BINDING POLYNOMIAL OF A GRAPH

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ABSTRACT

We introduce here a new concept called the binding polynomial of a graph G to cater the needs of addressing the local and global binding number of a graph. The binding polynomial of G , denoted by $b^p(G)$, is defined as $b^p(G) = \sum_{i=1}^n \text{bind}_{v_i}(G) x^{\deg(v_i)}$. The binding polynomials of some standard graphs have been established here.

KEYWORDS: Binding Number, Local Binding Number, Binding Degree of a Vertex

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INTRODUCTION

Throughout this paper, by a "graph" we mean a finite simple graph without loops as treated in F. Harary [3].

The binding number of a graph is a measure how well the edges are distributed over vertices. If a graph has lower binding number, then it conveys that, the edges are concentrated at some vertices, not uniformly over all. This nature of the binding number finds applications in network reliability. So the study of binding number has both theoretical as well as practical implications. Formally the binding number was defined by Woodall [9] in his seminal paper way back in 1947. It is considered to be one of the toughest graph parameters and hence find considerably less number of research papers in seven decades of study. Recently, it is extended for digraphs by Xu et al. [10]. For further details one can refer to [1], [6], [7], [8], [9].

In this paper we undertake a study on binding polynomial of a graph and related results, which we are introducing here. Till 2019 the study of binding number was considered as a single, major parameter for a graph. But the treatment got a new direction by Aslan [2], where it is considered and defined for each vertex of the graph under consideration and then the average was defined over all the vertices. This approach gives a local vulnerability test of the given graph or network. And this motivated us to define another new parameter namely the binding degree of a graph in [4], [5]. This works as a triple layered parameter combining the degree of a vertex and its local binding number, so that the duality of local connectedness and local vulnerability are addressed.

Going one more step ahead, we get the binding polynomial of a graph, which is the focal point of this article.

BASICS

Here we list some important definitions.

Definition: [9] The binding number of a graph G is defined as $d(G) = \min_{\{ \frac{|N(X)|}{|X|} : X \subseteq V(G), X \neq \emptyset \text{ and } N(X) \neq V(G) \}} \frac{|N(X)|}{|X|}$, where $N(X) = \{y: xy \in E(G) \forall x \in X\}$, is called the open neighbourhood set of X .

Definition : [2] For $v \in V(G)$, the local binding number of v is $bind_v(G) = \min_{S \in F_v(G)} \frac{|N(S)|}{|S|}$, where $F_v = \{S \subseteq V(G) : v \in S, S \neq \emptyset, N(S) \neq V(G)\}$.

Clearly, $bind(G) = \min_{v \in V(G)} \{bind_v(G)\}$. A local binding set of v in G is $S \in F_v(G)$, such that $bind_v(G) = \frac{|N(S)|}{|S|}$.

Definition : The average binding number of G is defined as $bind_{av}(G) = \frac{1}{n} \sum_{v \in V(G)} bind_v(G)$. where n is the number of vertices in graph G .

Definition : [4] The binding degree of G , denoted by $b^d(G)$, is defined as $b^d(G) = \sum_{i=1}^n deg(v_i) bind_{v_i}(G)$.

RESULTS

Motivated by all the above treatments of binding number we define here the binding polynomial of a graph. This is in line with any parameter related polynomial like domination polynomial. We formally define it here and prove some related results.

Definition : The binding polynomial of a graph G , denoted by $b^p(G)$, is defined as $b^p(G) = \sum_{i=1}^n bind_{v_i}(G) x^{deg(v_i)}$.

Illustration:

In G , we have $|V(G)| = 4$ and $|E(G)| = 4$.

Note that, $bind_{v_1}(G) = 1, deg(v_1) = 1, bind_{v_2}(G) = 3, deg(v_2) = 3, bind_{v_3}(G) = 1, deg(v_3) = 2$ and $bind_{v_4}(G) = 1, deg(v_4) = 2$ It follows that, $b^p(G) = x + 2x^2 + 3x^3$.

Proposition 3.1 : The binding polynomial of cycles, paths, complete graphs, complete bipartite graphs, wheels is given as follows:

(i). If G is a path of order $n, n \geq 3$, then $b^p(G) = \begin{cases} 2x + (n-2)x^2, & \text{if } n \text{ is even} \\ 2\left(\frac{n-1}{n+1}\right)x + \left(\frac{n-1}{2}\right)\left(\frac{2n^2-5n+5}{(n+1)(n-2)}\right)x^2, & \text{if } n \text{ is odd.} \end{cases}$

(ii). If G is a cycle of order $n, n \geq 3$, then $b^p(G) = \begin{cases} nx^2, & \text{if } n \text{ is even} \\ n\left(\frac{n-1}{n-2}\right)x^2, & \text{if } n \text{ is odd.} \end{cases}$

(iii). If G is a complete graph of order $n, n \geq 3$, then $b^p(G) = n(n-1)x^{n-1}$.

(iv). If G is a complete bipartite graph of order m and $n, m, n \geq 2$, then $b^p(G) = mx^m + nx^n$.

(v). For $n \geq 4$, the binding polynomial of a wheel is, $b^p(G) = \begin{cases} \frac{(n-1)^2}{n-3} + (n-1)x^{n-1}, & \text{if } n \text{ is even} \\ (n+1)x^3 + (n-1)x^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$

Proof : (i) Let P_n be a path with $n \geq 3$ vertices. If the vertices of P_n are labelled as $v_1, v_2, \dots, v_n$ then we know that $\deg(v_1) = \deg(v_n) = 1$ and $\deg(v_2) = \dots = \deg(v_{n-1}) = 2$. Here we consider two cases.

Case (a): n be even.

From [9] we know that $\text{bind}(P_n)=1$, whenever n is even. For $v_i \in V(P_n)$ the local binding number of v_i with $1 \leq i \leq n$ is $\text{bind}_{v_i}(P_n) = 1$. Therefore $b^p(G) = 2x + (n-2)x^2$.

Case (b): n be odd.

For $v_i \in V(P_n)$, the local binding number of $\{v_1, v_2, \dots, v_n\}$ is $\text{bind}_{v_i}(P_n) \frac{n-1}{n+1}$ and the local binding number of $\{v_2, v_4, \dots, v_{n-1}\}$ is $\text{bind}_{v_i}(P_n) \frac{n-1}{n-2}$. Therefore $b^p(G) = 2 \frac{n-1}{n+1}x + \left(\frac{n-3}{2}\right) \left(\frac{n-1}{n+1}\right)x^2 + \left(\frac{n-1}{2}\right) \left(\frac{n-1}{n-2}\right)x^2 = 2 \left(\frac{n-1}{n+1}\right)x + \left(\frac{n-1}{2}\right) \left(\frac{2n^2-5n+5}{(n+1)(n-2)}\right)x^2$. Hence the result.

(ii)

Let C_n be a cycle with $n \geq 3$ vertices. We know that C_n is a self-centred, regular graph of regularity 2. Here also we have two cases.

Case (a) : n be even.

Form [9] we know that $\text{bind}(C_n)=1$ if n is even. Therefore $b^p(G) = nx^2$

Case (b): n be odd.

from [9] we know that $\text{bind}(C_n) = \frac{n-1}{n-2}$ for n odd. Therefore $b^p(G) = n \left(\frac{n-1}{n-2}\right)x^2$. Hence the result.

(iii)

Since a complete graph K_n , for $n \geq 3$ is a regular graph of regularity $n-1$ and all vertices have local binding number $n-1$, we get $b^p(G) = n(n-1)x^{n-1}$.

(iv)

Let $K_{m,n}$ be a complete bipartite graph. Let vertices be labelled as $v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n$. For $v_i \in V(K_{m,n})$, the local binding number of v_i for $1 \leq i \leq m$, is $\text{bind}_{v_i}(K_{m,n}) = \frac{n}{m}$ and the local binding number of u_j for $1 \leq j \leq n$, is $\text{bind}_{v_j}(K_{m,n}) = \frac{m}{n}$. Therefore $b^p(G) = n \frac{m}{n}x^m + m \frac{n}{m}x^n = mx^m + nx^n$.

(v)

Let W_n be a wheel with $n \geq 4$. We know that $W_n = C_{n-1} + K_1$. If W_n has $\{v_1, v_2, \dots, v_n\}$ vertices, we know that $\deg(v_n) = n-1$ and $\deg(v_1) = \deg(v_2) = \dots = \deg(v_{n-1}) = 3$. Here we have two cases.

Case (a): n be even.

For $v_i \in V(W_n)$, from [2], the local binding number of v_i for $1 \leq i \leq n-1$ is $\text{bind}_{v_i}(W_n) = \frac{n-1}{n-3}$ and the local

binding number of v_n is $bind_{v_n}(W_n) = n - 1$. Therefore $b^p(G) = (n - 1) \frac{n-1}{n-3} x^3 + (n - 1)x^{n-1} = \frac{(n-1)^2}{n-3} x^3 + (n - 1)x^{n-1}$.

\Case (b): n be odd.

For $v_i \in V(W_n)$, from [2], the local binding number of v_i for $1 \leq i \leq n - 1$ is $bind_{v_i}(W_n) = \frac{n+1}{n-1}$ and the local binding number of v_n is $bind_{v_n}(W_n) = n - 1$. Therefore $b^p(G) = (n - 1) \frac{n+1}{n-1} x^3 + (n - 1)x^{n-1} = (n + 1)x^3 + (n - 1)x^{n-1}$.

Hence the result.

Proposition 3.2 : The binding polynomial of a fan graph and a double star is given as follows:

(i) For a fan graph F_n with $n \geq 4$ vertices,

$$b^p(F_n) = \begin{cases} 2x^2 + \frac{n^2-5n+7}{n-3}x^3 + (n-1)x^{(n-1)}, & \text{if } n \text{ is even} \\ 2\left(\frac{n-1}{n-2}\right)x^2 + \frac{n^2-4n+3}{n-2}x^3 + (n-1)x^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

$$(ii)] \text{ For a double star graph } S_{m,n} \text{ with } m, n \geq 2 \text{ vertices, } b^p(S_{m,n}) = \begin{cases} 2x + 2\frac{m+1}{2m+1}x^m, & \text{if } m = n, \\ \frac{3m+n-4}{m+n-2}x + \frac{m+1}{m+n-1}x^m + \frac{n+1}{m+n-1}x^n, & \text{if } m < n, \\ \frac{m+3n-4}{m+n-2}x + \frac{m+1}{m+n-1}x^m + \frac{n-1}{m+n-1}x^n, & \text{if } n < m. \end{cases}$$

Proof: (i)

Let F_n be a fan graph with $n \geq 4$, labelled $v_1, v_2, \dots, v_n$ vertices. We know that $\deg(v_n) = n-1$, $\deg(v_1) = \deg(v_{n-1}) = 2$ and $\deg(v_2) = \deg(v_3) = \dots = \deg(v_{n-2}) = 3$. Here we have two cases.

Case (a): n be even.

For $v_i \in V(F_n)$, $i=2, 4, \dots, n-1$, the local binding number is $bind_{v_i}(F_n) = 1$, the local binding number of $v_3, v_5, \dots, v_{n-2}$ is $bind_{v_i}(F_n) = \frac{n-1}{n-3}$ and the local binding number of v_n is $bind_{v_i}(F_n) = n-1$. Therefore $b^p(F_n) = 2x^2 + \left(\frac{n-4}{2}\right)x^3 + \left(\frac{n-2}{2}\right)\left(\frac{n-1}{n-3}\right)x^3 + (n-1)x^{n-1} = 2x^2 + \frac{n^2-5n+7}{n-3}x^3 + (n-1)x^{n-1}$.

Case (b): n be odd.

For $v_i \in V(F_n)$, the local binding number of v_i with $1 \leq i \leq n-1$ is $bind_{v_i}(F_n) = \frac{n-1}{n-2}$ and the local binding number of v_n is $bind_{v_n}(F_n) = n-1$. Therefore $b^p(F_n) = 2\left(\frac{n-1}{n-2}\right)x^2 + \frac{n^2-4n+3}{n-2}x^3 + (n-1)x^{n-1}$. Hence the result.

(ii)

Let $S_{m,n}$ be a double star with $m, n \geq 2$, and hence $S_{m,n} = \overline{K_{m-1}} + K_1 + K_1 + \overline{K_{n-1}}$. If $S_{m,n}$ has $v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n$ vertices with $\deg(v_1) = \deg(v_2) = \dots = \deg(v_{\{m-1\}}) = \deg(u_1) = \deg(u_2) = \dots = \deg(u_{n-1}) = 1$, $\deg(v_m) = m$ and $\deg(u_n) = n$, then we have three cases.

Case (a): $m=n$.

For $v_i \in S_{m,n}$, the local binding number of each vertex of $v_1, v_2, \dots, v_{\{m-1\}}$ is $bind_{v_i}(S_{m,n}) = \frac{1}{m-1}$ from [2], and that of each of $u_1, u_2, \dots, u_{\{m-1\}}$ as $bind_{u_i}(S_{m,n}) = \frac{1}{m-1}$, $bind_{v_m}(S_{m,n}) = \frac{m-1}{2m-1}$ for v_m and is $bind_{u_m}(S_{m,n}) = \frac{m+1}{2m-1}$ for u_m . Therefore, the binding polynomial of the double star is,

$$b^p(S_{m,m}) = 2x + 2 \frac{m+1}{2m+1} x^m.$$

Case (b): $m < n$.

For $v_i \in S_{m,n}$, the local binding number of each vertex $v_1, v_2, \dots, v_{m-1}$ is $bind_{v_i}(S_{m,n}) = \frac{2}{m+n-2}$ and that of $u_1, u_2, \dots, u_{\{n-1\}}$ $bind_{u_i}(S_{m,n}) = \frac{1}{n-1}$, $bind_{v_m}(S_{m,n}) = \frac{n+1}{m+n-1}$ for v_m and $bind_{u_m}(S_{m,n}) = \frac{n+1}{m+n-1}$ for u_m . Therefore

$$\begin{aligned} b^p(S_{m,n}) &= (n-1) \frac{1}{n-1} x + (m-1) \frac{2}{m+n-2} x + \frac{m+1}{m+n-1} x^m + \frac{n+1}{m+n-1} x^n \\ &= \frac{3m+n-4}{m+n-2} x + \frac{m+1}{m+n-1} x^m + \frac{n+1}{m+n-1} x^n. \end{aligned}$$

Case (c): $m > n$.

Similar to Case(b), we get,

$$\begin{aligned} b^p(S_{m,n}) &= (m-1) \frac{1}{m-1} x + (n-1) \frac{2}{m+n-2} x + \frac{m+1}{m+n-1} x^m + \frac{n+1}{m+n-1} x^n \\ &= \frac{m+3n-4}{m+n-2} x + \frac{m+1}{m+n-1} x^m + \frac{n+1}{m+n-1} x^n. \end{aligned}$$

Hence the result.

CONCLUSIONS

In this paper we have introduced the binding polynomial of a graph. We have determined the polynomial for some familiar class of graphs. Determining for many other classes, find the roots of these polynomials, their multiplicities, their relation to the local, average binding number, etc. are being explored further.

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