QUALITATIVE ANALYSIS ON SECOND ORDER NEUTRAL DELAY DIFFERENCE EQUATIONS

G. GOMATHI JAWAHAR

Department of Mathematics, Karunya Institute of Technology and Sciences, Coimbatore, Tamil Nadu, India

ABSTRACT

In this paper, the oscillatory and non oscillatory behavior of all solutions of the second order neutral delay difference equations of the form mentioned by the equation is dealt with.

$\Delta(a_n \Delta(x_n + p_n x_{n-l})) - q_n f(x_{n-m}) = 0, .$ Where $n \in N(n_0)$, $\{a_n\}, \{b_n\}, \{c_n\}$ are positive sequences and $l, m > 0$.

Examples are provided to illustrate the main results.

KEYWORDS: Oscillation, Non-Oscillation, Neutral, Delay & Difference Equations

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1. INTRODUCTION

Consider the second order neutral delay difference equation of the form

$$\Delta(a_n \Delta(x_n + p_n x_{n-l})) - q_n f(x_{n-m}) = 0, \quad (1.1)$$

Where, $n \in N(n_0)$, $\{a_n\}, \{b_n\}, \{c_n\}$ are positive sequences and $l, m > 0$. Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$. f is a continuous function such that $uf(u) \neq 0$ for $\forall u \neq 0$.

By a Solution of (1.1), we mean a real sequence $\{x_n\}$ which satisfies (1.1) for all $n \in N(n_0)$. A solution $\{x_n\}$ of (1.1) said to be oscillatory if the terms $\{x_n\}$ of the sequence are not eventually positive or not eventually negative. Otherwise, it is called non oscillatory.

In the past few years, there has been an increasing interest in the study of oscillatory and non oscillatory behavior of all solutions of (1.1). Followings this trend, in this paper, we obtain some sufficient conditions for the oscillation and non oscillation of all solutions of (1.1). Examples are provided to illustrate the main results.

2. MAIN RESULTS

Theorem 2.1

Let ρ_n be defined such that $\Delta \rho_n \geq 0$ and $\Delta(a_{n-m} \Delta \rho_n) \leq 0$ for $\forall n \in N(n_0)$. Also, assume that

$$A_1 : f_{nm} > f_n f_m,$$

$$A_2: \sum_{u=\varepsilon}^{\infty} \frac{u}{f(u)} < \infty, \text{ for some } \varepsilon > 0.$$

If $\lim_{n \rightarrow \infty} \sup \sum_{s=N_0}^n \rho_{s+1} q_s f(1-p_{s-m}) = \infty$, then every solution of equation (1.1) is oscillatory.

Proof: Suppose x_n be a non oscillatory solution of (1.1). Without loss of generality we may assume that

x_n is eventually positive.

$$\text{Define } z_n = x_n + p_n x_{n-1}, \quad (2.1)$$

Now equation (1.1) becomes,

$$\Delta(a_n \Delta z_n) - q_n (z_{n-m} - p_{n-m} x_{n-m-1}) = 0$$

$$\text{From } A_1, \Delta(a_n \Delta z_n) - q_n (1-p_{n-m}) f(z_{n-m}) < 0 \quad (2.2)$$

Define $w_n = \rho_n \{a_n \Delta z_n\} / f(z_{n-m})$ then $w_n > 0$.

Now, $w_n f(z_{n-m}) / \rho_n = a_n \Delta z_n$

Equation (2.2) becomes,

$$\Delta(a_n \Delta z_n) < q_n f(1-p_{n-m}) f(z_{n-m})$$

$$\Delta(w_n f(z_{n-m}) / \rho_n) < q_n f(1-p_{n-m}) f(z_{n-m})$$

$$\{\rho_n \Delta(w_n f(z_{n-m}) / \rho_n) - w_n f(z_{n-m}) \Delta \rho_n\} / \rho_n \rho_{n+1} < q_n f(1-p_{n-m}) f(z_{n-m})$$

$$\{\rho_n w_n f(z_{n+1-m}) - \rho_n w_n f(z_{n-m}) + \rho_n f(z_{n-m}) w_{n+1} - w_n f(z_{n-m}) \rho_{n+1}\} / \rho_n \rho_{n+1} < q_n f(1-p_{n-m}) f(z_{n-m})$$

$$\rho_n f(z_{n-m}) \Delta w_n < q_n f(1-p_{n-m}) f(z_{n-m}) \rho_n \rho_{n+1} - w_n (\rho_n f(z_{n+1-m}) - \rho_{n+1} f(z_{n-m}))$$

$$\Delta w_n < \rho_{n+1} q_n f(1-p_{n-m}) - \frac{\rho_n a_n \Delta z_n \{[\rho_n f(z_{n+1-m}) - \rho_{n+1} f(z_{n-m})] / \rho_n f(z_{n-m})\}}{f(z_{n-m})}$$

$$\Delta w_n < \rho_{n+1} q_n f(1-p_{n-m}) - a_n \Delta z_n \{[\rho_n f(z_{n+1-m}) - \rho_{n+1} f(z_{n-m})] / (f(z_{n-m}))^2\}$$

Since $\{a_n \Delta z_n\}$ is a decreasing sequence, we have

Sequence, $a_n \Delta z_n \leq a_{n-m} \Delta z_{n-m}$, for $n \geq N_0$.

$$\Delta w_n < \rho_{n+1} q_n f(1-p_{n-m}) - \{a_n \Delta z_n \rho_n f(z_{n+1-m}) / (f(z_{n-m}))^2\} + \rho_{n+1} a_n \Delta z_n / f(z_{n-m})$$

$$\Delta w_n < \rho_{n+1} q_n f(1-p_{n-m}) + [\rho_{n+1} a_{n-m} \Delta z_{n-m} / f(z_{n-m})]$$

$$\rho_{n+1} q_n f(1-p_{n-m}) < \frac{\Delta w_n + \rho_{n+1} a_{n-m} \Delta z_{n-m}}{f(z_{n-m})}$$

$\sum_{s=N_0}^n \rho_{s+1} q_s f(1-p_{s-m}) < \infty$, for $n \geq N_0$, which contradicts the given condition. Hence every solution of (1.1) is

oscillatory.

Theorem 2.2

Let x_n be a non oscillatory solution of (1.1) and assume

A_3 : $[F(u)/u] > \gamma > 0$, and $0 \leq p_n \leq 1$.

A_4 : $\sum_{s=n}^{\infty} q_s (1 - p_{s-m}) < \infty$. Suppose, the sequence $\{v_n\}$ is defined as, $v_{n_1} = Q_n$,

$$v_{n_{k+2}} = Q_n + \sum_{s=n}^{\infty} \{Q_s v_{s_{k+1}} / a_{s-m}\}$$

Then, $\lim_{n \rightarrow \infty} \sup \left(\left(\sum_{s=N_0}^{n-m} 1/a_s \right) (-Q_n + \sum_{s=n}^{\infty} (Q_s^2 / a_{s-m})) \right) \leq 1$.

Proof

Let x_n be a non oscillatory solution of (1.1). Without loss of generality assume $x_n > 0$, for $n \geq N_0$.

Let $z_n = x_n + p_n x_{n-1}$

Equation (1.1) becomes,

$$\Delta(a_n \Delta z_n) - q_n f(z_{n-m} - p_{n-m} x_{n-1-m}) = 0$$

$$\Delta(a_n \Delta z_n) = q_n f(z_{n-m} - p_{n-m} x_{n-1-m})$$

$$\text{From } A_3, \Delta(a_n \Delta z_n) - \gamma q_n f(z_{n-m} - p_{n-m} x_{n-1-m}) \leq 0$$

Since $z_n > x_n$,

$$\Delta(a_n \Delta z_n) - \gamma q_n (1 - p_{n-m}) p_{n-m} z_{n-m} \leq 0$$

Define $v_n = a_n \Delta z_n / z_{n-m}$, $v_n > 0$

$$\Delta v_n = z_{n-m} \Delta(a_n \Delta z_n) - a_n \Delta z_n \Delta(z_{n-m}) / z_{n+1-m} z_{n-m}$$

$$\{\Delta v_n z_{n-m} z_{n+1-m} + a_n \Delta z_n \Delta(z_{n-m})\} / z_n = \Delta(a_n \Delta z_n)$$

Equation (2.3) becomes,

$$\{\Delta v_n \cdot z_{n-m} z_{n+1-m} + a_n \Delta z_n \Delta z_{n-m}\} / z_{n-m} \leq \gamma q_n (1 - p_{n-m})$$

$$\Delta v_n z_{n+1-m} + [a_n \Delta z_n \Delta z_{n-m} / z_{n-m}] \leq \gamma q_n (1 - p_{n-m}) z_{n-m}$$

$$\Delta v_n \leq \gamma q_n (1 - p_{n-m}) - \{v_n^2 / a_{n-m}\} \quad (2.4)$$

This implies $\{v_n\}$ is an decreasing sequence and $\lim_{n \rightarrow \infty} v_n$ exists.

Taking summation of (2.4) from n to N ,

$$v_N - v_n \leq \sum_{s=n}^N q_s (1 - p_{s-m}) - \sum_{s=n}^{\infty} \frac{v_s^2}{a_{s-m}} \quad (2.5)$$

$$\text{Since } \lim_{n \rightarrow \infty} v_N \text{ exists, } \sum_{s=n}^{\infty} \frac{v_s^2}{a_{s-m}} < \infty$$

This implies $\lim_{n \rightarrow \infty} v_N = 0$

$$\text{Equation (2.5) becomes, } -v_n \leq Q_n - \sum_{s=n}^{\infty} v_s^2 / a_{s-m}$$

$$v_n \geq -Q_n + \sum_{s=n}^{\infty} v_s^2 / a_{s-m}, \quad n \geq N_0.$$

Since $v_n \geq Q_n$,

$$\sum_{s=N_0}^{\infty} Q_s v_s / a_{s-m} \leq \sum_{s=N_0}^{\infty} v_s^2 / a_{s-m} \quad (2.6)$$

Equation (2.6) becomes,

$$v_n \geq -Q_n + \sum_{s=n}^{\infty} Q_s v_s / a_{s-m}, \quad \text{for } n \geq N_0.$$

Also, $a_n \Delta z_n$ is decreasing sequence for $n \geq N_0$.

$$\text{Thus } \frac{z_{n-m}}{a_n \Delta z_n} = \frac{1}{v_n} = z_{N_0} + \frac{\sum_{s=N_0}^{n-m} \left(\frac{1}{as}\right) a_s \Delta z_s}{a_n \Delta z_n} \geq \left\{ a_s \Delta z_s \sum_{s=N_0}^{n-m} \frac{1}{a_s} \right\} / a_n = \sum_{s=N_0}^{n-m} \frac{1}{as}$$

$$\text{Hence } v_n \sum_{s=N_0}^{n-m} \frac{1}{as} \leq 1$$

$$\text{Since } a_n > 0, \quad \limsup_{n \rightarrow \infty} v_n \sum_{s=N_0}^{n-m} \frac{1}{as} \leq 1. \quad (2.7)$$

$$v_{n_2} = -Q_n + \sum_{s=n}^{\infty} Q_s v_{s_1} / a_{s-m},$$

since $v_{n_2} \leq v_n$, equation (2.7) becomes,

$$\lim_{n \rightarrow \infty} \sup v_{n_2} \sum_{s=N_0}^{n-m} \frac{1}{as} \leq 1 \quad (2.8)$$

Since $v_{s_1} = Q_s$, equation (2.8) becomes,

$$\lim_{n \rightarrow \infty} \sup \left(-Q_n + \sum_{s=n}^{\infty} Q_s^2 / a_{s-m} \right) \sum_{s=N_0}^{n-m} \frac{1}{as} \leq 1. \text{ This completes the proof.}$$

Theorem 2.3

Let x_n be a non oscillatory solution of (1.1) and $z_n = x_n + p_n x_{n-1}$. If $\{a_n\}$ is an increasing sequence, then $\{z_n \Delta z_n\}$ is eventually negative.

Proof

Without loss of generality, we may assume that $x_{n-m-1} > 0$ for $n \geq N_0$.

Then $z_n > 0$.

$$\text{Equation (1.1) becomes } \Delta (a_n \Delta z_n) = q_n f(x_{n-m}) < 0 \text{ for } n \geq N_0. \quad (2.9)$$

Hence, $a_n \Delta z_n$ is decreasing sequence.

To prove $\Delta z_n < 0$

Suppose to the contrary that there exists $n_1 > N_0$

Such that $\Delta z_{n_1} \geq 0$.

From (2.9), $\Delta (a_n \Delta z_n) < 0$.

This implies $\Delta (a_n (z_{n+1} - z_n)) < 0$.

$$\Rightarrow \Delta (a_n z_{n+1} - a_n z_n) < 0.$$

$$a_{n+1} z_{n+2} - a_n z_{n+1} - (a_{n+1} z_{n+1} - a_n z_n) < 0.$$

$$a_n z_n < a_n z_{n+1} + a_{n+1} z_{n+1} - a_{n+1} z_{n+2}.$$

$$z_n < a_n z_{n+1} - a_{n+1} \Delta z_{n+1}. \quad (3.0)$$

Hence $z_n < 0$.

This contradicts the fact that $z_n > 0$. Hence $\Delta z_n < 0$. This completes our proof.

3. EXAMPLES**Example 3.1**

Consider the difference equation

$$\Delta (n \Delta (x_n + n x_{n-1})) - (2n-1) x_{n-1}^3 = 0, \text{ for } n > 0 \quad (3.1)$$

Here $a_n = n$, $p_n = n$, $q_n = 2n-1$, $l = 1$, $m = 1$

All conditions of theorem (2.1) are satisfied. Hence all solutions of (3.1) are oscillatory. one such solution is $(-1)^n$.

Example 3.2

Consider the difference equation,

$$\Delta \left(\frac{1}{n} \Delta \left(x_n + \frac{1}{n-1} x_{n-1} \right) \right) - \frac{x_{n-1}}{n(n+1)(n-1)} = 0, \text{ for } n > 1 \quad (3.2)$$

Here, $a_n = 1/n$, $p_n = 1/n-1$, $q_n = 1/n(n+1)(n-1)$, $l=1$, $m=1$.

All conditions of theorem (2.2) are satisfied and n is a non-oscillatory solution of (3.2).

$$\text{Hence, } \lim_{n \rightarrow \infty} \sup \left(-Q_n + \sum_{s=n}^{\infty} Q_s^2 / a_{s-m} \right) \sum_{s=N_0}^{n-m} \frac{1}{as} \leq 1.$$

4. CONCLUSIONS

In this paper, by using summation averaging techniques and comparison principle some new oscillation criteria for second order neutral delay difference equation is obtained. Examples are provided to illustrate the results. Further none of the results in the papers [3-8] can be applied to Eq. (1.1) to yield any conclusion.

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