A CLASS OF UNIFORMLY HARMONIC FUNCTIONS OF TYPE K USING SALAGEAN OPERATORS

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ABSTRACT

This paper introduces a new generalized function $USHK(\mu, \nu, \alpha)$ which is a sub class of k -uniformly harmonic functions of order α based on Salagean operators. A sub function $UTHK(\mu, \nu, \alpha)$ of $USHK(\mu, \nu, \alpha)$ is also been defined and studied in this Paper.

KEYWORDS: Harmonic, Univalent, Salagean, Generalized

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1. INTRODUCTION

Suppose the class SH indicates the class of functions $F = H + \bar{G}$ which are defined in the open disk $D = \{Z: |Z| < 1\}$ of radius unity for which the normalization condition holds $F(0) = F(0) - 1 = 0$. The function F is univalent as well as harmonic and also F preserves the orientation and then for $F = H + \bar{G} \in SH$ the analytic functions H and G is defined as

$$(1a) \quad H(Z) = Z + \sum_{N=2}^{\infty} A_N Z^N, G(Z) = \sum_{N=1}^{\infty} B_N Z^N, |B_1| < 1.$$

Where H denotes analytic portion and G denotes the co analytic portion of F . Salagean [7] defined an operator given below:

For $H(Z) \in S$

$$D^0(Z) = H(Z), D^1 H(Z) = D(H(Z)) = ZH'(Z) \text{ and}$$

$$D^\nu H(Z) = D(D^{\nu-1} H(Z)) = Z(D^{\nu-1} H(Z))' = Z + \sum_{N=2}^{\infty} N^\nu A_N Z^N, \nu \in N = \{1, 2, \dots\}.$$

Where D is called the Salagean operator.

Jahangiri *et al.* [9,10] introduced the modified Salagean operator of harmonic functions F which is univalent, also as given

$$D^\nu F(Z) = D^\nu H(Z) + \underline{(-1)^\nu D^\nu G(Z)}, \nu \in N_0 = \{0, 1, 2, \dots\}$$

$$\text{where } D^\nu H(Z) = Z + \sum_{N=2}^{\infty} N^\nu A_N Z^N \text{ and } D^\nu G(Z) = \sum_{N=1}^{\infty} N^\nu B_N Z^N.$$

Definition: For $0 \leq \alpha < 1, 0 \leq k < \infty, \mu > \nu$, $USHK(\mu, \nu, \alpha)$ is the class in which function $F = H + G$ is defined and satisfying the following condition:

$$(1b) \left\{ (1 + ke^{i\varphi}) \frac{D^\mu F(z)}{D^\nu F(z)} - ke^{i\varphi} \right\} \geq \alpha, \phi \in \mathbb{R}.$$

Also, $UTHK(\mu, \nu, \alpha) \subseteq USHK(\mu, \nu, \alpha)$ is the class of harmonic functions $F_\mu = H + \bar{G}_\mu$ which are of negative coefficients so that

$$(1c) \quad H(Z) = Z - \sum_{N=2}^{\infty} |A_N| Z^N, G(Z) = (-1)^{\mu-1} \sum_{N=1}^{\infty} |B_N| Z^N, |B_1| < 1.$$

$$\text{Also } USHK(1, 0, \alpha) \equiv USHK(\alpha),$$

$USHK(2, 1, \alpha) \equiv UCHK(\alpha)$ and classes $UTHK(\alpha) \subseteq USHK(\alpha)$ and $THCVK(\alpha) \subseteq UCHK(\alpha)$ consist of functions F of the form (3).

$$\text{Also } 1 - USH(0) \equiv G_H, 1 - HCV(0) \equiv HCV.$$

Generalization: The class $UTHK(\mu, \nu, \alpha)$ generalizes several groups of harmonic functions which are univalent and also are defined earlier.

- 1) For $k=0, \mu = 1, \nu = 0$, this class reduces to $SH(\alpha)$ of order α which was studied by *Jahangiri* [9].
- 2) For $k = 0, \mu = 2, \nu = 1$, it reduces to the class $KH(\alpha)$, of order α which was studied by *Jahangiri* [10].
- 3) For $k=1, \mu = 1, \nu = 0$, this class reduces to $G_H(\alpha)$ which was studied by *Rosy et al.* [11].
- 4) For $k=1, \mu = \nu + 1$, this class reduces to $RS_H(\nu, \alpha)$ which was studied by *Yalcin et al.* [12].

2.SUFFICIENT COEFFICIENT CONDITION FOR $USHK(\mu, \nu, \alpha)$

Theorem 2.1: The function F be given by (1a). Furthermore, let

$$(2a) \quad \sum_{n=1}^{\infty} \{ \psi(\mu, \nu, \alpha) |A_n| + \theta(\mu, \nu, \alpha) |B_n| \} \leq 2$$

$$\text{where, } \psi(\mu, \nu, \alpha) = N^\nu + \frac{(N^\mu - N^\nu)(1+k)}{1-\alpha}$$

$$\theta(\mu, \nu, \alpha) = (-1)^{\nu-\mu} N^\nu + \frac{(N^\mu - (-1)^{\nu-\mu} N^\nu)(1+k)}{1-\alpha}$$

with $A_1 = 1, 0 \leq \alpha < 1, 0 \leq k < \infty, \mu \in N = \{1, 2, \dots\}, \nu \in N_o$ and $\mu > \nu$, then F is univalent, harmonic and preserve the orientation in D and $F \in USHK(\mu, \nu, \alpha)$. The result is sharp also.

Proof

For $|Z| \leq |Z_2| < 1$ preserving in D , this is because

$$\begin{aligned} |H'(z)| &\geq 1 - \sum_{N=2}^{\infty} N |A_N| |Z|^{N-1} \\ &\geq 1 - \sum_{N=2}^{\infty} N |A_N| \\ &\geq 1 - \sum_{N=2}^{\infty} \left[N^\nu + \frac{(N^\mu - N^\nu)(1+k)}{1-\alpha} \right] |A_N| \\ &\geq \alpha + \sum_{N=1}^{\infty} \left[(-1)^{\nu-\mu} N^\nu + \frac{(N^\mu - (-1)^{\nu-\mu} N^\nu)(1+k)}{1-\alpha} \right] |B_N| \text{ (using 2)} \end{aligned}$$

$$\begin{aligned}
 &\geq \sum_{N=1}^{\infty} [(-1)^{v-\mu} N^v + (N^\mu - (-1)^{v-\mu} N^v)(1+k)] |B_N| \\
 &\geq \sum_{N=1}^{\infty} N |B_N| \\
 &\geq \sum_{N=1}^{\infty} N |B_N| |Z|^{N-1} \\
 &\geq |G'(Z)|.
 \end{aligned}$$

It only needs to show that $F \in \text{USHK}(\mu, v, \alpha)$

That is

$$\left\{ \frac{(1+ke^{i\varphi})D^\mu F(Z) - (ke^{i\varphi} + \alpha)D^v F(Z)}{D^v F(Z)} \right\} \geq 0$$

or,

$$\left\{ \frac{(Z + \sum_{N=2}^{\infty} N^\mu A_N Z^N + (-1)^\mu \sum_{N=1}^{\infty} N^\mu B_N \bar{Z}^N)(1+ke^{i\varphi}) - (ke^{i\varphi} + \alpha)(Z + \sum_{N=2}^{\infty} N^v A_N Z^N + (-1)^v \sum_{N=1}^{\infty} N^v \bar{B}_N \bar{Z}^N)}{Z + \sum_{N=2}^{\infty} N^v A_N Z^N + (-1)^v \sum_{N=1}^{\infty} N^v \bar{B}_N \bar{Z}^N} \right\} \text{ For } Z = Re^{i\theta},$$

$$\begin{aligned}
 A(Re^{i\theta}) &= \sum_{N=2}^{\infty} [N^\mu(1+ke^{i\varphi}) - N^v(ke^{i\varphi} + \alpha)] A_N R^{N-1} e^{(N-1)\theta i} + (-1)^\mu \sum_{N=1}^{\infty} [N^\mu(1+ke^{i\varphi}) - (-1)^{v-\mu} N^v(ke^{i\varphi} \\
 &\quad + \alpha) \bar{B}_N R^{N-1} e^{-(N+1)\theta i}
 \end{aligned}$$

$$B(Re^{i\theta}) = \sum_{N=2}^{\infty} N^v A_N R^{N-1} e^{(N-1)\theta i} + (-1)^v \sum_{N=1}^{\infty} N^v \bar{B}_N R^{N-1} e^{-(N+1)\theta i}.$$

$$\text{Setting } \frac{(1-\alpha)+A(Z)}{1+B(Z)} = (1-\alpha) \frac{1+w(Z)}{1-w(Z)}$$

The proof will be completed if it can show $|w(Z)| \leq R < 1$.

Since,

$$\begin{aligned}
 |w(Z)| &= \left| \frac{A(Z) - (1-\alpha)B(Z)}{A(Z) + (1-\alpha)B(Z) + 2(1-\alpha)} \right| \\
 &= \left| \frac{\sum_{N=2}^{\infty} [N^\mu(1+ke^{i\varphi}) - N^v(ke^{i\varphi} + \alpha) - (1-\alpha)N^v] A_N R^{N-1} e^{(N-1)\theta i} \right. \\
 &\quad \left. + (-1)^\mu \sum_{N=1}^{\infty} [N^\mu(1+ke^{i\varphi}) - (-1)^{v-\mu}(ke^{i\varphi} + \alpha)N^v - (1-\alpha)(-1)^{v-\mu}N^v] \bar{B}_N R^{N-1} e^{-(N+1)\theta i} \right. \\
 &\quad \left. + 2(1-\alpha) + \sum_{N=2}^{\infty} [N^\mu(1+ke^{i\varphi}) - N^v(ke^{i\varphi} + \alpha) + (1+\alpha)N^v] A_N R^{N-1} e^{(N-1)\theta i} \right. \\
 &\quad \left. + (-1)^\mu \sum_{N=1}^{\infty} [N^\mu(1+ke^{i\varphi}) - (-1)^{-\mu+v}N^v(ke^{i\varphi} + \alpha) + (-1)^{v-\mu}N^v(1-\alpha)] \bar{B}_N R^{N-1} e^{-(N+1)\theta i} \right| \\
 &\leq \frac{\sum_{N=2}^{\infty} [N^\mu - N^v](1+k) |A_N| R^{N-1} + \sum_{N=1}^{\infty} [(N^\mu - (-1)^{v-\mu}N^v)(1+k)] |B_N| R^{N-1}}{2(1-\alpha) - \sum_{N=2}^{\infty} [(N^\mu - N^v)k + N^\mu - 2\alpha N^v + N^v] |A_N| R^{N-1} \\
 &\quad - \sum_{N=1}^{\infty} [(N^\mu - (-1)^{v-\mu}N^v)k + N^\mu - 2\alpha(-1)^{v-\mu}N^v \\
 &\quad + (-1)^{v-\mu}N^v] |B_N| R^{N-1}}
 \end{aligned}$$

$$\text{where, } C(\mu, v, \alpha) = (N^\mu - N^v)k + N^\mu + (1-2\alpha)N^v$$

$$\begin{aligned}
 D(\mu, v, \alpha) &= (N^\mu - (-1)^{v-\mu}N^v)k + N^\mu + (1-2\alpha)(-1)^{v-\mu}N^v \\
 &= \frac{\sum_{N=2}^{\infty} [(N^\mu - N^v)(1+k)] |A_N| + \sum_{N=1}^{\infty} [(N^\mu - (-1)^{v-\mu}N^v)(1+k)] |B_N|}{4(1-\alpha) - \sum_{N=1}^{\infty} \{C(\mu, v, \alpha) |A_N| + D(\mu, v, \alpha) |B_N|\}}
 \end{aligned}$$

≤ 1 (using 2a).

which proves the Theorem.

Sharpness of the Result

The function:

$$(2b) \quad F(Z) = Z + \sum_{N=2}^{\infty} \frac{1}{\psi(\mu, \nu, \alpha)} X_N Z^N + \sum_{N=1}^{\infty} \frac{1}{\theta(\mu, \nu, \alpha)} Y_N \bar{Z}^N$$

where $\mu \in N, \nu \in N_0, \mu > \nu$ and $\sum_{N=2}^{\infty} |X_N| + \sum_{N=1}^{\infty} |Y_N| = 1$, shows that the coefficient bound given by (2a) is sharp. The functions of the form (2a) are in USHK(μ, ν, α) because

$$\sum_{N=1}^{\infty} \{\psi(\mu, \nu, \alpha)|A_N| + \theta(\mu, \nu, \alpha)|B_N|\} = 1 + \sum_{N=2}^{\infty} |X_N| + \sum_{n=1}^{\infty} |Y_N| = 2.$$

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