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Q-Hypergeometric Series and Their Transformation Formulae

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Abstract: In this paper, making use of certain known summation formulae, an attempt has been made to establish transformation formulae, for q -hypergeometric series.

Keywords: Summation Formulae, Transformation Formulae, Hypergeometric Series, Identity, Inter-Series

I. INTRODUCTION

In 1972 Verma [1] established the following expansion formula

$$\sum_{n=0}^{\infty} \frac{(-x)^n q^{n(n-1)/2}}{(q, \gamma q^n; q)_n} \sum_{k=0}^{\infty} \frac{(\alpha, \beta; q)_{n+k}}{(q, \gamma q^{2n+1}; q)_k} B_{n+k} x^k \sum_{j=0}^n \frac{(q^{-n}, \gamma q^n; q)_j}{(q, \alpha, \beta; q)_j} A_j (wq)^j = \sum_{n=0}^{\infty} A_n B_n \frac{(xw)^n}{(q; q)_n} \quad (1.1)$$

In this paper, making use of (1.1) and certain known summation formulae, an attempt has been made to establish transformation formulae for q -hypergeometric series.

II. NOTATIONS AND DEFINITIONS

The generalized basic hypergeometric function is defined as

$${}_A\Phi_B \left[\begin{matrix} (a); q; z \end{matrix} \middle| \begin{matrix} (b); q^i \end{matrix} \right] = \sum_{r=0}^{\infty} q^{\frac{ir(r-1)}{2}} \frac{\prod_{j=1}^A (a_j; q)_r z^r}{\prod_{j=1}^B (b_j; q)_r (q; q)_r} \quad (2.1)$$

Where

$$(a; q)_r = (1-a)(1-aq) \dots (1-aq^{r-1}); (a; q)_0 = 1, i > 0, |q| < 1, |z| < \infty \quad (2.2)$$

and for $i = 0, \max(|q|, |z|) < 1$. Also stands for a sequences of A –parametrs of the form

$$a_1, a_2, \dots, a_A \text{ Type equation here.}$$

We shall make use of following known summations

$${}_4\Phi_3 \left[\begin{matrix} a^2, a^2q, e^4q^{2n}, q^{-2n}; q^2; q^2 \end{matrix} \middle| \begin{matrix} (-q; q)_n (e^2/a^2; q)_n a^{2n} \end{matrix} \right] = \frac{(-q; q)_n (e^2/a^2; q)_n a^{2n}}{(e^2; q)_n (-a^2q; q)_n} \quad (2.3)$$

$${}_4\Phi_3 \left[\begin{matrix} a^2, a^2q, e^4q^{2n}, q^{-2n}; q^2; q^2 \end{matrix} \middle| \begin{matrix} (-q; q)_n (e^2; q^2)_n (e^2q/a^2; q)_n a^{2n} \end{matrix} \right] = \frac{(-q; q)_n (e^2; q^2)_n (e^2q/a^2; q)_n a^{2n}}{(-a^2; q)_n (e^2; q)_n (e^2q^2; q^2)_n} \quad (2.4)$$

III. MAIN RESULTS

We shall establish our main results

$${}_{10}\Phi_9 \left[\begin{matrix} -e^2, eiq, -eiq, eq, -eq, e^2/a^2, \alpha, -\alpha, \beta, -\beta; q; -\frac{e^4a^2q^2}{\alpha^2\beta^2} \end{matrix} \middle| \begin{matrix} ei, -ei, -e, e, -a^2q, -e^2q/\alpha, e^2q/\alpha, -e^2q/\beta, e^2q/\beta; q^2 \end{matrix} \right]$$

$$= \frac{(e^4 q^2 / \alpha^2 \beta^2, e^4 q^2; q^2)_\infty}{(e^4 q^2 / \alpha^2, e^4 q^2 / \beta^2; q^2)_\infty} {}_4\Phi_3 \left[\begin{matrix} a^2, a^2 q, \alpha^2, \beta^2; q^2; \frac{e^4 q^2}{\alpha^2 \beta^2} \\ a^4 q^2, e^2, e^2 q \end{matrix} \right] \quad (3.1)$$

$${}_8\Phi_7 \left[\begin{matrix} -e^2, eiq, -eiq, e^2 q / \alpha^2, \alpha, -\alpha, \beta, -\beta; q; -\frac{e^4 a^2 q^2}{\alpha^2 \beta^2} \\ ei, -ei, -a^2, -e^2 q / \alpha, e^2 q / \alpha, -e^2 q / \beta, e^2 q / \beta; q^2 \end{matrix} \right] \\ = \frac{(e^4 q^2 / \alpha^2 \beta^2, e^4 q^2; q^2)_\infty}{(e^4 q^2 / \alpha^2, e^4 q^2 / \beta^2; q^2)_\infty} {}_4\Phi_3 \left[\begin{matrix} a^2, a^2 q, \alpha^2, \beta^2; q^2; \frac{e^4 q^2}{\alpha^2 \beta^2} \\ a^4, e^2 q, e^2 q^2 \end{matrix} \right] \quad (3.2)$$

Proof of (3.1) and (3.2)

Replacing q, α, β by q^2, α^2, β^2 respectively and then choosing

$$A_j = \frac{(a^2, a^2 q, \alpha^2, \beta^2; q^2)_j}{(a^4 q^2, e^2, e^2 q; q^2)_j}, \gamma = e^4, w = 1, B_n = 1,$$

$x = e^4 q^2 / \alpha^2 \beta^2$ in (1.1) and making use of (2.3) and Gauss's summation formula in order to sum the inner-series in the left hand side we get (3.1) after some simplifications.

Similarly, replacing q, α, β by q^2, α^2, β^2 respectively and then choosing

$$A_j = \frac{(a^2, a^2 q, \alpha^2, \beta^2; q^2)_j}{(a^4, e^2 q, e^2 q^2; q^2)_j}, w = 1, \gamma = e^4, B_n = 1, x = \frac{e^4 q^2}{\alpha^2 \beta^2}$$

In (1.1) and making use of use of (2.4) and Gauss's summation formula in order to sum the inner series in the left hand side we get (3.2) after some simplifications.

Taking $\alpha, \beta \rightarrow \infty$ in (3.1) we get

$$\sum_{r=0}^{\infty} \frac{(-e^2; q)_r (e^2 / a^2; q)_r \left(\frac{1 - e^4 q^{4r}}{1 - e^4} \right)}{(q; q)_r (-a^2 q; q)_r} q^{3r(r-1)} (-e^4 a^2 q^2)^r \\ = (e^4 q^2; q^2)_\infty \sum_{r=0}^{\infty} \frac{(a^2, a^2 q; q^2)_r e^{4r} q^{2r^2}}{(q^2, a^4 q^2, e^2, e^2 q; q^2)_r} \quad (3.3)$$

Taking $a = 1$ and $e^4 = 1$ in (3.3) we obtain

$$\sum_{r=-\infty}^{\infty} (-)^r q^{r(3r-1)} = (q^2; q^2)_\infty, \quad (3.4)$$

Which on replacing q^2 by q gives the Euler's pentagonal identity:

$$\sum_{r=-\infty}^{\infty} (-)^r q^{r(3r-1)/2} = (q; q)_\infty,$$

Taking $a = 1$ and $e^4 = q^2$ in (3.3) we get another identity:

$$\sum_{r=0}^{\infty} (-)^r (1 - q^{4r+2}) q^{r(3r+1)} = (q^2; q^2)_\infty. \quad (3.5)$$

Taking $a^2 = 1$ in (3.1) we obtain the following summation formula:

$${}_5\Phi_4 \left[\begin{matrix} e^4, e^2 q^2, -e^2 q^2, \alpha^2, \beta^2; q^2; -e^4 q^2 / \alpha^2 \beta^2 \end{matrix} \right] \\ = \frac{(e^4 q^2 / \alpha^2 \beta^2, e^4 q^2; q^2)_\infty}{(e^4 q^2 / \alpha^2, e^4 q^2 / \beta^2; q^2)_\infty}. \quad (3.6)$$

Taking $a = e$ and $\beta = eq^{1/2}$ in (3.1) we get the following summation formula:

$${}_4\Phi_3 \left[\begin{matrix} -e^2, eiq, -eiq, e^2/a^2; q; -a^2 q \end{matrix} \right] = \frac{(-e^2 q; q)_\infty}{(-a^2 q; q)_\infty}. \quad (3.7)$$

Taking $a \rightarrow 0$ in (3.7) we get:

$$\sum_{r=0}^{\infty} \frac{(-e^2; q)_r}{(q; q)_r} (1 + e^2 q^{2r}) e^{2r} q^{r(3r-1)/2} = (-e^2; q)_\infty \quad (3.8)$$

Which for $e^2 = q$ yields:

$$\sum_{r=0}^{\infty} \frac{(-q; q)_r}{(q; q)_r} (1 + q^{2r+1}) q^{r(3+1)/2} = (-q; q)_\infty \quad (3.9)$$

Taking $\alpha, \beta \rightarrow \infty$ in (3.2) we get:

$$\sum_{r=0}^{\infty} \frac{(-e^2; q)_r (1 + e^2 q^{2r}) (e^2 q / a^2; q)_r}{(q; q)_r (1 + e^2) (-a^2; q)_r} q^{3r(r-1)} (-e^4 a^2 q^2)^r \\ = (e^4 q^2; q^2)_\infty \sum_{r=0}^{\infty} \frac{(a^2, a^2 q; q^2)_r (e^4 q^2)^r q^{2r(r-1)}}{(q^2, a^4, e^2 q, e^2 q^2; q^2)_r} \quad (3.10)$$

For $a \rightarrow 1$, (3.10) gives:

$$\sum_{r=0}^{\infty} \frac{(-e^2, e^2 q; q)_r (1 + e^2 q^{2r})}{(q; q)_r (-1; q)_r (1 + e^2)} q^{3r(r-1)} (-e^4 q^2)^r \\ = (e^2 q^2; q^2)_\infty \left\{ 1 + \frac{1}{2} \sum_{r=1}^{\infty} \frac{(q; q^2)_r e^{4r} q^{2r^2}}{(q^2, e^2 q, e^2 q^2; q^2)_r} \right\} \quad (3.11)$$

Taking $e^2 = 1$ in (3.11) we find :

$$\sum_{r=0}^{\infty} (1 + q^{2r}) (-)^r q^{r(3r-1)} = (q^2; q^2)_\infty \left\{ 1 + \sum_{r=1}^{\infty} \frac{q^{2r^2}}{(q^2; q^2)_r^2} \right\},$$

Which by an appeal to Jacobi's triple product identity yields the well known identity (after replacing q^2 by q)

$$\sum_{r=0}^{\infty} \frac{q^{r^2}}{(q; q)_r^2} = \frac{1}{(q; q)_\infty} \quad (3.12)$$

Similarly, several results can also be obtained.

IV. CONCLUSIONS

In this paper, transformation formulae for q-hypergeometric series have been established by using certain known summation formulae. Eight important results have been derived including Euler's pentagonal identity and Jacobi's triple product identity.

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