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Integral Representation of Polynomial $X_n(x; a, b)$

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Abstract: In the present paper we have obtained fine difference formula, contour integral representation, real integral representation, infinite single integral representation, finite single integral representation, finite double integral representation, finite double integral representation of polynomial $X_n(x; a, b)$.

Keywords: Finite difference, single integral representation, contour integral representation, simple generating relation, double integral representation

I. INTRODUCTION

Bajpai, S.D.[1,1993] defined the classical polynomial

$$X_n(x; a, \alpha) = {}_2F_1\left[-n, a; -\frac{x}{\alpha}\right] \quad (1)$$

II. FINITE DIFFERENCE FORMULA

From (1) $X_n(x; a, \alpha) = {}_2F_1\left[-n, a; -\frac{x}{\alpha}\right]$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k \left(\frac{x}{\alpha}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (-n)_k (a)_k \left(\frac{x}{\alpha}\right)^k}{k!}$$

But $(-n)_k = \frac{(-1)^k n!}{(n-k)!} \quad (0 \leq k \leq n)$

$= 0 \quad k > n$

$$X_n(x; a, \alpha) = \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)!} \frac{(a)_k \left(-\frac{x}{\alpha}\right)^k}{k!} \quad (2)$$

$$X_n(x; a, \alpha) = \sum_{k=0}^n \frac{(-1)^n (-1)^{n-k} n!}{(n-k)!} \frac{(a)_k \left(-\frac{x}{\alpha}\right)^k}{k!}$$

$$= \sum_{k=0}^n \frac{(-1)^n (-1)^{n-k} n!}{(n-k)!} \frac{\Gamma(a+k)}{\Gamma a} \frac{\left(-\frac{x}{\alpha}\right)^k}{k!}$$

$$= \frac{(-1)^n}{\Gamma a} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \Gamma(a+k) \left(-\frac{x}{\alpha}\right)^k$$

Replacing a by $a+\lambda$ we get

$$X_n(x; a + \lambda, \alpha) = \frac{(-1)^n}{\Gamma a} \left(\frac{x}{\alpha}\right)^{-\lambda} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \Gamma(a+k) \left(-\frac{x}{\alpha}\right)^{k+\lambda}$$

$$= \frac{(-1)^n}{\Gamma(a+\lambda)} \left(\frac{x}{\alpha}\right)^{-\lambda} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \Gamma(a+\lambda+k) \left(-\frac{x}{\alpha}\right)^{k+\lambda}$$

$$= \frac{(-1)^n}{\Gamma(a+\lambda)} \left(\frac{x}{\alpha}\right)^{-\lambda} \Delta_\lambda^n \left[\Gamma(a+\lambda) \left(-\frac{x}{\alpha}\right)^{k+\lambda} \right]$$

Hence we have proved that

$$X_n(x; a+\lambda, \alpha) = \frac{(-1)^n}{\Gamma(a+\lambda)} \left(\frac{x}{\alpha}\right)^{-\lambda} \Delta_\lambda^n \left[\Gamma(a+\lambda) \left(-\frac{x}{\alpha}\right)^{k+\lambda} \right] \quad (3)$$

III. SIMPLE GENERATING RELATION

A. By equation (2)

$$\sum_{n=0}^{\infty} \frac{X_n(x; a, \alpha) t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)!} \frac{(a)_k \left(-\frac{x}{\alpha}\right)^k}{k!} \frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{X_n(x; a, \alpha) t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{(n)!} \frac{(a)_k \left(-\frac{x}{\alpha}\right)^k}{k!} \frac{t^{n+k}}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(a)_k \left(-\frac{xt}{\alpha}\right)^k}{k!} \frac{t^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(a)_k \left(-\frac{xt}{\alpha}\right)^k}{k!} \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(a)_k \left(-\frac{xt}{\alpha}\right)^k}{k!} e^t$$

$$= e^t \sum_{k=0}^{\infty} \frac{(a)_k \left(\frac{xt}{\alpha}\right)^k}{k!}$$

$$= e^t {}_1F_0 \left[a; -; \frac{xt}{\alpha} \right]$$

$$f(t) = e^t {}_1F_0 \left[a; -; \frac{xt}{\alpha} \right]$$

B. By using Maclaurin's theorem

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0) t^n}{n!}$$

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{t^{n+1}}^{(+0)} \frac{f(t)}{t^{n+1}} dt \quad \forall n = 0, 1, 2, \dots$$

$$\text{If } e^t {}_1F_0 \left[a; -; \frac{xt}{\alpha} \right] = \sum_{n=0}^{\infty} \frac{X_n(x; a, \alpha) t^n}{n!}, \text{ then}$$

$$X_n(x; a, \alpha) = \frac{n!}{2\pi i} \int_{t^{n+1}}^{(+0)} t^{-n-1} e^t {}_1F_0 \left[a; -; \frac{xt}{\alpha} \right] dt \quad (4)$$

IV. REAL INTEGRAL REPRESENTATION

Put $t = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$) in (4)

$$dt = ie^{i\theta} d\theta$$

$$X_n(x; a, \alpha) = \frac{n!}{2\pi i} \int_0^{2\pi} (e^{i\theta})^{-n-1} \exp(e^{i\theta}) {}_1F_0\left[a; -; \frac{xe^{i\theta}}{\alpha}\right] ie^{i\theta} d\theta$$

$$\begin{aligned} X_n(x; a, \alpha) &= \frac{n!}{2\pi} \int_0^{2\pi} (e^{-in\theta}) \exp(e^{i\theta}) \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \left(\frac{xe^{i\theta}}{\alpha}\right)^k d\theta \\ &= \frac{n!}{2\pi} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \int_0^{2\pi} (e^{i(k-n)\theta}) \exp(e^{i\theta}) \left(\frac{x}{\alpha}\right)^k d\theta \\ &= \frac{n!}{2\pi} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \int_0^{2\pi} (e^{i(k-n)\theta}) \sum_{s=0}^{\infty} \frac{(e^{i\theta})^s}{s!} \left(\frac{x}{\alpha}\right)^k d\theta \\ &= \frac{n!}{2\pi} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{1}{s!} \frac{(a)_k}{k!} \left(\frac{x}{\alpha}\right)^k \int_0^{2\pi} (e^{i(k+s-n)\theta}) d\theta \\ &= \frac{n!}{2\pi} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{1}{s!} \frac{(a)_k}{k!} \left(\frac{x}{\alpha}\right)^k \int_0^{2\pi} \text{Cis}(k+s-n)\theta d\theta \\ &\quad \text{where } \text{Cis}\varphi = \cos\varphi + i\sin\varphi \\ &= \frac{n!}{2\pi} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_k}{k! s!} \left(\frac{x}{\alpha}\right)^k \int_0^{2\pi} [\cos\varphi + i\sin\varphi] d\theta \end{aligned}$$

$$\text{where } \varphi = (k+s-n)\theta$$

$$X_n(x; a, \alpha) = \frac{n!}{2\pi} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_k}{k! s!} \left(\frac{x}{\alpha}\right)^k \int_0^{2\pi} [\cos\varphi + i\sin\varphi] d\theta \quad (5)$$

$$\text{where } \varphi = (k+s-n)\theta.$$

V. SINGLE INFINITE INTEGRAL REPRESENTATION

From (1) $X_n(x; a, \alpha) = {}_2F_1\left[-n, b; -\frac{x}{\alpha}\right]$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k \left(-\frac{x}{\alpha}\right)^k}{k!} \\ &= \sum_{k=0}^n \frac{(-n)_k}{k!} \frac{\Gamma(a+k) \left(-\frac{x}{\alpha}\right)^k}{\Gamma a} \\ &= \sum_{k=0}^n \frac{(-n)_k}{k! \Gamma a} \frac{\Gamma\left(a+k+\frac{1}{2}\right) - \frac{1}{2} \left(-\frac{x}{\alpha}\right)^k}{\Gamma a} \\ &= \sum_{k=0}^n \frac{(-n)_k}{k! \Gamma a} \left(-\frac{x}{\alpha}\right)^k \int_{-\infty}^{\infty} \exp(-t^2) t^{2(a+k-\frac{1}{2})} dt \\ &= \frac{1}{\Gamma a} \int_{-\infty}^{\infty} \exp(-t^2) t^{2a-1} \sum_{k=0}^n \frac{(-n)_k}{k!} \left(-\frac{xt^2}{\alpha}\right)^k dt \\ &= \frac{1}{\Gamma a} \int_{-\infty}^{\infty} \exp(-t^2) t^{2a-1} {}_1F_0\left[-n; -; -\frac{xt^2}{\alpha}\right] dt \quad (6) \end{aligned}$$

VI. FINITE DOUBLE INTEGRAL REPRESENTATION

Srivastava, H.M. and Karlsson, P.W.[5, P.275]

$$\iint_D u^{a-1} v^{b-1} (1-u-v)^{c-1} du dv = \frac{\Gamma a \Gamma b \Gamma c}{\Gamma(a+b+c)}$$

Where D is bounded by the lines $u \geq 0, v \geq 0$ and $u + v \leq 1$.

$$X_n(x; a, c) = {}_2F_1\left[-n, b; -\frac{x}{c}\right]$$

$$X_n(x; a, c) = \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k \left(\frac{x}{c}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k \Gamma(a+k) \left(-\frac{x}{c}\right)^k}{\Gamma a k!}$$

$$= \frac{\Gamma a}{\Gamma a \Gamma b \Gamma(\alpha-a-b)} \sum_{k=0}^{\infty} \frac{(-n)_k \Gamma b \Gamma(\alpha-a-b) \Gamma(a+k) (a)_k \left(-\frac{x}{c}\right)^k}{\Gamma(\alpha+k) k!}$$

$$= \frac{\Gamma a}{\Gamma a \Gamma b \Gamma(\alpha-a-b)} \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k}{k!} \iint_D u^{a+k-1} v^{b-1} (1-u-v)^{\alpha-a-b-1} \left(-\frac{x}{c}\right)^k du dv$$

$$= \frac{\Gamma a}{\Gamma a \Gamma b \Gamma(\alpha-a-b)} \iint_D u^{a-1} v^{b-1} (1-u-v)^{\alpha-a-b-1} \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k}{k!} \left(-\frac{xu}{c}\right)^k du dv$$

$$= \frac{\Gamma a}{\Gamma a \Gamma b \Gamma(\alpha-a-b)} \iint_D u^{a-1} v^{b-1} (1-u-v)^{\alpha-a-b-1} {}_2F_1\left[-n, b; -\frac{xu}{c}\right] du dv \quad (7)$$

VII. INFINITE SINGLE INTEGRAL REPRESENTATION

From equation (1), $X_n(x; a, \alpha) = {}_2F_1\left[-n, b; -\frac{x}{\alpha}\right]$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k \left(-\frac{x}{\alpha}\right)^k}{k!}$$

$$= \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)!} \frac{(a)_k \left(-\frac{x}{\alpha}\right)^k}{k!}$$

$$= \sum_{k=0}^n \frac{(-n)_k}{(k)!} \frac{\Gamma(a+k) \left(-\frac{x}{\alpha}\right)^k}{\Gamma a}$$

$$= \frac{1}{\Gamma a} \int_0^{\infty} \sum_{k=0}^n \frac{(-n)_k}{(k)!} \frac{\left(-\frac{x}{\alpha}\right)^k}{k!} e^{-t} t^{a+k-1} dt$$

$$= \frac{1}{\Gamma a} \int_0^{\infty} \sum_{k=0}^n \frac{(-n)_k}{(k)!} \frac{\left(-\frac{xt}{\alpha}\right)^k}{k!} e^{-t} t^{a-1} dt$$

$$= \frac{1}{\Gamma a} \int_0^{\infty} \left(1 + \frac{xt}{\alpha}\right)^n e^{-t} t^{a-1} dt$$

$$= \frac{1}{\alpha^n \Gamma a} \int_0^{\infty} e^{-t} t^{a-1} (\alpha + xt)^n dt \quad (8)$$

VIII. INFINITE DOUBLE INTEGRAL REPRESENTATION

From [2 ,P.177(16)],

$$\begin{aligned}
 \int_0^\infty \int_0^\infty \varphi(x+y) x^\alpha y^\beta dx dy &= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \int_0^\infty \varphi(z) z^{\alpha+\beta+1} dz \\
 \int_0^\infty \int_0^\infty u^{\alpha+1/2} v^\alpha (1-u-v)^{-3/2} X_n(x; a, 4uvc) du dv \\
 &= \int_0^\infty \int_0^\infty u^{\alpha+1/2} v^\alpha (1-u-v)^{-3/2} \sum_{k=0}^\infty \frac{(-n)_k (a)_k \left(-\frac{x}{4uvc}\right)^k}{k!} du dv \\
 &= \sum_{k=0}^\infty \frac{(-n)_k (a)_k}{k!} \int_0^\infty \int_0^\infty u^{\alpha+1/2} v^\alpha 2^{-2k} u^{-k} v^{-k} (1-u-v)^{-3/2} \left(-\frac{x}{c}\right)^k du dv \\
 &= \sum_{k=0}^\infty \frac{(-n)_k (a)_k 2^{-2k}}{k!} \left(-\frac{x}{c}\right)^k \frac{\Gamma(\alpha-k+3/2)\Gamma(\alpha-k+1)}{\Gamma(2\alpha-2k+5/2)} \int_0^\infty z^{2\alpha-2k+3/2} (1-z)^{-3/2} dz \\
 &= \sum_{k=0}^\infty \frac{(-n)_k (a)_k 2^{-2k}}{k!} \left(-\frac{x}{c}\right)^k \frac{\Gamma(\alpha-k+3/2)\Gamma(\alpha-k+1)}{\Gamma(2\alpha-2k+5/2)} \beta\left(2\alpha+2k+\frac{5}{2}, 2k-2\alpha-1\right) \\
 &= \sum_{k=0}^\infty \frac{(-n)_k (a)_k 2^{-2k}}{k!} \left(-\frac{x}{c}\right)^k \frac{\Gamma(\alpha-k+3/2)\Gamma(\alpha-k+1)}{\Gamma(2\alpha-2k+5/2)} \frac{\Gamma(2\alpha+2k+\frac{5}{2})\Gamma(2k-2\alpha-1)}{\Gamma(3/2)} \\
 &= \sum_{k=0}^\infty \frac{(-n)_k (a)_k 2^{-2k}}{k!} \left(-\frac{x}{c}\right)^k \frac{\Gamma(\alpha-k+3/2)\Gamma(\alpha-k+1)}{\Gamma(2\alpha-2k+5/2)} \frac{\Gamma(2k-2\alpha-1)}{\Gamma(3/2)} \\
 &= \sum_{k=0}^\infty \frac{(-n)_k (a)_k 2^{-2k}}{k!} \left(-\frac{x}{c}\right)^k \frac{\Gamma(\alpha-k+1)\Gamma(\alpha-k+1+\frac{1}{2})\Gamma(2k-2\alpha-1)}{\frac{1}{2}\sqrt{\pi}} \\
 &= \sum_{k=0}^\infty \frac{(-n)_k (a)_k 2^{-2k}}{k!} \frac{\left(-\frac{x}{c}\right)^k \sqrt{\pi} \Gamma(2\alpha-2k+2)\Gamma(2k-2\alpha-1)}{\frac{1}{2}\sqrt{\pi} 2^{2\alpha-2k+2-1}} \\
 &= \sum_{k=0}^\infty \frac{(-n)_k (a)_k}{k!} \frac{\left(-\frac{x}{c}\right)^k \Gamma(2\alpha-2k+2)\Gamma(2k-2\alpha-1)}{2^{2\alpha}} \\
 &= \sum_{k=0}^\infty \frac{(-n)_k (a)_k}{k!} \frac{\left(-\frac{x}{c}\right)^k \Gamma(2\alpha-2k+2)\Gamma(1-(2\alpha-2k+2))}{2^{2\alpha}} \\
 &= \sum_{k=0}^\infty \frac{(-n)_k (a)_k \left(-\frac{x}{c}\right)^k}{k! 2^{2\alpha}} \frac{\pi}{\sin[\pi(2\alpha-2k+2)]} \\
 &= \frac{\pi}{2^{2\alpha} \sin(2\pi\alpha)} \sum_{k=0}^\infty \frac{(-n)_k (a)_k \left(-\frac{x}{c}\right)^k}{k!} \\
 &= \frac{\pi}{2^{2\alpha} \sin(2\pi\alpha)} X_n(x; a, c)
 \end{aligned}$$

Thus we arrive at

$$X_n(x; a, c) = \frac{2^{2\alpha} \sin(2\pi\alpha)}{\pi} \int_0^\infty \int_0^\infty u^{\alpha+1/2} v^\alpha (1-u-v)^{-3/2} X_n(x; a, 4uvc) du dv \quad (9)$$

The equations (4), (5), (6), (7), (8), (9) are not in literature.

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