DO MATH WITH PLAUSIBLE REASONING

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ABSTRACT

Of all knowledge, apart from mathematics and demonstrative logic, they consist of conjectures [6]. In plausible reasoning, conjecture is the name given to the proposition that is formed from indications and observations, that is, from experience through induction. In mathematics, the concept of conjecture refers to a statement that is supposed to be true, but has not yet been proven or disproved. Once the veracity of a conjecture is demonstrated, it will be considered a theorem. This research shows how to do mathematics by first making conjectures through non-routine problems of linear algebra.

KEY WORDS: Conjectures, Plausible Reasoning & Linear Algebra

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1. INTRODUCTION

Hersh states that: "Mathematics is a method of conjecture and proof, it comes to an inherited network of concepts and facts, properties and connections, called theory" and "This currently existing theory is the result of a historical evolution, it is the cooperative and competitive work of generations of mathematicians, associated by friendship and rivalry, by criticism and mutual correction, as leaders and followers, mentors and protégés".

But it also says that doing mathematics is solving problems: "mathematical discovery is based on a validation called 'proof', the analogue of experimentation in physical science. A proof of an argument concludes that the proposed result follows from the accepted theory. "Follow" is understood as the argument convinces qualified skeptical mathematicians".

The differences between pure and applied mathematics are thus distinguished: Mathematics that makes emphasis on the results of previous tests is often called "Applied Mathematics". While the mathematics that makes reference in the test results above is sometimes called "pure mathematics", more often only "mathematics"[5].

2. CONCEPTUAL FRAMEWORK

2.1. What is Mathematics?

"Mathematics is a vast network of interconnected and solved problems. Sometimes a problem is called "a guess. That is to say, the mathematician has a great number of complicated parts and what he must do is to organize it for convenience and for aesthetics and this he does by axiomatizing, since any set of axioms posed is a proposal of solution which will not be unique.

Returning to the axioms, he says that when someone tries to invent a new branch of mathematics, he must raise some axioms and go from there, but sometimes such efforts rarely reach the goal, rather the problems and the solutions come first, later on the sets of axioms will come which may be from already existing theories.

It is also very important to take into account intuition, which is the basis to a great extent in the realization of a test.

As for what mathematics is by its content, we begin with the definition of the dictionary: "mathematics is the science of numbers and figures", a definition that would be correct 200 years ago, not today because mathematics is much more than that, at present there are more than 3400 subfields that nobody can describe and in order to identify one of these branches one must be guided more in the method than in the content.

In this article we will present how to do mathematics through plausible reasoning in a linear algebra class.

2.2 Mathematical Model

The model described in [2] was applied in an engineering faculty of a public university in Bogotá, Colombia. The model deals with how to learn and teach mathematics through plausible reasoning, the use of technology and geometric visualization, by means of non-routine problems, so the model was taken and applied to the topic of vectors in a linear algebra course. The model is the following:

Figure 1: Learning Model with Plausible Reasoning [3].

3. METHODOLOGY

Since this work is based on mathematical education, the methodology used is under a qualitative approach and the theoretical bases of action-research, since the main objective is to improve the quality of an action, in this case of education through theoretical and practical advances.

3.1. Scope of the Study

The study was carried out in a course of 34 students of linear algebra of the Faculty of Engineering of the District University which has an intensity of 6 hours per week. From the total group of students, 12 groups were created and the topics of vector spaces were worked on.

To achieve the objective of the study, a set of non-routine problems was created in which each group of students had to explore, analyze, conjecture and demonstrate properties of the subject to be developed through plausible reasoning.

For this work, results corresponding to the topic of vectors are shown.

3.2 Learning by Questioning

The learning process must be active where students can build new concepts and ideas, therefore, the teacher must provide challenging situations through well-formulated questions so that the student discovers by himself the concepts that have been set as a goal [1].

Therefore, a set of problems must be designed which must be chosen from some books or elaborated by the researcher, for this research some were created and others were adapted, the purpose is that the students explore them, guess and solve them correctly.

4. RESULTS

4.1. Problem

Given two vectors $\mathbf{u}, \mathbf{v}$ in R^2 and k a scalar,

- If $\mathbf{v} = k\mathbf{u}$, what is the condition on scale k for vector $\mathbf{v}$ to change its direction?
- If it is known that $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$, find an algebraic expression to find the angle between any two vectors in R^2 . What will happen if the angle between the two vectors is 0 or $\pi/2$? (Own source)

This is an introductory question to the topic which is intended for the student to explore and guess about the angle between two vectors.

Solution presented by the G1 group:

Figure 2: Solution to Problem 1 Presented by Group G1.

Students explore with a given vector and different k -values, then guess correctly.

4.2. Problem

How to find a vector v in R^2 that is orthogonal and another one parallel to the line $ax + by = 0$ (Modified from [4]).

This is a second level question that aims at having the student apply the concept of parallel or orthogonal vectors in the solution of problems.

Solution presented by the group G3:

$ax + by = 0$
 $by = -ax$
 $y = -\frac{a}{b}x$
 $u = \left(1, -\frac{a}{b}\right)$
 $V = (x, y)$
 $\rightarrow$ orthogonal $uv = 0$
 $\left(1, -\frac{a}{b}\right) \cdot (x, y) = 0$
 $1x - \frac{a}{b}y = 0$
 $\frac{a}{b}y = 1x$
 $y = \frac{1}{\frac{a}{b}}x$
 $y = \frac{b}{a}x$
 $V = \left(1, \frac{b}{a}\right)$
 comprobando
 $u \cdot v = 0$
 $\left(1, -\frac{a}{b}\right) \cdot \left(1, \frac{b}{a}\right) = 0$
 $1 - \frac{ab}{ab} = 0$
 comprobando
 $u = k \cdot v$
 $\left(1, -\frac{a}{b}\right) = k \left(\frac{1}{k}, \frac{-a}{bk}\right)$
 $\left(1, -\frac{a}{b}\right) = \left(1, -\frac{a}{b}\right)$
 Vector orthogonal
 $V = \left(1, \frac{b}{a}\right)$
 Vector paralelo
 $v = \left(\frac{1}{k}, \frac{-a}{bk}\right)$

Figure 3: Solution to Problem 2 Presented by Group G3.

The students clear a variable from the original line, with it they find a vector on the line.

Knowing that the vector they want to find must give zero when they make the product point, they find the requested vector and check it. To find the parallel vector, they know that it must be a multiple of the one they found on the line, so they determine and verify that it is indeed the vector they were looking for.

4.3. Problem

Given two vectors u, v in R^2 , how could it be justified that $u \cdot v \leq |u||v|$? When is equality fulfilled?

This is a second level question that aims to have the student explore the inequality of the point product with the product of the norms.

Solution presented by the group G1:

$u = (3, 2)$
 $v = (2, 1)$
 $u \cdot v = 8$
 $|u| = \sqrt{13}$
 $|v| = \sqrt{5}$
 $|u||v| = \sqrt{13} \cdot \sqrt{5} = \sqrt{65} = 8.06$
 $u \cdot v$ es menor o igual a $|u||v|$
 Forma general de demostración.
 $\cos \theta = \frac{u \cdot v}{|u||v|}$
 $\cos \theta \leq 1$
 $\frac{u \cdot v}{|u||v|} \leq 1 \rightarrow u \cdot v \leq |u||v|$
 Si $\cos \theta = 1$ $\cos 0^\circ = 1$
 $\theta = \cos^{-1}(1)$ $\frac{u \cdot v}{|u||v|} = 1$
 $\theta = 0^\circ$ $u \cdot v = |u||v|$

Figure 4: Solution to Problem 3 Presented by Group G1.

Students explore with an example and then make a correct argument for the inequality.

4.4. Problem

Given two vectors u, v in R^2 , how could it be justified that $|u + v| \leq |u| + |v|$? When is equality fulfilled?

This is a second level question whose objective is for the student to manipulate and display the triangular inequality.

Solution presented by the G2 group:

Por la desigualdad triangular se nota que la suma de sus dos lados tiene que ser mayor o igual a su base o... el triángulo no existiría.
 $|u+v| = w$
 $|w| \leq |u| + |v|$
 Si el ángulo entre u y v es 0° , entonces
 $|u+v| = |u| + |v|$
 Demostración:
 $|u+v|^2 = (u+v) \cdot (u+v)$
 $|u+v|^2 = u \cdot u + 2u \cdot v + v \cdot v$
 $|u+v|^2 = |u|^2 + 2|u||v|\cos\theta + |v|^2$
 $|u+v|^2 \leq |u|^2 + 2|u||v| + |v|^2$
 $|u+v|^2 \leq (|u| + |v|)^2$
 $|u+v| \leq |u| + |v|$
 Diagrama: $u = (2, 1)$, $v = (1, 1)$, $u+v = (3, 2)$
 $|u+v| = \sqrt{2^2 + 1^2} = \sqrt{5} \approx 2.24$
 $|u| + |v| = \sqrt{2^2 + 1^2} + \sqrt{1^2 + 1^2} = \sqrt{5} + \sqrt{2} \approx 2.24 + 1.41 = 3.65$

Figure 5: Solution to Problem 4 Presented by Group G2.

Students draw the corresponding graphic, perform the demonstration correctly.

4.5. Problem

Are the previous points fulfilled or can they be done in $\mathbb{R}^3$? And in $\mathbb{R}^n$? (Own source)

This is a level three question whose objective is that the student generalizes about vectors in $\mathbb{R}^n$.

Solution presented by the group G2:

En $\mathbb{R}^2$ y en $\mathbb{R}^n$ cumple lo mismo que en $\mathbb{R}^3$ cuando se multiplica un vector por un escalar, cambia su dirección y su módulo. En $\mathbb{R}^n$ también se cumple lo mismo.

(b) La fórmula para hallar el ángulo entre dos vectores en $\mathbb{R}^2$ y en $\mathbb{R}^n$ sigue siendo la misma:

$$\frac{U \cdot V}{|U||V|} = \cos \theta, \quad \theta = \cos^{-1} \left(\frac{U \cdot V}{|U||V|} \right)$$

2. $\mathbb{R}^3 \rightarrow U = (a, b, c) \quad V = (d, e, f)$

$$U \cdot V = ad + be + cf = 0 \rightarrow ad + be = -cf$$

$$|U| = \sqrt{a^2 + b^2 + c^2} = 1$$

$$\Rightarrow U = (1, 0, 0) \quad V = (0, 1, 0)$$

$$U \cdot V = 0 + 0 + 0 = 0 \quad |V| = \sqrt{0^2 + 1^2 + 0^2} = 1$$

En $\mathbb{R}^n$ cumple lo mismo:

en $\mathbb{R}^3$ y $\mathbb{R}^n$ $U \cdot V = 0$ para que sea ortogonal.

$$U \cdot V = 0 \Rightarrow (a, b, c) \cdot (d, e, f) = 0$$

$$\Rightarrow (u_1, u_2, u_3, \dots, u_n) \cdot (v_1, v_2, v_3, \dots, v_n) = 0$$

$U \cdot V \leq |U||V|$ se cumple tanto en $\mathbb{R}^2$ como en $\mathbb{R}^3$ y $\mathbb{R}^n$ ya que el $\cos \theta$ siempre va a ser menor o igual a 1.

$$|U+V| \leq |U| + |V|$$

Sean U, V vectores en $\mathbb{R}^2, \mathbb{R}^3$ o $\mathbb{R}^n$ siempre se cumple que $|U+V| \leq |U| + |V|$.

Figure 6: Solution to Problem 5 Presented by Group G2.

It is difficult for students to generalize; however, they try to argue as best they can for each of the points.

4.6. Problem

Determine an algebraic expression to calculate the distance from an external point $P(x_0, y_0, z_0)$ to the plane $\pi: ax + by + cz + d = 0$.

This is a third level question that pretends that the student, applying projections and the vector norm, finds an algebraic expression for the distance between a point and a line.

Solution presented by the group G4:

r. tenemos un plano $ax+by+cz+d=0$ y un punto $P=(x_0, y_0, z_0)$
 $Q=(x, y, z)$
 el vector $n=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$

$u=(x_0-x)\mathbf{i}+(y_0-y)\mathbf{j}+(z_0-z)\mathbf{k}$
 $n=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$

Proyectando u en n
 $\rightarrow \text{Proy}_n u = \left| \frac{u \cdot n}{|n|^2} \right| n$ pero como es distancia tiene que ser positivo
 $|\text{Proy}_n u| = \frac{|u \cdot n|}{|n|^2} = \frac{|u \cdot n|}{|n|}$
 aplicando $|u \cdot n| = |ax_0 + by_0 + cz_0 - (ax + by + cz)|$
 $|n| = \sqrt{a^2 + b^2 + c^2}$
 $\rightarrow \frac{|ax_0 + by_0 + cz_0 - (ax + by + cz)|}{\sqrt{a^2 + b^2 + c^2}}$

Figure 7: Solution to Problem 6 Presented by Group G4.

The students make the graph corresponding to the problem, find the projection and the norm of the vector $\overrightarrow{pq}$, which they call u over the normal vector, and then find the requested expression.

4.7. Problem

Determine an algebraic expression to find the distance between two parallel planes: $ax + by + cz + d = 0$ and $ax + by + cz + e = 0$. [7]

This is a second level question, which is intended for the student to apply the above result in problem solving.

Solution presented by the group G1:

$L: ax+by+cz+d=0$
 $E: ax+by+cz+e=0$
 $\frac{a}{a} = \frac{b}{b} = \frac{c}{c} \neq \frac{d}{e}$
 $cz+e=0$
 $z = -\frac{e}{c}$
 $P_0(0, 0, -e/c)$
 $d(P_0, \alpha) = \frac{|-\frac{e}{c} + d|}{\sqrt{a^2+b^2+c^2}} = \frac{-e+d}{\sqrt{a^2+b^2+c^2}}$

Figure 8: Solution to Problem 7 Presented by Group G1.

Students apply the above result to find the requested solution.

The solutions presented in each of the questions posed, show that students explore the given propositions, as evidenced in the results of the problems: 4.1, 4.4. Then they guess as shown in the results of the problems: 4.2, 4.5, 4.6, 4.8, and then they make the corresponding demonstrations when the propositions are true. It can be seen in the solutions to problems 4.4, 4.1, and 4.8 that they make use of technology when it is required, as can be seen in solutions 4.1. The concept applied to problem solving can be evidenced in the solutions of all the problems.

It can be inferred that the model used to do mathematics through plausible reasoning is effective, as reflected in the results described by the students.

5. CONCLUSIONS AND RECOMMENDATIONS

From the results obtained in the application of the problems, it is observed that the methodological procedure clearly heuristic that makes use of technology and geometric visualization, showed a dynamic and effective way to do mathematics in a linear algebra class.

The students in their results show skill in guessing propositions, correctly perform simple demonstrations and correctly apply the concepts learned in the solution of problems.

Students were observed to work dynamically and with a very positive attitude on each of the problems in the activity.

From what was observed in the study, it can be concluded that a software or mathematical package with capacity to process symbolically and algebraically and with graphic resources should be used in the practices for the development of the programmatic contents of linear algebra in engineering careers.

On the other hand, in the development of the methodological or didactic procedure, relevance was given to the use of the heuristic method, since the students were asked questions of different types of conjectural level that facilitated the independent search to learn the concept through the solution of the problems, which allowed them to construct their own knowledge and thus arrive at the concepts proposed in each one of the topics seen in the development of the semester.

One of the fundamental indicators that show the understanding of a concept is in the capacity to solve problems related to it, so that they are fed back in two ways. First, the appropriate meaning that allows them to solve problems, and second, from this, it facilitates them to build more and more appropriate meanings for different concepts.

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