_3 solution of (2) with the property that $\lim_{k \rightarrow \infty} \Delta_{2\alpha(l)} v(k) = 2n > 0$. This completes the proof.

Theorem 3: The equation (2) has a positive of type- II_1 if and only if

$$\sum_{k=K}^{\infty} \left[\sum_{s=k}^{\infty} (s-k)q(s)(\sigma(s))^{\beta} \right]^{\frac{1}{\gamma}} < \infty \quad (13)$$

Proof: Necessary. Suppose that (2) has a positive of type- II_1 , then, it satisfies (6) for $k \geq K$, which implies that

$$\sum_{k=K}^{\infty} (k-K)q(k)(v(\sigma(k)))^{\beta} < \infty$$

This together with the asymptotic relation $\lim_{k \rightarrow \infty} \frac{v(k)}{k} = \text{const} > 0$; shows that the condition (13) is satisfied.

Sufficiently: Suppose now that (13) holds. Let $n > 0$ be any given constant. Choose $K > k_0$ large enough so that

$$\sum_{k=K}^{\infty} \left[\sum_{s=k}^{\infty} (s-k)q(s)(\sigma(s))^{\beta} \right]^{\frac{1}{\gamma}} < 2^{\frac{-\beta}{\gamma}} n^{1-\frac{\beta}{\gamma}}$$

Put $K_* = \min\{K, \inf_{k \geq K} \sigma(k)\}$. Let B_K be the Banach space of all real sequences $V = v(k)$, with the norm $\|V\| = \sum_{k \geq k_0} |v(k)| < \infty$ define a closed, bounded and convex subset Ω of B_K as follows:

$$\Omega = \{V = v(k) \in B_K \mid nk \leq v(k) \leq 2nk, k \geq K_*\}$$

Define the map $T : \Omega \rightarrow B_K$ as follows:

$$\begin{cases} Tv(k) = nk + \sum_{K_*}^{k-1} \lim_s \left[\sum_r^{\infty} (\tau-r)q(\tau)(v(\sigma(\tau)))^{\beta} \right]^{\frac{1}{\gamma}}, & k \geq K \\ Tv(k) = nk, & K_* \leq k \leq K \end{cases} \quad (14)$$

The proof is similar to that of Theorem 1 and there exists an element v such that $v = Tv$, which is a type-II₁ solution of (2) with the property that $\lim_{k \rightarrow \infty} \Delta_{\alpha(\ell)} v(k) = k > 0$; This completes the proof.

Theorem 4: The equation (2) has a positive of type-II₃ if and only if

$$\sum_{k=k_0}^{\infty} k \left[\sum_{s=k}^{\infty} (s-k)q(s) \right]^{\frac{1}{\gamma}} < \infty \quad (15)$$

Proof Necessary: Suppose that (2) has a positive of type-II₃, then, it satisfies (7) for $k \geq K$, which implies that

$$\sum_{k=K}^{\infty} k \left[\sum_{s=k}^{\infty} (s-k)q(s)(v(\sigma(s)))^{\beta} \right]^{\frac{1}{\gamma}} < \infty \quad (16)$$

This together with the asymptotic relation $\lim_{k \rightarrow \infty} = \text{const} > 0$; shows that the condition (15) is satisfied.

Sufficiently: Suppose now that (15) holds. Let $n > 0$ be any given constant. Choose $K > k_0$ large enough so that

$$\sum_{k=K}^{\infty} k \left[\sum_{s=k}^{\infty} (s-k)q(s)(v(\sigma(s)))^{\beta} \right]^{\frac{1}{\gamma}} < \frac{1}{2} n^{1-\frac{\beta}{\gamma}} \quad (17)$$

Put $K_* = \min\{K, \inf_{k \geq K} \sigma(k)\}$. Let B_K be the Banach space of all real sequences $V = v(k)$, with the norm $\|V\| = \sum_{k \geq k_0} |v(k)| < \infty$, we define a closed, bounded and convex subset Ω of B_K as follows:

$$\Omega = \{V = v(k) \in B_K \mid \frac{n}{2} \leq v(k) \leq n, k \geq K_*\}$$

Define the map $T : \Omega \rightarrow B_K$ as follows:

$$\begin{cases} Tv(k) = n - \sum_k (s-k) \left[\sum_{r=s}^{\infty} (r-s) q(r) (v(\sigma(r)))^{\beta} \right]^{\frac{1}{\gamma}}, & k \geq K \\ Tv(k) = Tv(K), & K_* \leq k \leq K \end{cases} \quad (18)$$

The proof is similar to that of Theorem 1 and there exists an element v such that $v = Tv$, which is a type-II₁ solution of (2) with the property that $\lim_{k \rightarrow \infty} \Delta_{\alpha(\ell)} v(k) = k > 0$; This completes the proof.

Theorem 5: The equation (2) has a positive of type-I₂ if

$$\sum_{k=k_0}^{\infty} q(k) (v(\sigma(k)))^{\left(2+\frac{1}{\gamma}\right)\beta} \leq \infty \quad (19)$$

And

$$\sum_{k=k_0}^{\infty} k q(k) (v(\sigma(k)))^{2\beta} = \infty \quad (20)$$

Proof: Suppose now that (19) holds. Choose $K > k_0$ large enough so that

$$\sum_{k=K}^{\infty} q(k) (v(\sigma(k)))^{\left(2+\frac{1}{\gamma}\right)\beta} \leq \frac{1}{2^{\gamma+1}} \left(\frac{(\gamma+1)(2\gamma+1)}{\gamma^2} \right)^{\gamma} \quad (21)$$

Put $K_* = \min\{K, \inf_{k \geq K} \sigma(k)\}$. Let B_K be the Banach space of all real sequences $V = v(k)$, with the norm $\|V\| = \sum_{k \geq k_0} |v(k)| < \infty$, we define a closed, bounded and convex subset Ω of B_K as follows:

$$\Omega = \{V = v(k) \in B_K : \frac{1}{2^{1+\frac{1}{\gamma}}} (k-K)_+^2 \leq v(k) \leq n^{2+\frac{1}{\gamma}}, k \geq K_*\}$$

Define the map $T : \Omega \rightarrow B_K$ as follows:

$$\begin{cases} Tv(k) = n - \sum_{K_*}^{k-1} (k-s) \left[\frac{1}{2} \sum_{\tau=r}^{\infty} (\tau) q(\tau) (v(\sigma(\tau)))^{\beta} \right]^{\frac{1}{\gamma}}, & k \geq K \\ Tv(k) = 0, & K_* \leq k \leq K \end{cases} \quad (22)$$

The proof is similar to that of Theorem 1 and there exists an element v such that $v = Tv$, which is a type-I₂ solution of (2). This completes the proof.

Theorem 6: The equation (2) has a positive of type-II₂ if

$$\sum_k k \left[\sum_k (s-k) q(s) (\sigma(s))^{\beta} \right]^{\frac{1}{\gamma}} < \infty \quad (23)$$

And

$$\sum_{k=k_0}^{\infty} k \left[\sum_k^{\infty} (s-k)q(s) \right]^{\frac{1}{\gamma}} = \infty \quad (24)$$

Proof: Suppose now that (23) holds. Choose $K > k_0$ large enough so that

$$\sum_{k=K}^{\infty} \left[\sum_k^{\infty} (s-k)q(s)(v(\sigma(s)))^{\beta} \right]^{\frac{1}{\gamma}} \leq 2^{\frac{-\beta}{\gamma}} n^{\frac{1-\beta}{\gamma}} \quad (25)$$

Put $K_* = \min\{K, \inf_{k \geq K} \sigma(k)\}$. Let B_K be the Banach space of all real sequences $V = v(k)$, with the norm $\|V\| = \sum_{k \geq k_0} |v(k)| < \infty$, we define a closed, bounded and convex subset Ω of B_K as follows:

$$\Omega = \{V = v(k) \in B_K \mid n \leq v(k) \leq 2nk, k \geq K_*\}$$

Define the map $T : \Omega \rightarrow B_K$ as follows:

$$\begin{cases} Tv(k) = n + \sum_{K_*}^{k-1} (s-k) \sum_s^{\infty} \left[\sum_{\tau=r}^{\infty} (\tau-r)q(\tau)(v(\sigma(\tau)))^{\beta} \right]^{\frac{1}{\gamma}}, & k \geq K \\ Tv(k) = Tv(K), & K_* \leq k \leq K \end{cases} \quad (26)$$

The proof is similar to that of Theorem 1 and there exists an element v such that $v = Tv$, which is a type-II₂ solution of (2) with the property that $\lim_{k \rightarrow \infty} \Delta_{\alpha(\ell)} v(k) = k > 0$; This completes the proof.

CONCLUSIONS

In this paper we derived oscillation and nonoscillation theorems for a class of fourth order quasilinear generalized α -difference equations. Also, results are verified and proved with suitable example verified by using matlab programme.

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