ANALYSIS OF SOLIDIFICATION THICKNESS DURING PHASE CHANGE USING HEAT BALANCE INTEGRAL METHOD

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ABSTRACT

The objective of the current research work is to predict the time required for the completion of the solidification process and also to predict the temperature distribution in the solidified region of a liquid kept inside and outside cylindrical and spherical vessels. Whose walls are kept at constant sub melting temperature? The heat balance integral technique incorporating special subdivisions has been used and the obtained systems of equations have been solved by fourth order Runge-kutta method. Although a number of assumption have been made in the problem specification and change of flux has been approximated by the discontinues change in the adjacent profile gradients. The acceptable estimates of both the temperature and the flux have been obtained by using small sub-divisions ($n=7$) and incremental time (initially $t=2 \times 10^{-12}$). An approximate analytical approach has been made to predict solid-liquid interface position at any time. Results obtained by both numerical and analytical methods, have been compared and an estimate of the errors has been established, assuming heat balance integral (numerical) method to be the more correct. The results show that the percentage of error while using approximate analytical method is less for higher values of latent parameters and high for lower values. The percentage error $\varepsilon=0\%$ at $\alpha=9.4$ and $\varepsilon=0\%$ at $\alpha=12.3$ for inside solidification process of cylindrical and spherical vessel respectively. For the only case of solidification outside the spherical vessel, the percentage error is up to a maximum of 25.425% even for the lowest value of $\alpha=0$.

KEYWORDS: Temperature Distribution, Sub Melting Temperature, Profile Gradients, Solid-Liquid Interface Position, 7 Heat Balance Integral Method

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INTRODUCTION

There is a considerable interest in the group of problems known as the Stefan problem. This is comparatively little information of an analytical or approximate technique available and so the development of numerical methods to deal with the Stefan problems is highly desirable. Transient heat-conduction problems involving melting or solidification generally referred to as “Phase-change” or “moving boundary” problems are important in many engineering applications such as in the making of ice, freezing of food, the solidification of metal in casting, and the cooling of large masses of igneous rock.

Heat transfer problem involving phase-change can be of two types,

- When whole liquid is at its melting temperature that is no convection phenomenon in the liquid region

- The liquid is at a higher temperature than its melting point and convection phenomenon is included.

T. R. Goodman and Boston, Mass (1) presented an approximate mathematical method for solving heat-transfer problems utilizing the heat-balance integral and applied to five problems involving a change of phase. Analytical expressions are driven when (i) Boundary temperature is fixed (ii) Heat flux at the boundary is given (iii) Heat flux is generated aerodynamically or by radiations (iv) Heat flux at the boundary is given and the melt is completely removed (v) Heat flux at the boundary is given and at time t_0 melt begins to vaporize. Graham E. Bell (2) predicted the temperature distribution and the rate of removal of heat by a coolant for the process of solidification of a liquid about a cold isothermal pipe. The heat balance integral method incorporating special sub-divisions together with a piecewise linear profile is used. W. W. Yuen (3) applied the heat-balance integral method to melting problems with initial sub-cooling. A refinement of the heat-balance integral method as described by the G. E. Bell (4) has been successfully applied to the problem of the solidification of a cylindrical pipe. The author in (5) has demonstrated how the incorporation of special sub-division overcomes the sensitivity previously observed in the heat-balance integral method. N. K. Samria (6) studied solidification of liquid inside a spherical vessel using heat-balance integral method incorporating special sub-division and compared his results with an approximate method with simplified assumptions.

METHODOLOGY

Approximate Method

The following governing equation for temperature fields can be given to represent the solidification process for cylindrical 'a'.

For the liquid region

$$\frac{\partial^2 T_l(r,t)}{\partial r^2} + \frac{r}{r} \frac{\partial T_l(r,t)}{\partial r} = \frac{1}{a_l} \frac{\partial T_l(r,t)}{\partial t} \quad (1)$$

$0 < r < R(t)$, $t > 0$ for inside solidification

$R(t) < r < \infty$, $t > 0$ for outside solidification

For solidified region

$$\frac{\partial^2 T_s(r,t)}{\partial r^2} + \frac{r}{r} \frac{\partial T_s(r,t)}{\partial r} = \frac{1}{a_s} \frac{\partial T_s(r,t)}{\partial t} \quad (2)$$

$R(t) < r < a$, $t > 0$ for inside solidification

$R(t) < r < \infty$, $t > 0$ for outside solidification

$\Gamma = 1$ for cylindrical case.

The boundary conditions for the general case for the above solidification problem will be:

For inside solidification problem

$$\frac{\partial T_l(r,t)}{\partial r} = 0, \text{ at } r = 0, t > 0. \quad (3)$$

$$T_s(r, t) = T_l(r, t) = T_m \text{ at } r = R(t), t > 0$$

$$T_s(r, t) = T_w \text{ at } r = a, t > 0$$

For outside solidification problem

$$\frac{\partial T_l(r,t)}{\partial r} = 0, \text{ at } r \rightarrow \infty, t > 0$$

$$T_s(r,t) = T_l(r,t) = T_m \text{ at } r = R(t), t > 0$$

$$T_s(r,t) = T_w \text{ at } r = a, t > 0$$

At solid-liquid interface,

$$K_s \frac{\partial T_s(r,t)}{\partial r} - K_l \frac{\partial T_l(r,t)}{\partial r} = \rho_s L_s \frac{dR(t)}{dt}. \quad (4)$$

At $r = R(t)$, $t > 0$ for both inside and outside cases of solidification.

The initial conditions are given below.

$$\text{At } t=0, T_l(r,t) = T_0 = T_m \text{ for all } r, t = 0, R(0) = a$$

The above problem has been simplified by considering that the initial temperature T_m of the liquid is equal to constant, so the variation of temperature in the liquid region becomes zero. The equation of solidified region is applicable given by the simplified statement of the problem as:

$$\frac{\partial^2 T_s(r,t)}{\partial r^2} + \frac{r}{r} \frac{\partial T_s(r,t)}{\partial r} = \frac{1}{a_s} \frac{\partial T_s(r,t)}{\partial t}. \quad (5)$$

$R(t) < r < a$, $t > 0$ for inside solidification

$a < r < R(t)$, $t > 0$ for outside solidification

$$T_s(r,t) = T_w \text{ at } r = a, t > 0 \text{ for both inside and outside solidification problems.}$$

$$T_s(r,t) = T_m \text{ at } r = R(t), t > 0$$

for both inside and outside solidification problems.

$$K_s \frac{\partial T_s(r,t)}{\partial r} = \rho_s L_s \frac{dR(t)}{dt}. \quad (6)$$

At $r = R(t)$, $t > 0$ for both inside and outside cases of solidification.

The initial conditions are given by

$$T_l(r,t) = T_m \text{ at } t = 0, \text{ at all } r.$$

$$R(0) = a$$

The following non-dimensional parameters have been adopted in order to generalize the problem:

$$u(z, \tau) = \frac{T_s(r,t) - T_w}{T_m - T_w}$$

$$z = \frac{r}{a}, z(\tau) = \frac{R(t)}{a}, \tau = \frac{a_s t}{a^2}$$

After being non-dimensionalized, the above equations take the form:

$$\frac{\partial^2 u(z, \tau)}{\partial z^2} + \frac{z}{z} \frac{\partial u(z, \tau)}{\partial z} = \frac{\partial u(z, \tau)}{\partial \tau}. \quad (7)$$

$Z(\tau) < z < 1, \tau > 0$ for inside solidification.

$1 < z < Z(\tau), \tau > 0$ for outside solidification

The boundary conditions for both cases of inside and outside solidification process are given as,

$$u(z, \tau) = 0 \text{ at } z = 1, \tau > 0 \quad (8)$$

$$u(z, \tau) = 1 \text{ at } z = Z(\tau), \tau > 0 \quad (9)$$

$$\frac{\partial u(z, \tau)}{\partial z} = \alpha \frac{\partial Z(\tau)}{\partial \tau} \text{ at } z = Z(\tau), \tau > 0. \quad (10)$$

Where,

$$\alpha = \frac{1}{Ste} = \frac{L_s}{C_s(T_m - T_w)} = \text{Non-dimensional latent heat parameter, with initial condition given as, } u(z, 0) = 1 \text{ at}$$

$\tau = 0$ for all z .

For the approximate solution of the problem, we assume that the time dependence of the temperature in the solidified region can be neglected. So the temperature profile in the solidified region can be assumed to be stationary for the time being. This enables us to derive the analytical formulae for the temperature distribution in a radial direction. From this temperature profile we get the value of temperature gradient $\partial u(z)/\partial z$ at $z=Z(\tau)$, and also we can get an expression for τ in terms of $Z(\tau)$.

$$\text{Since } \frac{\partial u(z)}{\partial \tau} \ll \frac{\partial u(z)}{\partial z}$$

$$\therefore \frac{\partial u(z)}{\partial \tau} \cong 0$$

Applying this approximation, we obtain,

$$\frac{\partial^2 u(z, \tau)}{\partial z^2} + \frac{r}{z} \frac{\partial u(z, \tau)}{\partial z} = 0. \quad (11)$$

With boundary conditions

$$u(z) = 0 \text{ at } z=1, \tau > 0,$$

$$u(z) = 1 \text{ at } z=Z(\tau), \tau > 0,$$

$$\frac{\partial u(z, \tau)}{\partial z} = \alpha \frac{\partial Z(\tau)}{\partial \tau} \text{ at } z = Z(\tau), \tau > 0$$

Initial conditions,

$$u(z) = 1 \text{ at } z=0 \text{ for all } z$$

$$Z(0) = 1$$

HEAT BALANCE INTEGRAL TRANSFORMATION TECHNIQUE

In this technique the non-dimensional temperature u from $u=0$ to 1 is sub-divided into an equal intervals such that at i_{th} interval:

$$u_i = i/n, i = 0, 1, \dots, n$$

A penetration variable is associated with each isotherm created by the sub-division. The position of each isotherm u_i is denoted by penetration variable $Z_i(\tau)$ such that $Z_0(\tau)=1$ constant for all τ , and $Z_n(\tau) = Z(\tau)$ is the position of solid liquid interface.

$$\frac{\partial^2 u}{\partial z^2} + \frac{\Gamma}{z} \frac{\partial u}{\partial z} = \frac{\partial u}{\partial \tau} \quad (12)$$

$$\therefore u_{i+1} - u = \frac{1}{n} \frac{Z_{i+1} - Z}{Z_{i+1} - Z_i}$$

or

$$u = u_{i+1} - \frac{1}{n} \frac{Z_{i+1} - Z}{Z_{i+1} - Z_i}$$

$$= \frac{i+1}{n} - \frac{1}{n} \frac{Z_{i+1} - Z}{Z_{i+1} - Z_i}$$

or

$$u = \frac{i}{n} + \frac{z - Z_i}{n(Z_{i+1} - Z_i)}$$

or

$$uz = \frac{iz}{n} + \frac{z(z - Z_i)}{n(Z_{i+1} - Z_i)}$$

$$\begin{aligned} \therefore \int_{Z_i}^{Z_{i+1}} (uz) dz &= \int_{Z_i}^{Z_{i+1}} \left[\frac{iz}{n} + \frac{z^2 - Z_i z}{n(Z_{i+1} - Z_i)} \right] dz \\ &= \frac{i}{2n} [Z_{i+1}^2 - Z_i^2] + \frac{1}{6n} [Z_{i+1}^3 - Z_i^3 - Z_{i+1}Z_i] \\ \therefore \int_{Z_i}^{Z_{i+1}} \frac{\partial}{\partial \tau} (uz) dz &= \frac{i}{2n} [2\dot{Z}_{i+1}Z_{i+1} - 2\dot{Z}_iZ_i] + \frac{1}{6n} [4\dot{Z}_{i+1}Z_{i+1} - 2\dot{Z}_iZ_i - \dot{Z}_{i+1}Z_i - Z_{i+1}\dot{Z}_i] \\ \therefore \int_{Z_i}^{Z_{i+1}} \frac{\partial}{\partial \tau} (uz) dz &= \frac{\partial}{\partial \tau} \int_{Z_i}^{Z_{i+1}} (uz) dz + \frac{i}{n} \dot{Z}_iZ_i - \frac{i+1}{n} \dot{Z}_{i+1}Z_{i+1} \\ &= \frac{1}{n} [\dot{Z}_{i+1}Z_{i+1} - \dot{Z}_iZ_i] + \frac{1}{6n} [4\dot{Z}_{i+1}Z_{i+1} - 2\dot{Z}_iZ_i - \dot{Z}_{i+1}Z_i - Z_i\dot{Z}_{i+1}] + \frac{i}{n} \dot{Z}_iZ_i - \frac{i+1}{n} \dot{Z}_{i+1}Z_{i+1} \\ \therefore \int_{Z_i}^{Z_{i+1}} \frac{\partial}{\partial \tau} (uz) dz &= -\frac{1}{6n} [2\dot{Z}_{i+1}Z_{i+1} + 2\dot{Z}_iZ_i + \dot{Z}_{i+1}Z_i + \dot{Z}_iZ_{i+1}]. \end{aligned} \quad (13)$$

At the interface,

$$z = Z_n = Z(\tau)$$

$$u_n = 1, Z_0 = 1$$

$$\left. \frac{\partial u}{\partial z} \right|_{z=Z_n} = \alpha \frac{\partial Z_n}{\partial \tau} = \alpha \dot{Z}_n$$

$$\int_{Z_{n-1}}^{Z_n} \frac{\partial}{\partial \tau} (uz) dz = \frac{\partial}{\partial \tau} \int_{Z_{n-1}}^{Z_n} (uz) dz + \dot{Z}_{n-1}u_{n-1}Z_{n-1} - \dot{Z}_n u_n Z_n$$

We know that,

$$\frac{u_n - u_{n-1}}{u_n - u} = \frac{Z_n - Z_{n-1}}{Z_n}$$

or

$$u_n - u = \frac{1}{n} \frac{Z_n - Z}{Z_n - Z_{n-1}}$$

or

$$u = u_n - \frac{1}{n} \frac{Z_n - Z}{Z_n - Z_{n-1}}$$

or

$$uz = z - \frac{Z_n z - z^2}{n(Z_n - Z_{n-1})}$$

or

$$\begin{aligned} \int_{Z_{n-1}}^{Z_n} (uz) dz &= \int_{Z_{n-1}}^{Z_n} \left[z - \frac{Z_n z - z^2}{n(Z_n - Z_{n-1})} \right] dz \\ &= \frac{1}{2} [Z_n^2 - Z_{n-1}^2] - \frac{1}{n(Z_n - Z_{n-1})} \left[\frac{Z_n^2 - Z_{n-1}^2}{2} - \frac{Z_n^3 - Z_{n-1}^3}{3} \right] \end{aligned}$$

or

$$\begin{aligned} \frac{\partial}{\partial \tau} \int_{Z_{n-1}}^{Z_n} (uz) dz &= \frac{1}{2} [2\dot{Z}_n Z_n - 2\dot{Z}_{n-1} Z_{n-1}] - \frac{1}{6n} [2\dot{Z}_n Z_n + \dot{Z}_{n-1} Z_n + \dot{Z}_n Z_{n-1} - 4\dot{Z}_{n-1} Z_{n-1}] \\ &= [\dot{Z}_n Z_n - \dot{Z}_{n-1} Z_{n-1}] - \frac{1}{6n} [2\dot{Z}_n Z_n + \dot{Z}_n Z_{n-1} + \dot{Z}_{n-1} Z_n - 4\dot{Z}_{n-1} Z_{n-1}] \\ \therefore \int_{Z_{n-1}}^{Z_n} \frac{\partial}{\partial \tau} (uz) dz &= \frac{\partial}{\partial \tau} \int_{Z_{n-1}}^{Z_n} (uz) dz + \dot{Z}_{n-1} Z_{n-1} u_{n-1} - \dot{Z}_n Z_n u_n \\ &= [\dot{Z}_n Z_n - \dot{Z}_{n-1} Z_{n-1}] - \frac{1}{6n} [2\dot{Z}_n Z_n + \dot{Z}_{n-1} Z_n + \dot{Z}_n Z_{n-1} - 4\dot{Z}_{n-1} Z_{n-1}] + \frac{n-1}{n} \dot{Z}_{n-1} Z_{n-1} - \dot{Z}_n Z_n \\ \therefore \int_{Z_{n-1}}^{Z_n} \frac{\partial}{\partial \tau} (uz) dz &= -\frac{1}{6n} [2\dot{Z}_n Z_n + 2\dot{Z}_{n-1} Z_{n-1} + \dot{Z}_n Z_{n-1} + \dot{Z}_{n-1} Z_n]. \end{aligned} \quad (14)$$

STATEMENT OF THE PROBLEM

We have chosen cylindrical vessel of radius $a = 28 \text{ cm.} = 0.28 \text{ m}$ and have considered its unit length (1m). The radius of spherical vessels has also been taken as $a = 0.28 \text{ m}$. Water has been chosen as liquid at 0°C and the problem of ice formation has been tackled for five different values of wall temperatures i.e., $T_w = -8^\circ\text{C}, -10^\circ\text{C}, -20^\circ\text{C}, -30^\circ\text{C}$ and -40°C .

The properties of ice at 0°C has been taken as:

$$K_s = 2.22 \times 10^{-3} \text{ KW/mK}$$

$$\rho_s = 913 \text{ Kg/m}^3$$

$$C_s = 1.93 \text{ KJ/kgK}$$

$$a_s = 1.26 \times 10^{-6} \text{ m}^2/\text{sec}$$

$$L_s = 334.88 \text{ KJ/kg}$$

From these properties of ice, the five different values of non-dimensional latent heat parameters have been calculated for five different values of wall temperatures ' T_w ' by the formula $\alpha = L_s / (C_s(T_m - T_w))$. We have sub-divided

$u=0$ to 1 into 7 equal intervals.

For the purpose of solving the problems of cylindrical and spherical vessels, the starting values of Z_7 has been taken as 0.95 for inside solidification and $Z_7=1.05$ for outside solidification. The below tables show percentage errors at five different values of α for all the inside (outside) cylindrical (spherical) cases. Plots of percentage errors versus latent heat parameter have been drawn.

Table 1: Starting Values for Numerical (Runge-Kutta) Method for Solidification Inside Cylindrical Vessel

α	21.689	17.351	8.675	5.784	4.338
τ_0	0.026208	0.020966	0.010482	0.006989	0.005242
Z_0	1	1	1	1	1
Z_1	0.992537	0.992537	0.992537	0.992537	0.992537
Z_2	0.985185	0.985185	0.985185	0.985185	0.985185
Z_3	0.977941	0.977941	0.977941	0.977941	0.977941
Z_4	0.970803	0.970803	0.970803	0.970803	0.970803
Z_5	0.963768	0.963768	0.963768	0.963768	0.963768
Z_6	0.956835	0.956835	0.956835	0.956835	0.956835
Z_7	0.95	0.95	0.95	0.95	0.95

Table 2: Starting Values for Numerical (Runge-Kutta) Method for Solidification Inside Cylindrical Vessel

α	21.689	17.351	8.675	5.784	4.338
τ_0	0.028015	0.022412	0.011205	0.007471	0.005603
Z_0	1	1	1	1	1
Z_1	1.006849	1.006849	1.006849	1.006849	1.006849
Z_2	1.013793	1.013793	1.013793	1.013793	1.013793
Z_3	1.020833	1.020833	1.020833	1.020833	1.020833
Z_4	1.027972	1.027972	1.027972	1.027972	1.027972
Z_5	1.035211	1.035211	1.035211	1.035211	1.035211
Z_6	1.042553	1.042553	1.042553	1.042553	1.042553
Z_7	1.05	1.05	1.05	1.05	1.05

Table 3: Variation of Thickness of Solidified Region With Time and Rate of Heat Out through the Wall of Cylindrical Vessel for Inside Solidification for ($\alpha=8.675$)

Approximate Method			Runge-Kutta Method		
$Z(\tau)$	τ	Q(watts)	$Z(\tau)$	τ	Q(watts)
0	2.61875		0.01675	2.18486	143.17
0.1	2.04719	122.7979	0.09023	2.08486	165.26
0.2	1.80276	175.684	0.21339	1.78486	219.59
0.3	1.50356	234.8496	0.29681	1.53486	259.15
0.4	1.18584	308.5838	0.40702	1.18486	341.27
0.5	0.87493	407.9256	0.50142	0.88986	434.87
0.6	0.59035	553.5206	0.60082	0.60486	551.1
0.7	0.34799	792.7455	0.70151	0.35486	824.14
0.8	0.1613	1267.133	0.80098	0.16486	1311.37
0.9	0.04189	2683.666	0.90014	0.04286	2752.12
1	0		1	0	

Table 4: Variation of Thickness of Solidified Region with Time and Rate of Heat Out Through the Wall of Cylindrical Vessel for Inside Solidification for ($\alpha = 5.784$)

Approximate Method			Runga-Kutta Method		
Z(τ)	τ	Q(watts)	Z(τ)	τ	Q(watts)
0	1.486		0.01252	1.5083	229.17
0.1	1.36495	184.1755	0.09291	1.4313	261.79
0.2	1.20198	263.4956	0.21002	1.2313	337.64
0.3	1.00249	352.2338	0.30147	1.0413	414.69
0.4	0.79065	462.8224	0.40191	0.8213	524.37
0.5	0.58335	611.8179	0.50205	0.6063	675.08
0.6	0.39361	830.1852	0.60094	0.4113	895.1
0.7	0.23202	1188.981	0.70182	0.2413	1266.34
0.8	0.10755	1900.38	0.80042	0.1128	1988.31
0.9	0.02793	4025.035	0.90067	0.0288	4151.15
1	0		1	0	

Table 5: Variation of Thickness of Solidified Region with Time and Rate of Heat out Through the Wall of Cylindrical Vessel for Inside Solidification for ($\alpha = 4.338$)

Approximate method			Runga-Kutta Method		
Z(τ)	τ	Q(watts)	Z(τ)	τ	Q(watts)
0	1.0845		0.01095	1.16748	321.75
0.1	1.02371	245.5674	0.10039	1.09748	368.5
0.2	0.90149	351.3275	0.20171	0.96148	453.88
0.3	0.75187	469.6451	0.30124	0.80148	563.57
0.4	0.59299	617.0966	0.4004	0.63348	707.51
0.5	0.43752	815.7573	0.50186	0.46348	911.02
0.6	0.29521	1106.914	0.60215	0.31348	1210.15
0.7	0.17402	1585.308	0.70115	0.18548	1699.38
0.8	0.08066	2533.974	0.80033	0.08648	2670.74
0.9	0.02095	5366.713	0.90075	0.02188	5619.69
1	0		1	0	

ERROR ANALYSIS

In tables (1) to (2) show, complete solidification times by both approximate and numerical method at five different values of latent heat parameter (α) for inside and outside solidification process of cylindrical and spherical problems while in the table (3) to (5) shows a variation of the thickness of solidified region. A heat balance integral method is more correct than the approximate analytical method.

RESULTS AND DISCUSSIONS

Variation of Thickness of Solidification Region with Time

The position of the interface was found by the approximate solution given by equation (3) for a cylindrical vessel at different times for five different values of latent heat parameter (α). The systems of equations (12) to (15) for cylindrical vessel obtained by the heat balance integral method has been solved numerically using fourth order Runga-Kutta method for five different values of α . The procedure adopted in solution by a numerical method is followed by subdividing temperature into seven equal divisions. Starting values of $Z_0, Z_1, \dots, Z_7$ was supplied for $Z_7=0.95$ (inner solidification) and for $Z_7=1.05$ (outer solidification) given in tables (2) to (4). Initially, we have started with a time increment of $\Delta\tau = 2 \times 10^{-12}$ for the purpose of stabilizing the results.

Once the results are stabilized a time increment of $\Delta\tau = 2 \times 10^{-8}$ gave stable results up to 0.9. After the lapse of time $t=1.2$ a time increment of $\tau = 2 \times 10^{-5}$ gave quite accurate results.

Inside Solidification

The results are given by approximate method show higher values of complete solidification times as compared to those given by numerical method for $\alpha = 21.689$ and 17.351 and lower values of complete solidification times for $\alpha = 8.675, 5.784$ and 4.338 (tables to 3 to 5). By numerical method τ_s corresponding to $Z(\tau) = 0.02$ could have been calculated after which the results show the negative values of $Z(\tau)$.

Outside Solidification

The times for complete solidification have been calculated upto $Z(\tau)=3.0$. The solidification times obtained by approximate methods as compared to numerical method show higher values for $\alpha = 21.689$ and 17.351 and lower values for $\alpha = 8.675, 5.784$ and 4.338 . Figures show that the results obtained by both methods are similar for cylindrical upto $Z(\tau)=2.0$.

Heat Flow Rate at Various Times Through the Wall of Vessel

The amount of heat flow rates has been calculated for an approximate method by equations (5) and (11) and numerical method by equations (13) at different dimensionless times.

Inside Solidification

In inside solidification, there is a decreasing character of heat flow out through the walls of the cylinder (spherical) vessels with $\sqrt{t}$. The result for both approximate and numerical method are identical to each other. The rate of heat flow decreases fastly upto $\sqrt{t} = 0.5$ and after this decreasing tendency is reduced to a remarkable extent.

CONCLUSIONS

The main aim of this work is to predict the time required for the completion of the solidification process and also to predict the temperature distribution in the solidified region of a liquid kept inside and outside cylindrical and spherical vessels. Whose walls are maintained at constant sub melting temperatures. The heat balance integral technique incorporating special subdivisions has been used and the obtained systems of equations have been solved by fourth order Runge-kutta method. Although a number of assumption have been made in the problem specification and change of flux has been approximated by the dis-continues change in the adjacent profile gradients. The acceptable estimates of both the temperature and the flux have been obtained by using small sub-divisions ($n=7$) and incremental time (initially $t=2 \times 10^{-12}$). An approximate analytical approach has been made to predict solid-liquid interface position at any time. Results obtained by both numerical and analytical methods, have been compared and an estimate of the errors has been established, assuming heat balance integral (numerical) method to be the more correct.

The results show that the percentage of error while using approximate analytical method is less for higher values of latent parameters and high for lower values. The percentage error $\varepsilon = 0\%$ at $\alpha = 9.4$ and $\varepsilon = 0\%$ at $\alpha = 12.3$ for the inside solidification process of cylindrical and spherical vessel respectively. The percentage error becomes infinite when α is reduced below 4.0 for cylindrical (inside and outside) and spherical (inside) cases. For the only case of a solidification outside spherical vessel, the percentage error is upto a maximum of 25.425%, even for the lowest value of $\alpha = 0$. Thus, an

approximate analytical solution can be adopted for higher values of alpha with the maximum errors encountered:

- $\varepsilon = -5.4\%$ when $\alpha > 9.4$ for inside the cylindrical case.
- $\varepsilon = +7.225\%$ when $\alpha \geq 24.0$ for an outside cylindrical case.
- $\varepsilon = -2.3\%$ when $\alpha > 12.2$ for inside spherical case and,
- $\varepsilon = +25.42\%$ for outside spherical case.

(Applicable for both higher and lower value of α .)

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