Generalized Birkhoff Center of Almost Distributive Lattices

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Abstract. In this paper we define sectional Birkhoff center for an Almost Distributive Lattice not necessarily with maximal elements; as a Birkhoff center of its principal ideals. Moreover, we extend this result and define the generalized Birkhoff center of an ADL not necessarily with maximal elements. We give a necessary and sufficient condition for an ADL to be relatively complemented in terms of its sectional Birkhoff centers. Also we define and characterize sectionally complemented ideals and sectional factor congruences using sectional Birkhoff centers

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1. Introduction

Several algebraic structures have been studied by different scholars as a generalization for distributive lattices for instance near lattices (see [9]), skew lattices (see [10]), 0-distributive and 1-distributive lattices (see [2, 11, 16]), 0-distributive near lattices (see [8, 5, 6]), 0-distributive and 1-distributive semilattices (see [1, 7]), almost distributive lattices (see [12,15]), etc.

The concept of almost distributive lattices (or shortly an ADL) was first introduced by Swamy and Rao [13] in 1980. An ADL is an algebra with two binary operations $\wedge$ and $\vee$ which satisfies most of the properties of a distributive lattice with smallest element 0 except the commutativity of the binary operations $\wedge$ and $\vee$.

It is well known that the Birkhoff centre of a bounded partially ordered set P is a Boolean algebra in which the operations are l.u.b. and g.l.b. in P . In [14], Swamy, Rao, Ravi Kumar and Pragathi have extended the above concept for a general partially ordered set P and proved that $B(P)$ is a relatively complemented distributive lattice in which the operations are l.u.b. and g.l.b. in P (provided $B(P)$ is non-empty). In [15], U.M.Swamy, and Murthy, studied Boolean center of algebras in a more general context; in universal algebra. Later in 2009 Swamy and Ramesh [12] have been introduced the concept of Birkhoffcentre $B(L)$ of an Almost Distributive Lattice L with maximal elements and they have proved that $B(L)$ is a relatively complemented almost distributive lattice. Also they

have proved that an ADL L with maximal elements is relatively complemented if and only if $B(L) = L$. Moreover they have obtained one-to-one correspondences between the Birkhoff centre $B(L)$ of L and the set of complemented ideals of L and between the Birkhoff center of L and the set of factor-congruences on L . The question in this case is that what can we say about the Birkhoff center of an ADL with no maximal elements? In this paper we answer this question in a positive way by defining sectional Birkhoff centers of an Almost Distributive Lattice L (not necessarily with maximal elements). We also prove certain results analogous to those results in [12]. Mainly we give a set of equivalent conditions for an ADL (not necessarily with maximal elements) to be relatively complemented in terms of its sectional Birkhoff centers. In addition we define sectionally complemented ideals (respectively sectional factor congruences) of ADLs and we characterize them using sectional Birkhoff centers. Finally we extend the result and define the generalized Birkhoff center of an ADL L not necessarily with maximal elements and we give some characterizations.

2. Preliminaries

In this section, we recall some necessary definitions and results on ADLs which will be used in the paper. We refer to [3,4] for those standard concepts on distributive lattices and general universal algebras.

Definition 2.1. [4] *An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice, abbreviated as ADL, if it satisfies the following axioms;*

- (1) $a \vee 0 = a$
- (2) $0 \wedge a = 0$
- (3) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
- (4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- (5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- (6) $(a \vee b) \wedge b = b$

Example 2.2. [4] *Let X be a nonempty set. Fix $x_0 \in X$. For any $x, y \in X$ define $\wedge$ and $\vee$ as follows*

$$x \wedge y = \begin{cases} x_0 & \text{if } x = x_0 \\ y & \text{if } x \neq x_0 \end{cases} \text{ and } x \vee y = \begin{cases} y & \text{if } x = x_0 \\ x & \text{if } x \neq x_0 \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL with x_0 as its 0.

From now onwards we write simply L for an ADL $(L, \vee, \wedge, 0)$ unless and otherwise mentioned.

Lemma 2.3. [4] *For any $a \in L$, we have*

- (1) $a \wedge 0 = 0$
- (2) $0 \vee a = a$
- (3) $a \wedge a = a$
- (4) $a \vee a = a$

Lemma 2.4. [4] *For any $a, b \in L$, we have*

- (1) $(a \wedge b) \vee b = b$
- (2) $a \vee (a \wedge b) = a = a \wedge (a \vee b)$
- (3) $a \vee (b \wedge a) = a = (a \vee b) \wedge a$

Corollary 2.5. [4] *For any $a, b \in L$, we have*

- (1) $a \vee b = a$ if and only if $a \wedge b = b$
- (2) $a \vee b = b$ if and only if $a \wedge b = a$

In view of the above corollary, we give the following

Definition 2.6. [4] *For any $a, b \in L$, we say that a is less than or equals to b and we write $a \leq b$ if $a \wedge b = a$ or equivalently $a \vee b = b$.*

Theorem 2.7. [4] *For any $a, b \in L$, the following are equivalent*

- (1) $(a \wedge b) \vee a = a$
- (2) $a \wedge (b \vee a) = a$
- (3) $(b \wedge a) \vee b = b$
- (4) $b \wedge (a \vee b) = b$
- (5) $a \wedge b = b \wedge a$
- (6) $a \vee b = b \vee a$
- (7) *The supremum of a and b exists in L and equals $a \vee b$*
- (8) *There exists $x \in L$ such that $a \leq x$ and $b \leq x$*
- (9) *The infimum of a and b exists in L and equals $a \wedge b$*

Theorem 2.8. [4] *For any $a, b, c \in L$, we have*

- (1) $(a \vee b) \wedge c = (b \vee a) \wedge c$
- (2) $\wedge$ is associative in L and
- (3) $a \wedge b \wedge c = b \wedge a \wedge c$

From the above theorem it follows that for any $x \in L$, the set $\{a \wedge x : a \in L\}$ forms a bounded distributive lattice and in particular we have $((a \wedge b) \vee c) \wedge x = ((a \vee c) \wedge (b \vee c)) \wedge x$ for all $a, b, c, x \in L$.

An element $m \in L$ is said to be maximal in L if $m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$ for all $x \in L$.

Definition 2.9. [4] *A nonempty set I in an ADL L is called an ideal of L , if:*

- (1) $a, b \in I \Rightarrow a \vee b \in I$
- (2) $a \in I$ and $x \in L \Rightarrow a \wedge x \in I$

If I is an ideal of L and $a \in I$, then $x \wedge a \in I$ for all $x \in L$; For, $x \wedge a = x \wedge a \wedge a = a \wedge x \wedge a \in I$.

Definition 2.10. [4] *Let L be an ADL. For any $a \in L$, the set $(a) = \{a \wedge x : x \in L\}$ is called a principal ideal of L generated by a and the set $[a] = \{x \vee a : x \in L\}$ is called the principal filter of L generated by a .*

Note that the principal ideal (a) of L is also a subADL of L with maximal elements (since a itself is maximal) and an element b is maximal in (a) if and only if $a \wedge b = b$ and $b \wedge a = a$ or equivalently if $(a) = (b)$.

3. Sectional Birkhoff center

In this section we define sectional Birkhoff center $B_a(L)$ of an Almost Distributive Lattice L (not necessarily with maximal elements) determined by each elements $a \in L$.

Definition 3.1. Given an ADL L and $a \in L$, define

$$B_a(L) = \{x \in L : \exists y \in (a] \text{ such that } x \wedge y = 0 \text{ and } x \vee y \text{ is maximal in } (a]\}$$

For each $a \in L$, we call $B_a(L)$ the sectional Birkhoff center of L determined by a . In other words, this $B_a(L)$ is the Birkhoff center of a subADL $(a]$ of L .

For any $a \in L$ we have $a \wedge 0 = 0 = 0 \wedge a$ and $a \vee 0 = a = 0 \vee a$. Thus $0, a \in B_a(L)$ and hence $B_a(L)$ is nonempty for all $a \in L$.

Lemma 3.2. If m is a maximal element in L , then $B_m(L) = B(L)$; the Birkhoff center of L introduced in [3].

Lemma 3.3. For any $a, b \in L$ we have the following

- (1) $B_a(L) \subseteq (a]$
- (2) $B_a(L) = \{0\}$ if and only if $a = 0$.
- (3) $B_a(L) = B_b(L)$ if and only if $(a] = (b]$

Proof:

- (1) $x \in B_a(L) \Rightarrow$ there exists $y \in (a]$ such that $x \wedge y = 0$ and $x \vee y$ is maximal in $(a]$; that is, $(x \vee y) \wedge a = a$ and $a \wedge (x \vee y) = (x \vee y)$. Now from $(x \vee y) \wedge a = a$ we have $a \wedge x = ((x \vee y) \wedge a) \wedge x = (a \wedge (x \vee y)) \wedge x = (x \vee y) \wedge x = x$. Therefore $x \in (a]$.
- (2) Since $a \in B_a(L)$, $B_a(L) = \{0\} \Rightarrow a = 0$. On the otherhand if $a = 0$ since $B_a(L) \subseteq (a] = \{0\}$, then we get that $B_a(L) = \{0\}$.
- (3) Let $B_a(L) = B_b(L)$, then we have $a \in B_a(L) = B_b(L) \subseteq (b] \Rightarrow a \in (b]$ and $b \in B_b(L) = B_a(L) \subseteq (a] \Rightarrow b \in (a]$. Therefore $(a] = (b]$. The converse is clear.

Theorem 3.4. Let L be an ADL and $a \in L$. Then $x \in B_a(L)$ if and only if there exists two ADLs L_1 and L_2 with maximal elements m_1 and m_2 respectively and an isomorphism $f: (a] \rightarrow L_1 \times L_2$ such that $f(x) = (m_1, 0)$.

Proof: Suppose that $x \in B_a(L)$. Then there exists $y \in (a]$ such that $x \wedge y = 0$ and $x \vee y$ is maximal in $(a]$; that is, $(x \vee y) \wedge a = a$ and $a \wedge (x \vee y) = (x \vee y)$. Put $L_1 = (x] = \{x \wedge z : z \in L\}$ and $L_2 = (y] = \{y \wedge z : z \in L\}$. Then both L_1 and L_2 are ADLs with maximal elements x and y respectively. Now define $f: (a] \rightarrow L_1 \times L_2$ by $f(a \wedge z) = (x \wedge z, y \wedge z)$ for all $z \in L$. Then $f(x) = f(a \wedge x) = (x \wedge x, y \wedge x) = (x, 0)$. We show that f is an isomorphism. Clearly f is a homomorphism and To prove f is one-one, let $z_1, z_2 \in L$ such that $f(a \wedge z_1) = f(a \wedge z_2)$

$$\begin{aligned} &\Rightarrow (x \wedge z_1, y \wedge z_1) = (x \wedge z_2, y \wedge z_2) \\ &\Rightarrow x \wedge z_1 = x \wedge z_2 \text{ and } y \wedge z_1 = y \wedge z_2 \end{aligned}$$

Now consider the following;

$$\begin{aligned} a \wedge z_1 &= ((x \vee y) \wedge a) \wedge z_1 \\ &= (a \wedge (x \vee y)) \wedge z_1 \\ &= (x \vee y) \wedge z_1 \end{aligned}$$

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$$\begin{aligned}
 &= (x \wedge z_1) \vee (y \wedge z_1) \\
 &= (x \wedge z_2) \vee (y \wedge z_2) \\
 &= (x \vee y) \wedge z_2 \\
 &= (a \wedge (x \vee y)) \wedge z_2 \\
 &= ((x \vee y) \wedge a) \wedge z_2 \\
 &= a \wedge z_2
 \end{aligned}$$

This says that $f(a \wedge z_1) = f(a \wedge z_2) \Rightarrow a \wedge z_1 = a \wedge z_2$. Hence f is one-one. It remains to show that f is onto. Let $u \in L_1 \times L_2$, then $u = (u_1, u_2)$ such that $u_1 \in L_1 = [x]$ and $u_2 \in L_2 = [y]$. Then $u_1 = x \wedge z_1$ for some $z_1 \in L$ and $u_2 = y \wedge z_2$ for some $z_2 \in L$. So that $u = (x \wedge z_1, y \wedge z_2)$ for some z_1 and $z_2 \in L$. Put $z = (x \wedge z_1) \vee (y \wedge z_2)$. Then consider;

$$\begin{aligned}
 x \wedge z &= x \wedge ((x \wedge z_1) \vee (y \wedge z_2)) \\
 &= (x \wedge (x \wedge z_1)) \vee (x \wedge (y \wedge z_2)) \\
 &= (x \wedge z_1) \vee (x \wedge y \wedge z_2) \\
 &= (x \wedge z_1) \vee 0 \\
 &= x \wedge z_1
 \end{aligned}$$

Similarly;

$$\begin{aligned}
 y \wedge z &= y \wedge ((x \wedge z_1) \vee (y \wedge z_2)) \\
 &= (y \wedge (x \wedge z_1)) \vee (y \wedge (y \wedge z_2)) \\
 &= (y \wedge x \wedge z_1) \vee (y \wedge z_2) \\
 &= 0 \vee (y \wedge z_2) \\
 &= y \wedge z_2
 \end{aligned}$$

Therefore $u = (x \wedge z_1, y \wedge z_2) = (x \wedge z, y \wedge z)$ where $z = (x \wedge z_1) \vee (y \wedge z_2) \in L$. Thus $u = f(a \wedge z)$ for some $a \wedge z \in [a]$ and hence f is onto. Hence f is an isomorphism. Conversely suppose that there exists two ADLs L_1 and L_2 with maximal elements and an isomorphism $f: [a] \rightarrow L_1 \times L_2$ such that $f(x) = (m_1, 0)$ where m_1 is a maximal element in L_1 . Choose a maximal element m_2 in L_2 . Since f is an isomorphism there exists $y \in [a]$ such that $f(y) = (0, m_2)$. Consider; $f(x \wedge y) = f(x) \wedge f(y) = (m_1, 0) \wedge (0, m_2) = (m_1 \wedge 0, 0 \wedge m_2) = (0, 0)$ (a zero element in $L_1 \times L_2$). Since f is an isomorphism it follows that $x \wedge y = 0$. Also $f(x \vee y) = f(x) \vee f(y) = (m_1, 0) \vee (0, m_2) = (m_1 \vee 0, 0 \vee m_2) = (m_1, m_2)$ which is maximal in $L_1 \times L_2$. Since f is an isomorphism we get $x \vee y$ to be maximal in $[a]$ which implies that y is the complement of x in $[a]$. So that $x \in B_a(L)$. Hence the result. ■.

Theorem 3.4. For any a in an ADL L , $B_a(L)$ is relatively complemented ADL.

Proof: Clearly we have $0 \in B_a(L)$ and hence $B_a(L) \neq \emptyset$.

Let $x_1, x_2 \in B_a(L)$. Then there exists y_1 and y_2 in $[a]$ such that $x_1 \wedge y_1 = 0, x_1 \vee y_1 = m_1$ and $x_2 \wedge y_2 = 0, x_2 \vee y_2 = m_2$ where m_1 and m_2 are maximal elements in $[a]$; that is, $(m_1] = [a] = (m_2]$.

Now consider;

$$\begin{aligned}
 (x_1 \wedge x_2) \wedge (y_1 \vee y_2) &= [(x_1 \wedge x_2) \wedge y_1] \vee [(x_1 \wedge x_2) \wedge y_2] \\
 &= (x_1 \wedge x_2 \wedge y_1) \vee (x_1 \wedge x_2 \wedge y_2) \\
 &= (x_2 \wedge x_1 \wedge y_1) \vee (x_1 \wedge x_2 \wedge y_2) \\
 &= (x_2 \wedge 0) \vee (x_1 \wedge 0) \\
 &= 0
 \end{aligned}$$

Also for any $t \in (a]$ we have;

$$\begin{aligned} [(x_1 \wedge x_2) \vee (y_1 \vee y_2)] \wedge t &= [(x_1 \vee y_1 \vee y_2) \wedge (x_2 \vee y_1 \vee y_2)] \wedge t \\ &= [(m_1 \vee y_2) \wedge (m_2 \vee y_1)] \wedge t \\ &= (m_1 \wedge m_2) \wedge t \\ &= m_2 \wedge t \\ &= t \end{aligned}$$

This says that $(x_1 \wedge x_2) \vee (y_1 \vee y_2)$ is maximal in $(a]$. Therefore $y_1 \vee y_2$ is the complement of $x_1 \wedge x_2$ in $(a]$ and hence $x_1 \wedge x_2 \in B_a(L)$. Similarly we get that $y_1 \wedge y_2$ is the complement of $x_1 \vee x_2$ in $(a]$ and hence $x_1 \vee x_2 \in B_a(L)$. Thus $B_a(L)$ is a sub ADL of L .

Next we prove that $B_a(L)$ is relatively complemented. Let $x_1, x_2 \in B_a(L)$. Then there exists y_1 and y_2 in $(a]$ such that $x_1 \wedge y_1 = 0$, $x_1 \vee y_1 = m_1$ and $x_2 \wedge y_2 = 0$, $x_2 \vee y_2 = m_2$ where m_1 and m_2 are maximal elements in $(a]$; that is, $(m_1] = (a] = (m_2]$. Put $u = y_1 \wedge x_2$ and $v = x_1 \vee y_2$. Since x_1, x_2, y_1 and $y_2 \in (a]$ then both u and v are in $(a]$.

Now consider;

$$\begin{aligned} u \wedge v &= (y_1 \wedge x_2) \wedge (x_1 \vee y_2) \\ &= (y_1 \wedge x_2 \wedge x_1) \vee (y_1 \wedge x_2 \wedge y_2) \\ &= 0 \end{aligned}$$

Also for any $t \in (a]$ we have;

$$\begin{aligned} (u \vee v) \wedge t &= [(y_1 \wedge x_2) \vee (x_1 \vee y_2)] \wedge t \\ &= [(y_1 \vee x_1 \vee y_2) \wedge (x_2 \vee x_1 \vee y_2)] \wedge t \\ &= [(m_1 \vee y_2) \wedge (m_2 \vee x_1)] \wedge t \\ &= (m_1 \wedge m_2) \wedge t \\ &= m_2 \wedge t \\ &= t \end{aligned}$$

This says that $u \vee v$ is maximal in $(a]$. Therefore v is a complement of u in $(a]$ so that $u \in B_a(L)$. Also, $x_1 \wedge u = x_1 \wedge y_1 \wedge x_2 = 0 \wedge x_2 = 0$ and $x_1 \vee u = x_1 \vee (y_1 \wedge x_2) = (x_1 \vee y_1) \wedge (x_1 \vee x_2) = m_1 \wedge (x_1 \vee x_2) = x_1 \vee x_2$. Therefore u is an element in $(a]$ such that $x_1 \wedge u = 0$ and $x_1 \vee u = x_1 \vee x_2$. That is $u = x_1^{x_2}$ and hence $B_a(L)$ is relatively complemented. ■

Theorem 3.5. *Let L be an ADL. Then the following are equivalent*

- (1) *L is relatively complemented*
- (2) *$(a] = B_a(L)$ for all $a \in L$*
- (3) *For any a and b in L we have $a \in B_{a \vee b}(L)$.*

Proof: (1) $\Rightarrow$ (2) suppose that L is relatively complemented and let $a \in L$ be any element.

Claim: $(a] = B_a(L)$

Clearly we have $B_a(L) \subseteq (a]$. So that it remains to show that $(a] \subseteq B_a(L)$. Let $x \in (a]$ then, $a \wedge x = x$. Since L is relatively complemented there exists an element $y = x^a$ such that $x \wedge y = 0$ and $x \vee y = x \vee a$. That is, $x \wedge y = 0$ and $x \vee y$ is maximal in $(a]$. This says that y is a complement of x in $(a]$ which implies that $x \in B_a(L)$. Therefore $(a] = B_a(L)$.

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(2) $\Rightarrow$ (3). Suppose that $(a] = B_a(L)$ for all $a \in L$. Let $a, b \in L$, then $a \vee b \in L$ and hence by our assumption $(a \vee b] = B_{a \vee b}(L)$. By one of the absorption laws, since $(a \vee b) \wedge a = a$, the $a \in (a \vee b] = B_{a \vee b}(L)$. Thus $a \in B_{a \vee b}(L)$. Hence the result.

(3) $\Rightarrow$ (1). Suppose that $a \in B_{a \vee b}(L)$ for all $a, b \in L$. Then there exists $x \in (a \vee b]$ such that $a \wedge x = 0$ and $a \vee x = m$ where m is a maximal element in $(a \vee b]$ (i.e. $m \wedge (a \vee b) = a \vee b$ and $(a \vee b) \wedge m = m$).

Now $(a \vee x) \wedge (a \vee b) = m \wedge (a \vee b)$

$$\begin{aligned} &\Rightarrow [a \wedge (a \vee b)] \vee [x \wedge (a \vee b)] = a \vee b \\ &\Rightarrow a \vee [x \wedge (a \vee b)] = a \vee b \end{aligned}$$

Put $y = x \wedge (a \vee b)$ then $y \in (a \vee b]$ such that; $a \wedge y = a \wedge x \wedge (a \vee b) = 0 \wedge (a \vee b) = 0$ and $a \vee y = a \vee [x \wedge (a \vee b)] = a \vee b$. Therefore $y = a^b$. Hence L is relatively complemented. ■

Theorem 3.6. Let L_1 and L_2 be ADLs and $a_1 \in L_1$ and $a_2 \in L_2$.

Then $B_{(a_1, a_2)}(L_1 \times L_2) = B_{a_1}(L_1) \times B_{a_2}(L_2)$

Let L be an ADL. The relation $\varphi = \{(x, y) \in L \times L : x \wedge y = y \text{ and } y \wedge x = x\}$ is a congruence relation on L and is the smallest such that L/φ is a distributive lattice. [6]

Theorem 3.7. Let L be an ADL and $a \in L$. Then $B_{\varphi(a)}(L/\varphi)$ is isomorphic to $B_a(L)/\varphi \cap (B_a(L) \times B_a(L))$

Proof: Define $f: B_a(L) \rightarrow B_{\varphi(a)}(L/\varphi)$ by $f(x) = \varphi(x)$ for all $x \in B_a(L)$. We first show that this f is well defined. For; let $x \in B_a(L)$ then there exists $y \in (a]$ such that $x \wedge y = 0$ and $x \vee y$ is maximal in $(a]$; that is, $(x \vee y) \wedge a = a$ and $a \wedge (x \vee y) = (x \vee y)$

$$\begin{aligned} &\Rightarrow x \wedge y = 0 \text{ and } (x \vee y, a) \in \varphi \\ &\Rightarrow \varphi(x \wedge y) = \varphi(0) \text{ and } \varphi(x \vee y) = \varphi(a) \\ &\Rightarrow \varphi(x) \wedge \varphi(y) = \varphi(0) \text{ and } \varphi(x) \vee \varphi(y) = \varphi(a) \\ &\Rightarrow \varphi(y) \text{ is the complement of } \varphi(x) \text{ in } (\varphi(a)] \\ &\Rightarrow \varphi(x) \in B_{\varphi(a)}(L/\varphi) \\ &\Rightarrow f(x) \in B_{\varphi(a)}(L/\varphi) \end{aligned}$$

Therefore f is well defined. It is also clear that f is a homomorphism. Now we show that f is an epimorphism. Let $\varphi(x) \in B_{\varphi(a)}(L/\varphi)$ for some $x \in L$, then there exists $\varphi(y) \in (\varphi(a)]$ such that; $\varphi(x) \wedge \varphi(y) = \varphi(0)$ and $\varphi(x) \vee \varphi(y) = \varphi(a)$.

$$\begin{aligned} &\Rightarrow \varphi(x \wedge y) = \varphi(0) \text{ and } \varphi(x \vee y) = \varphi(a) \\ &\Rightarrow (x \wedge y, 0) \in \varphi \text{ and } (x \vee y, a) \in \varphi \\ &\Rightarrow x \wedge y = 0 \text{ and } (x \vee y) \wedge a = a \text{ and } a \wedge (x \vee y) = (x \vee y); \text{ that is, } x \vee y \text{ is} \\ &\text{maximal in } (a] \end{aligned}$$

$$\begin{aligned} &\Rightarrow y \text{ is the complement of } x \text{ in } (a] \\ &\Rightarrow x \in B_a(L) \text{ such that } f(x) = \varphi(x) \end{aligned}$$

Therefore f is an epimorphism. Also consider its kernel;

$$\begin{aligned} \ker f &= \{(x, y) \in B_a(L) \times B_a(L) : f(x) = f(y)\} \\ &= \{(x, y) \in B_a(L) \times B_a(L) : \varphi(x) = \varphi(y)\} \\ &= \{(x, y) \in B_a(L) \times B_a(L) : (x, y) \in \varphi\} \end{aligned}$$

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$$= \varphi \cap (B_a(L) \times B_a(L))$$

Therefore by the fundamental theorem of homomorphisms; $B_a(L)/\ker f$ is isomorphic to

$$B_{\varphi(a)}(L/\varphi); \text{ that is, } B_a(L)/\varphi \cap (B_a(L) \times B_a(L)) \text{ is isomorphic to } B_{\varphi(a)}(L/\varphi). \blacksquare$$

It is known that an ideal I of an ADL L is complemented there exists an ideal J of L such that $I \cap J = (0)$ and $I \vee J = L$. In the following we define sectionally complemented ideals:

Definition 3.8. An ideal I in an ADL L is said to be sectionally complemented relative to an ideal J of L if there exists an ideal K of L such that $I \cap K = (0)$ and $I \vee K = I \vee J$.

It observed in [4] that an ideal I of an ADL L is complemented if and only if $I = (a)$ for some $a \in L$. And later it has been proved in [3] that an ideal I of L is complemented if and only if $I = (a)$ for some $a \in B(L)$. Analogously we have the following:

Theorem 3.9. Given ideals I and J of an ADL L . I is sectionally complemented relative to J whenever $I = (a]$ and $J = (b]$ for some a and b in L such that $a \in B_{a \vee b}(L)$

In the next theorem we characterize complemented ideals as sectionally complemented ideals:

Theorem 3.10. I is complemented if and only if I is sectionally complemented relative to each ideal J of L .

Corollary 3.11. An ADL L is relatively complemented if and only if the class $PI(L)$ of all principal ideals of L is relatively complemented distributive lattice.

Given any filter F of an ADL, define

$$\phi_F := \{(a, b) \in L \times L : x \wedge a = x \wedge b \text{ for some } x \in F\}$$

Then ϕ_F is a congruence on L . we write $\phi_x := \{(a, b) \in L \times L : x \wedge a = x \wedge b\}$ for any $x \in L$. It is observed in [3] that for any $x \in L$, $\phi_x = L \times L$ if and only if $x = 0$. A congruence θ on an ADL L is said to be factor congruence if there exists a congruence ϕ on L such that $\theta \cap \phi = \Delta$ and $\theta \circ \phi = L \times L$ [3]. In the following we extend this and we define sectional factor congruences.

Definition 3.12. Let θ_1 and θ_2 congruence relations on an ADL L . θ_1 is said to be sectional factor congruence relative to θ_2 if θ_1 is a factor congruence in the interval $[\theta_1 \cap \theta_2, L \times L]$. That is if there exists a congruence ψ on L such that $\theta_1 \cap \psi = \theta_1 \cap \theta_2$ and $\theta_1 \circ \psi = L \times L$.

It is proved in [3] that a congruence θ on L is a factor-congruence if and only if $\theta = \phi_a$ for some $a \in B(L)$. In the next theorem we extend this result to the case of sectional factor congruence relations:

Generalized Birkhoff Center of ADLs

Theorem 3.13. *Given congruence relations θ_1 and θ_2 on L . θ_1 is a sectional factor congruence relative to θ_2 , whenever $\theta_1 = \phi_a$ and $\theta_2 = \phi_b$ for some a and $b \in L$ such that $a \in B_{a \vee b}(L)$.*

In the rest part of this section we define the generalized Birkhoff center $GB(L)$ of an almost distributive lattice L not necessarily with maximal elements and we give some characterizing properties analogous to those results in [3].

Definition 3.14. *Given an ADL L , define*

$$GB(L) = \{a \in L : a \in B_{a \vee b}(L) \text{ for all } b \in L\}$$

Not that, in the case L has maximal elements, the generalized Birkhoff center $GB(L)$ of L coincides with the Birkhoff center $B(L)$ of L . It is also easy to verify that the generalized Birkhoff center $GB(L)$ of L is a relatively complemented ADL under the operations induced by those of operations in L . Moreover as a result of Theorem 3.5 it follows that L is relatively complemented if and only if $GB(L) = L$.

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