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Forbidden 3-Colored Posets of Cover-Incomparable Line Graphs

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Abstract: The cover-incomparability graph of a poset P is the edge-union of the covering and the incomparability graph of P . As a continuation of the study 3-colored diagrams we characterize some forbidden $\triangleleft$ -preserving subposets of the posets whose cover-in comparability graphs are not line graphs is proved.

Index Terms: Cover-incomparability graph, Line graph, Poset.

I. INTRODUCTION AND PRELIMINARIES

Cover-incomparability graphs of posets, or shortly C-I graphs, were introduced in [2] as the underlying graphs of the standard interval function or transit function on posets (for more on transit functions in discrete structures .[3, 4, 5, 6, 11]). On the other hand, C-I graphs can be defined as the edge-union of the covering and incomparability graph of a poset; in fact, they present the only non-trivial way to obtain an associated graph as unions and/or intersections of the edge sets of the three standard associated graphs (i.e. covering, comparability and incomparability graph). In the paper that followed [9], it was shown that the complexity of recognizing whether a given graph is the C-I graph of some poset is in general NP-complete. In [1] the problem was investigated for the classes of split graphs and block graphs, and the C-I graphs within these two classes of graphs were characterized. This resulted in linear-time recognition algorithms for C-I block and C-I split graphs. It was also shown in [1] that whenever a C-I graph is a chordal graph, it is necessarily an interval graph, however a structural characterization of C-I interval graphs (and thus C-I chordal graphs) is still open. C-I distance-hereditary graphs have been characterized and shown to be efficiently recognizable [10].

Let $P = (V; \leq)$ be a poset. If $u \leq v$ but $u \neq v$, then we write $u < v$. For $u, v \in V$ we say that v covers u in P if $u < v$ and there is no w in V with $u < w < v$. If $u \leq v$ we will sometimes say that u is below v , and that v is above u . Also, we will write $u \triangleleft v$ if v covers u ; and $u \triangleleft\triangleleft v$ if u is below v but not covered by v . By $u \parallel v$ we denote that u and v are incomparable. Let V' be a nonempty subset of V . Then there is a natural poset $Q = (V'; \leq')$, where $u \leq' v$ if and only if $u \leq v$ for any $u, v \in V'$. The poset Q is called a subposet of P and its notation is simplified to $Q = (V'; \leq)$. If, in addition, together with any two comparable elements u and v of Q , a chain of shortest length between u and v of P is also in Q , we say that Q is an isometric sub poset of P . Recall that a poset P is dual to a poset Q if for any $x, y \in P$ the following holds: $x \leq y$ in P if and only if $y \leq x$ in Q . Given a poset P , its cover-incomparability graph G_P has V as its vertex set, and uv is an edge of G_P if $u \triangleleft v$, $v \triangleleft u$, or u and v are incomparable. A graph that is a cover-incomparability graph of some poset P will be called a C-I graph.

Lemma 1 [2] Let P be a poset and G_P its C-I graph. Then

- (i). G_P is connected;
- (ii). vertices in an independent set of G_P lie on a common chain of P ;
- (iii). an antichain of P corresponds to a complete subgraph in G_P ;
- (iv). contains no induced cycles of length greater than 4.

II. 3-Colored Diagram

A 3-coloured diagram Q ; we consider normal edges to represent vertices in a covering relation and red edges to represent incomparable vertices or vertices in a covering relation and dashed lines to represent a chain of length three and thus constitute the 3-colors and hence the name 3-colored diagram. The idea of 3-colored diagrams is explained as follows. Let G be a C-I graph and H be an induced subgraph of G . We note that there can be different $\triangleleft$ -preserving subposets Q_i of some posets with G_{Q_i} isomorphic to the subgraph H . Let u, v, w be an induced path in the direction from u to v in H . There are four possibilities in which u, v and w can be related in the $\triangleleft$ -preserving subposets. It is possible to have $u \triangleleft v$, $u \parallel v$, $v \triangleleft w$ and $v \parallel w$. Each case will appear as a $\triangleleft$ -preserving subposet of four different posets. If $u \triangleleft v$ and $v \triangleleft w$ in a sub poset, then $u \triangleleft v \triangleleft w$ is a chain in the subposet and u, v, w is an induced path in H . If there is either $u \parallel v$ or $v \parallel w$ in a subposet Q , then there should be another chain from u to w in Q in

order to have u, v, w an induced path in H . We try to capture this situation using the idea of 3-colored diagram. Suppose in $\triangleleft$ -preserving subposet Q of a poset P , there exists two elements u, v which is always connected by some chain of length three in Q . Let w be an element in Q such that either both uw and vw are red edges or any one of them is a red edge. Then in order to have a chain between u and v , there must exist an element x in Q so that u, x, v form a chain in Q . When both edges are normal, then we have the chain u, w, v in Q and hence the chain u, x, v is not required in this case. We denote the chain u, x, v by dashed lines between ux and xv in order to specify that it is possible to have the presence or absence of the chain u, x, v in Q . The presence of the chain u, x, v implies that either both of the edges uw and wv are red edges or one of them is a red edge. The absence of the chain implies that both uw and wv are normal edges in Q . We call posets having the above mentioned diagrams as 3-colored diagrams. Thus a 3-colored diagram contains normal edges, red edges and dashed lines, in which the dashed line between elements u and v will vanish, when there is a chain between u and v using normal or red edges. We can define 3-colored subposets in a similar way as discussed above. All subposets of the poset P that we consider in this paper are 3-colored diagrams. Thus by a single 3-colored diagram, we represent a collection of $\triangleleft$ -preserving subposets to be forbidden for a poset. We sometimes use the term 3-colored subposets instead of 3-colored diagrams in this paper. In a similar way the dual of a 3-colored diagram is also meaningful and represents a collection of $\triangleleft$ -preserving dual subposets.

Theorem 2: (Theorem 1,[8]) Let G be a class of graphs with a forbidden induced subgraphs characterization. Let $P = \{P \mid P \text{ is a poset with } G_{T_P} \in G\}$. Then P has a characterization by forbidden $\triangleleft$ -preserving subposets.

Theorem 3: (Theorem 7.1.8, [7]) Let G be a graph. Then G is a line graph if and only if G contains none of the nine forbidden graphs of Figure 1 as an induced subgraph.

Figure 1: Nine Forbidden induced subgraphs of line graphs

Theorem 4: (Theorem 4.1,[12]) Let P be a poset. Then G_P is cograph if and only if P contains none of $T_1, \dots, T_7$, depicted in Figure 2, and no duals of T_2 and T_5 as $\triangleleft$ -preserving subposet.

Figure 2: Forbidden $\triangleleft$ - preserving subsets for C-I cographs

Theorem 5: (Theorem 4,[13]) If P is a poset, then G_P is cograph if and only if P does not contain T_1 from Figure 1 and no 3-colored diagram Q_C from Figure 3 and its dual are $\triangleleft$ - preserving subsets .

Figure 3: Forbidden $\triangleleft$ - preserving 3-colored subsets for C-I cographs

We consider 3-colored subsets to be forbidden so that its C-I graphs belong to the graph family $F(G_3)$ of G_3 in Figure1

III. 3-Colored $\triangleleft$ - preserving subsets of posets whose c-i graphs belong to the family $F(G_3)$

We have the following theorem regarding the graph family $F(G_3)$.

Theorem 6: If P is a poset, then G_P belongs to $F(G_3)$ if and only if P contains the 3-colored diagrams Q_i ; $i = 2, 3, 4$ from Figure 4 and their duals.

Figure 4:Forbidden 3-colored diagrams for posets whose C-I graphs contains G_3 , depicted in Figure 1 (e).

Proof: Suppose P contains the 3-colored diagrams Q_i ; $i = 2, 3, 4$. Then clearly G_P contains the graph from Figure1 (e) as an induced subgraph.

Conversely, suppose $G_P \in F(G_3)$. Then G_P contains an induced subgraph G_3 shown in Figure 1(e), with vertices labeled by u, v, w, x, y and z . From the graph G_3 (Figure 1(e)), it follows that the vertex sets $\{u, v, w, x\}$, $\{u, v, w, z\}$, $\{u, v, y, x\}$ and $\{u, v, y, z\}$ respectively induce a P_4 . Since the vertices w, y, x, z induce a K_4 , without loss of generality, we consider a P_4 induced by any of the above four sets, say the P_4 induced by the vertices u, v, w and x . We have already identified the $\triangleleft$ - preserving subsets T_i , $i = 1, 2$

,...,7 in Theorem 4 which correspond to an induced P_4 in its C-I graph. More clearly, by Theorem 4 and Theorem 5, the chain of height 4 isomorphic to T_1 and the 3-colored poset Q_C induce a P_4 in its corresponding C-I graph.

Now we consider two cases.

Case (1): The P_4 in G_3 induced by the vertices u, v, w and x is formed by the chain $u \triangleleft v \triangleleft w \triangleleft x$ in the poset P .

Case (2): The P_4 in G_3 induced by the vertices u, v, w and x is formed by two chains of length 3 as in the poset P as shown in Figure3.

We first consider Case (1).

Since w and y are adjacent in the graph G_3 , either w and y are in a covering relation or these vertices are incomparable in P . The covering relation between w and y in P contradict the fact that y and v are adjacent or y and x are adjacent in G_3 . Hence $w \parallel y$.

Since x and z are adjacent in G_3 we have two possibilities, that is, $x \parallel z$ or $z \triangleleft x$ ($x \triangleleft z$ is not possible since w and z are adjacent).

Subcase (1.1): $x \parallel z$.

Consider the vertex y in G_3 . There are two possibilities for y with respect to v .

Either $v \triangleleft y$ or $v \parallel y$ ($y \triangleleft v$ is not possible as w and y are adjacent). If $v \triangleleft y$ then y and u are connected by a path of length 2 follows.

Now consider z . Since z is adjacent to both w and y , there are two possibilities, either $w \triangleleft z$ ($z \triangleleft w$ is not possible as z is adjacent to x) or $w \parallel z$. Similarly $y \triangleleft z$ or $y \parallel z$.

This situation can be described into the following cases.

Subcase (1.1.1): $v \triangleleft y, w \triangleleft z$ and $y \triangleleft z$.

Subcase (1.1.2): $v \triangleleft y, w \triangleleft z$ and $y \parallel z$.

Subcase (1.1.3): $v \triangleleft y, w \parallel z$ and $y \triangleleft z$.

Subcase (1.1.4): $v \triangleleft y, w \parallel z$ and $y \parallel z$.

In the posets described by the subcases (1.1.1), (1.1.2) and (1.1.3), all the relations among the vertices u, v, w, x, y and z in the graph G_3 are captured and we are done. In the poset described by the subcase (1.1.4), since there is no path of length 2 between v and z , we conclude that there must be a dashed line between v and z representing a chain of length 3 between v and z . Here y and x can have both possibilities, namely $y \triangleleft x$ or $y \parallel x$ and hence the edge xy can also be represented by a red edge in the poset P . Therefore we can represent the edges between w and z, y and z by red lines and the posets described in all the four subcases are captured by the 3-colored diagram Q'_2 represented in Figure 4. It is also possible that $v \parallel y$. Then the poset described by the 3-colored diagram Q'_2 representing all the subcases (1.1.1), (1.1.2), (1.1.3) and (1.1.4) holds if we allow a dashed line between u and y . This situation is represented in the 3-colored diagram Q_2 shown in Figure 4.

Subcase (1.2): $z \triangleleft x$.

Since z is adjacent to w and y in G_3 and $u \triangleleft v \triangleleft w \triangleleft x$ in P , it follows that $w \parallel z$ and $y \parallel z$. Since v and z are nonadjacent in G_3 , there is a chain of length 3 between v and z defined by normal edges in P . Since u and y are nonadjacent in G_3 , there arises two possibilities according as $v \triangleleft y$ (the case $y \triangleleft v$ is not possible as w and y are adjacent in G_3) or $v \parallel y$ in P .

Subcase (1.2.1): $v \triangleleft y$.

The poset Q_3 describes this situation without dashed lines between u and y . Subcase (1.2.2): $v \parallel y$. Here, there must be a chain of length 3 between u and y in P . Both subcases are represented by the three colored diagram Q_3 . Case (2): The P_4 in G_3 induced by the vertices u, v, w and x is formed by two chains of length 3 as in the poset P as shown in Figure 3. By Theorem 5, the set $\{u, v, w, x\}$ will form the 3-colored diagram Q_C in Figure 3. Now we consider the vertices y and z in G_3 and find all the possibilities that these vertices can appear in the 3-colored diagram Q_C . Since there is a path of length 2 from u to y and a path of length three from u to z in G_3 , there must be a chain of length 3 from u to y and u to z in P . If both these chains pass through a in Q_C , then both the vertices are in a covering relation with a ($y \triangleleft a$ and $z \triangleleft a$, since y and z are adjacent with w). Otherwise, there must be a dashed line between u and y , and u and z representing a chain of length 3 between u and y , and u and z respectively. Similar is the case between v and z in the graph G_3 . Therefore, there must be a chain of length 3 from v to z in P . If the chain passes through b , then there is a covering relation between b and z ($z \triangleleft b$, since x and z are adjacent in G_3). Otherwise, there must be a dashed line between v and z representing a chain of length 3 between v and z . Since vw and vy are edges in G_3 , there are two cases, either $v \triangleleft w$ or $v \parallel w$ and $v \triangleleft y$ or $v \parallel y$ and hence these edges are red. From the above discussion, analyzing all the possibilities in which the vertices y and z can be related with the 3-colored diagram Q_C , it can be verified easily that we obtain the 3-colored diagram Q_4 in Figure 4, which is an extension of Q_C . Thus we have completed all the cases in which vertices of the graph G_3 can appear in the poset P , which completes the proof of the theorem

Figure 5: $\triangleleft$ - preserving subposets corresponding to Q_2 Figure 6: $\triangleleft$ - preserving subposets corresponding to Q_3

Figure 5: $\prec$ - preserving subposets corresponding to Q_4

II. REMARKS

The number of forbidden $\prec$ - preserving subposets of a poset P is such that its C-I graph G_P belongs to a graph possessing a forbidden induced subgraph characterization as instances of the Theorem 2 is in general very large compared to the number of forbidden induced subgraphs. Here we characterize forbidden $\prec$ - preserving subposets of G_3 in Figure1 and introduce the idea of 3-colored diagrams to minimize the list of subposets.

REFERENCES

- [1] Brešar, M. Changat, T. Gologranc, J. Mathew, A. Mathews, Cover incomparability graphs and chordal graphs, Discrete Appl. Math. 158 (2010) 1752-1759
- [2] B. Brešar, M. Changat, S. Klavžar, M. Kovše, J. Mathew, A. Mathews, Cover incomparability graphs of posets, Order 25 (2008) 335-347.
- [3] B. Brešar, M. Changat, S. Klavžar, J. Mathew, A. Mathews, Prasanth G. N., Characterizing posets for which their natural transit functions coincide, Ars Math. Contemp. 2 (2009) 27-33.
- [4] M. Changat, S. Klavžar and H. M. Mulder, The all-paths transit function of a graph, Czech. Math. J. 51(126) (2001) 439-448
- [5] M. Changat and J. Mathews, Induced path transit function, monotone and Peano axioms, Discrete Math. 286 (2004) 185-194
- [6] A. Mathews and J. Mathews, Transit functions on posets and lattices. In: Convexity in Discrete Structures (M. Changat, S. Klavžar, H.M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 105-116.
- [7] A. Brandstädt, V. B. Le, J. P. Spinrad: Graph Classes: A Survey, SIAM Monographs on Discrete Mathematics and Applications, 3, 1999.
- [8] J. Book, J. Maxová, Characterizing subclasses of cover-incomparability graphs by forbidden subposets, Order manuscript
- [9] J. Maxová, P. Pavliková, D. Turzik, On the complexity of cover-incomparability graphs of posets, Order 26 (2009), 229-236.
- [10] J. Maxová, D. Turzik, Which distance-hereditary graphs are cover-incomparability graphs?, Discrete Appl. Math. 161(2013), 2095-2100.
- [11] M. Mulder, Transit functions on graphs (and posets). In: Convexity in Discrete Structures (M. Changat, S. Klavžar, H.M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 117-130.
- [12] B. Brešar, M. Changat, TanjaGlorance, BaijuSukumaran, Cographs which are cover-incomparability graphs of posets, Order DOI 10.1007/s11083-014-9324-X.
- [13] Baiju Sukumaran "2-Colored and 3-Colored Diagrams of Posets in Cover-incomparability Graphs", International Journal of Mathematics Trends and Technology (IJMTT). V53(4):267-269 January 2018. ISSN:2231-5373.

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