SOME REMARKS ABOUT STIRLING NUMBERS OF THE SECOND KIND

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ABSTRACT

In this paper we give a representation of Stirling numbers of the second kind, we obtain explicit formulas for some cases of Stirling numbers of the second kind and illustrate a method for founding other such formulas.

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Key words: *Combinatorics sum, Stirling numbers of the second kind.*

PRELIMINARIES

In Combinatorial Theory (Aigner, 1979) are given the properties and the values of combinatorics sum

$$S(n, l) := \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^l \quad (n, l \in \mathbb{N} \cup \{0\}).$$

Especially, we define that $S(0,0) := 1$. It is a known fact that $S(n,n) = n!$ for every $n \in \mathbb{N}$, and $S(n,l) = 0$ for $l < n$: For $n < l$; the sum $S(n,l)$ is interesting because in combinatorics it represents the number of all possible permutations with repetitions of a set of l elements with n classes ([2], page 216), i.e., for $n < l$;

$$S(n, l) = \sum_{\substack{k_1, \dots, k_n \in \mathbb{N} \\ k_1 + \dots + k_n = l}} \frac{l!}{k_1! \dots k_n!} = l! \sum_{\substack{k_1, \dots, k_n \in \mathbb{N} \\ k_1 + \dots + k_n = l}} \frac{1}{k_1! \dots k_n!}$$

Also, there exists a relationship for the sum $S(n,l)$ using Stirling numbers of the second kind. Namely, for

$$S(n, l) = n! \cdot S_2(l, n) \quad (1)$$

where with $S_2(l,n)$ we denote Stirling numbers of the second kind (Velja, 2001, pp. 100-101). The Stirling numbers of the second kind $S_2(l,n)$ describe the number of ways a set with l elements can be partitioned into n disjoint non-empty subsets. In Combinatorial Theory (Aigner, 1979) it is proved the following representation of $S(n,l)$ for $n < l$.

Theorem 1 (Theorem 7. in Combinatorial Theory (Aigner, 1979)) *For every $k \in \mathbb{N}$ there exists a polynomial $f_k(n) = a_1 n^k + \dots + a_k n$ with rational coefficients and with property that for every $n \in \mathbb{N} \cup \{0\}$*

$$S(n, n+k) = f_k(n)(n+k)! \quad (2)$$

Also, the following holds

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$$\sum_{i=1}^k a_i = \frac{1}{(k+1)!} \quad (3)$$

Especially, for every $n \in \mathbb{N} \cup \{0\}$ are obtained the following concrete formulas (formulas (7)-(11) in [1]):

$$S(n, n+1) = \frac{n}{2}(n+1)! \quad (4)$$

$$S(n, n+2) = \frac{3n^2 + n}{24}(n+2)! \quad (5)$$

$$S(n, n+3) = \frac{n^3 + n^2}{48}(n+3)! \quad (6)$$

$$S(n, n+4) = \frac{15n^4 + 30n^3 + 5n^2 - 2n}{5760}(n+4)! \quad (7)$$

$$S(n, n+5) = \frac{9n^5 + 30n^4 + 15n^3 - 6n^2}{34560}(n+5)! \quad (8)$$

These formulas hold for every $n \in \mathbb{N} \cup \{0\}$. Therefore, the coefficients of the polynomial $f_k(n)$ depend only of k .

RESULTS

From (1) and (2) we conclude that Stirling numbers of the second kind $S_2(n+k, n)$ have the following representation:

$$S_2(n+k, n) = \frac{(n+k)!}{n!} \cdot f_k(n) = \binom{n+k}{k} \cdot g_k(n) \quad (9)$$

with $g_k(n) := k! \cdot f_k(n)$ for $n, k \in \mathbb{N} \cup \{0\}$. We want to see how the coefficients of polynomials $f_k(n)$ and $g_k(n)$ depend of k : From theorem 2. in Combinatorial Theory (Aigner, 1979) we have the recurrence relation:

$$S(n, n+k) = n \cdot S(n-1, n+k-1) + n \cdot S(n, n+k-1)$$

Using (2) this means

$$f_k(n) \cdot (n+k)! = n \cdot f_k(n-1) \cdot (n-1+k)! + n \cdot f_{k-1}(n) \cdot (n+k-1)!$$

Now, if we divide the last relation with $(n+k-1)!$, then we obtain the equality

$$f_k(n) \cdot (n+k) = n \cdot f_k(n-1) + n \cdot f_{k-1}(n) \quad (10)$$

Denote :

$$f_k(n) = a_1(k)n^k + a_2(k)n^{k-1} + \dots + a_k(k)n$$

Then, from (10) we have further

$$\begin{aligned} (n+k) \cdot [a_1(k)n^k + a_2(k)n^{k-1} + \dots + a_k(k)n] &= \\ = n \cdot [a_1(k)(n-1)^k + a_2(k)(n-1)^{k-1} + \dots + a_k(k)(n-1)] &+ \\ + n \cdot [a_1(k-1)n^{k-1} + a_2(k-1)n^{k-2} + \dots + a_{k-1}(k-1)n] \end{aligned} \quad (11)$$

Observing the coefficients by n^k in (11) we have

$$k \cdot a_1(k) + a_2(k) = -k \cdot a_1(k) + a_2(k) + a_1(k-1),$$

i.e.,

$$a_1(k) = \frac{1}{2k} \cdot a_1(k-1) \quad (*)$$

Because of $a_1(0) = 1$, we conclude that

$$a_1(k) = \frac{1}{2^k \cdot k!} \quad (k \in \mathbb{N} \cup \{0\}) \quad (12)$$

Now, we observe the coefficients by n^{k-1} in (11), and we get

$$k \cdot a_2(k) + a_3(k) = \binom{k}{2} a_1(k) - \binom{k-1}{1} a_2(k) + a_3(k) + a_2(k-1),$$

i.e.,

$$(2k-1)a_2(k) = \binom{k}{2} a_1(k) + a_2(k-1).$$

Using (12) we have further,

$$(2k-1)a_2(k) = a_2(k-1) + \frac{\binom{k}{2}}{2^k \cdot k!} \quad (**)$$

By mathematical induction we want to prove that

$$a_2(k) = \frac{1}{3} \frac{\binom{k}{2}}{2^k \cdot k!} \quad (k \in \mathbb{N} \cup \{0\}) \quad (13)$$

Because of $a_2(0) = a_2(1) = 0$ and $a_2(2) = \frac{1}{24}$, we conclude that (13) holds for $k=1,2$.

If we assume that

$$a_2(k-1) = \frac{1}{3} \frac{\binom{k-1}{2}}{2^{k-1} \cdot (k-1)!}$$

then from (**) we have

$$(2k-1)a_2(k) = \frac{\frac{1}{3}\binom{k-1}{2}}{2^{k-1} \cdot k \cdot 1!} + \frac{\binom{k}{2}}{2^k \cdot k!} = \frac{\frac{2k}{3}\binom{k-1}{2} + \binom{k}{2}}{2^k \cdot k!} =$$

$$= \frac{\binom{k}{2}}{2^k \cdot k!} \cdot \left[1 + \frac{2}{3}(k-2)\right] = \frac{\binom{k}{2}}{2^k \cdot k!} \cdot \frac{2k-1}{3}.$$

Hence, (13) holds. Observe now the coefficients by n^{k-2} in (11). Then, we have

$$k \cdot a_3(k) + a_4(k) = -\binom{k}{3}a_1(k) + \binom{k-1}{2}a_2(k) - \binom{k-2}{2}a_3(k) + a_4(k) + a_3(k-1)$$

Hence,

$$(2k-2)a_3(k) = a_3(k-1) - \binom{k}{3} \cdot \frac{1}{2^k \cdot k!} + \binom{k-1}{2} \cdot \frac{\frac{1}{3}\binom{k}{2}}{2^k \cdot k!}$$

i.e.,

$$(2k-2)a_3(k) = a_3(k-1) + \frac{2\binom{k}{4}}{2^k \cdot k!} \quad (***)$$

By mathematical induction we want to prove that

$$a_3(k) = \frac{\frac{1}{3}\binom{k}{4}}{2^k \cdot k!} \quad k \in \mathbb{N} \cup \{0\} \quad (14)$$

It is clear that $a_3(k) = 0$ for $k \leq 3$, i.e. (14) holds for $k \leq 3$. If we assume that

$$a_3(k-1) = \frac{\frac{1}{3}\binom{k-1}{4}}{2^{k-1} \cdot (k-1)!}$$

then from (***) we have

$$(2k-2)a_3(k) = \frac{\frac{1}{3}\binom{k-1}{4}}{2^{k-1} \cdot (k-1)!} + \frac{2\binom{k}{4}}{2^k \cdot k!} = \frac{\frac{2k}{3}\binom{k-1}{4} + 2\binom{k}{4}}{2^k \cdot k!} =$$

$$= \frac{\binom{k}{4}}{2^k \cdot k!} \cdot \left[2 + \frac{2}{3}(k-4)\right] = \frac{\binom{k}{4}}{2^k \cdot k!} \cdot \frac{2k-2}{3}.$$

Hence, (14) holds. Note that formulas (13) and (14) also we can obtain using the substitution

$$a_2(k) = \frac{A\binom{k}{2}}{2^k \cdot k!}, \quad a_3(k) = \frac{B\binom{k}{4}}{2^k \cdot k!} \quad (k \in \mathbb{N} \cup \{0\})$$

in (**) and (***) and finding $A = B = \frac{1}{3}$. Observe now the coefficients by n^{k-3} in (11). Then, we have

$$k \cdot a_4(k) + a_5(k) = \binom{k}{4}a_1(k) - \binom{k-1}{3}a_2(k) + \binom{k-2}{2}a_3(k) - \binom{k-3}{1}a_4(k) + a_4(k-1) + a_5(k),$$

i.e.,

$$(2k-3)a_4(k) = a_4(k-1) + \frac{1}{2^k \cdot k!} \left[\binom{k}{4} - \frac{1}{3}\binom{k}{2}\binom{k-1}{3} + \frac{1}{3}\binom{k}{4}\binom{k-2}{2} \right].$$

Since

$$\begin{aligned} & \binom{k}{4} - \frac{1}{3}\binom{k}{2}\binom{k-1}{3} + \frac{1}{3}\binom{k}{4}\binom{k-2}{2} = \\ & = \binom{k}{4} \cdot \left[1 - \frac{2}{3}(k-1) + \frac{1}{6}(k-2)(k-3) \right] \\ & = \binom{k}{4} \cdot \frac{1}{6}(k^2 - 9k + 16) = \frac{1}{6}\binom{k}{4} \cdot [(k-4)(k-5) - 4] \\ & = 5\binom{k}{6} - \frac{2}{3}\binom{k}{4}, \end{aligned}$$

we have the recurrence relation

$$(2k-3)a_4(k) = a_4(k-1) + \frac{5\binom{k}{6} - \frac{2}{3}\binom{k}{4}}{2^k \cdot k!} \quad (****)$$

Look for the solution of the last equation in the form

$$a_4(k) = \frac{A\binom{k}{6} + B\binom{k}{4}}{2^k \cdot k!}$$

for some rational constants A and B: Then, from (****) we have

$$(2k-3) \cdot \left[A\binom{k}{6} + B\binom{k}{4} \right] = 2k \cdot \left[A\binom{k-1}{6} + B\binom{k-1}{4} \right] + 5\binom{k}{6} - \frac{2}{3}\binom{k}{4}.$$

Fruther,

$$(2k-3) \cdot A\binom{k}{6} = 2k \cdot A\binom{k-1}{6} + 5\binom{k}{6}$$

implies $(2k-3) \cdot A = 2(k-6) \cdot A + 5$

Hence, $A = \frac{5}{9}$. Also, from

$$(2k-3) \cdot B \binom{k}{4} = 2k \cdot B \binom{k-1}{4} - \frac{2}{3} \binom{k}{4}$$

we obtain

$$(2k-3) \cdot B = 2(k-4) \cdot B - \frac{2}{3}$$

Therefore, $B = -\frac{2}{15}$. Hence,

$$a_4(k) = \frac{\frac{5}{9} \binom{k}{6} - \frac{2}{15} \binom{k}{4}}{2^k \cdot k!} \quad (k \in \mathbb{N} \cup \{0\}) \quad (15)$$

Observe now the coefficients by n^{k-4} in (11). Then, we have

$$k \cdot a_5(k) + a_6(k) = -\binom{k}{5} a_1(k) + \binom{k-1}{4} a_2(k) - \binom{k-2}{3} a_3(k) + \binom{k-3}{2} a_4(k) - \binom{k-4}{1} a_5(k) + a_5(k-1) + a_6(k),$$

i.e.,

$$(2k-4)a_5(k) = a_5(k-1) + \frac{1}{2^k \cdot k!} \left[-\binom{k}{5} + \frac{1}{3} \binom{k}{2} \binom{k-1}{4} \right] - \frac{1}{3} \binom{k}{4} \binom{k-2}{3} + \binom{k-3}{2} \left(\frac{5}{9} \binom{k}{6} - \frac{2}{15} \binom{k}{4} \right).$$

Because

$$\begin{aligned} & -\binom{k}{5} + \frac{1}{3} \binom{k}{2} \binom{k-1}{4} - \frac{1}{3} \binom{k}{4} \binom{k-2}{3} + \binom{k-3}{2} \left(\frac{5}{9} \binom{k}{6} - \frac{2}{15} \binom{k}{4} \right) = \\ & = \binom{k}{5} \cdot \frac{1}{108} (5k^3 - 90k^2 + 439k - 570) \\ & = \binom{k}{5} \cdot \frac{1}{108} \cdot [5(k-5)(k-6)(k-7) - 96(k-5)] \\ & = \frac{140}{9} \binom{k}{8} - \frac{48}{9} \binom{k}{6} \end{aligned}$$

we have the recurrence relation

$$(2k-4)a_5(k) = a_5(k-1) + \frac{\frac{140}{9} \binom{k}{8} - \frac{48}{9} \binom{k}{6}}{2^k \cdot k!} \quad (*****)$$

Let us search the solution of the last equation in the form

$$a_5(k) = \frac{A \binom{k}{8} + B \binom{k}{6}}{2^k \cdot k!}$$

for some rational constants A and B. Then, from (*****) we have

$$(2k-4) \cdot \left[A \binom{k}{8} + B \binom{k}{6} \right] = 2k \cdot \left[A \binom{k-1}{8} + B \binom{k-1}{6} \right] + \frac{140}{9} \binom{k}{8} - \frac{48}{9} \binom{k}{6}.$$

Further,

$$(2k-4) \cdot A \binom{k}{8} = 2k \cdot A \binom{k-1}{8} + \frac{140}{9} \binom{k}{8}$$

implies

$$(2k-4) \cdot A = 2(k-8) \cdot A + \frac{140}{9}.$$

Hence, $A = \frac{35}{27}$. Also, from

$$(2k-4) \cdot B \binom{k}{6} = 2k \cdot B \binom{k-1}{6} - \frac{48}{9} \binom{k}{6}$$

we obtain

$$(2k-4) \cdot B = 2(k-6) \cdot B - \frac{48}{9}.$$

Therefore, $B = -\frac{2}{3}$. Hence,

$$a_5(k) = \frac{\frac{35}{27}\binom{k}{8} - \frac{2}{3}\binom{k}{6}}{2^k \cdot k!} \quad (k \in \mathbb{N} \cup \{0\}) \quad (16)$$

Observing the coefficients by n^{k-5} in (11) we obtain

$$(2k-5)a_6(k) = a_6(k-1) + \frac{1}{2^k \cdot k!} \cdot \left[\binom{k}{6} - \frac{1}{3}\binom{k}{2}\binom{k-1}{5} + \frac{1}{3}\binom{k}{4}\binom{k-2}{4} \right] - \left(\binom{k-3}{3} \left(\frac{5}{9}\binom{k}{6} - \frac{2}{15}\binom{k}{4} \right) + \binom{k-4}{2} \left(\frac{35}{27}\binom{k}{8} - \frac{2}{3}\binom{k}{6} \right) \right)$$

Because

$$\begin{aligned} & \left(\binom{k}{6} - \frac{1}{3}\binom{k}{2}\binom{k-1}{5} + \frac{1}{3}\binom{k}{4}\binom{k-2}{4} \right) - \left(\binom{k-3}{3} \left(\frac{5}{9}\binom{k}{6} - \frac{2}{15}\binom{k}{4} \right) + \binom{k-4}{2} \left(\frac{35}{27}\binom{k}{8} - \frac{2}{3}\binom{k}{6} \right) \right) = \\ & = \binom{k}{6} \cdot \frac{1}{432} \cdot [5k^4 - 150k^3 + 1411k^2 - 4818k + 4800] \\ & = \binom{k}{6} \cdot \frac{1}{432} \cdot [5(k-6)(k-7)(k-8)(k-9) - 264(k-6)(k-7) + 768] \\ & = \frac{175}{3}\binom{k}{10} - \frac{308}{9}\binom{k}{8} + \frac{768}{432}\binom{k}{6} \end{aligned}$$

we have the recurrence relation

$$(2k-5)a_6(k) = a_6(k-1) + \frac{\frac{175}{3}\binom{k}{10} - \frac{308}{9}\binom{k}{8} + \frac{768}{432}\binom{k}{6}}{2^k \cdot k!} \quad (*****)$$

If we search the solution of the last equation in the form

$$a_6(k) = \frac{A\binom{k}{10} + B\binom{k}{8} + C\binom{k}{6}}{2^k \cdot k!}$$

for some rational constants A, B and C; then, from (*****) we have

$$\begin{aligned} & (2k-5) \cdot \left[A\binom{k}{10} + B\binom{k}{8} + C\binom{k}{6} \right] = \\ & = 2k \cdot \left[A\binom{k-1}{10} + B\binom{k-1}{8} + C\binom{k-1}{6} \right] + \frac{175}{3}\binom{k}{10} - \frac{308}{9}\binom{k}{8} + \frac{768}{432}\binom{k}{6} \end{aligned}$$

Solving this equation, we obtain :

$$A = \frac{35}{9}, \quad B = -\frac{28}{9}, \quad C = \frac{16}{63}.$$

Hence,

$$a_6(k) = \frac{\frac{35}{9}\binom{k}{10} - \frac{28}{9}\binom{k}{8} + \frac{16}{63}\binom{k}{6}}{2^k \cdot k!} \quad (k \in \mathbb{N} \cup \{0\}) \quad (17)$$

If we continue the described method, we can obtain explicit formulas for coefficients $a_7(k)$, $a_8(k)$ and other coefficients of the polynomial $f_k(n)$. Finally, using Theorem 1, formulas (9), (12)-(17), we have proved the following theorem.

Theorem 2 For every $k \in \mathbb{N} \cup \{0\}$ there exists a unique polynomial $g_k(n) = b_1(k)n^k + b_2(k)n^{k-1} + \dots + b_k(k)n$ with rational coefficients and with property that for every $n \in \mathbb{N} \cup \{0\}$,

$$S_2(n+k, n) = \binom{n+k}{k} g_k(n)$$

Also, the following holds

$$\sum_{i=1}^k b_i(k) = \frac{1}{k+1}.$$

The first six coefficients of the polynomial $g_k(n)$ are equal:

$$\begin{aligned} b_1(k) &= \frac{1}{2^k}, \quad b_2(k) = \frac{\frac{1}{3}\binom{k}{2}}{2^k}, \quad b_3(k) = \frac{\frac{1}{3}\binom{k}{4}}{2^k}, \quad b_4(k) = \frac{\frac{5}{9}\binom{k}{6} - \frac{2}{15}\binom{k}{4}}{2^k}, \\ b_5(k) &= \frac{\frac{35}{27}\binom{k}{8} - \frac{2}{3}\binom{k}{6}}{2^k}, \quad b_6(k) = \frac{\frac{35}{9}\binom{k}{10} - \frac{28}{9}\binom{k}{8} + \frac{16}{63}\binom{k}{6}}{2^k} \quad (k \in \mathbb{N} \cup \{0\}). \end{aligned}$$

Corollary 3 For every $n \in \mathbb{N} \cup \{0\}$ the following formulas hold:

$$S_2(n+1, n) = \binom{n+1}{1} \cdot \frac{n}{2} \quad (18)$$

$$S_2(n+2, n) = \binom{n+2}{2} \cdot \frac{3n^2 + n}{12} \quad (19)$$

$$S_2(n+3, n) = \binom{n+3}{3} \cdot \frac{n^3 + n^2}{8} \quad (20)$$

$$S_2(n+4, n) = \binom{n+4}{4} \cdot \frac{15n^4 + 30n^3 + 5n^2 - 2n}{240} \quad (21)$$

$$S_2(n+5, n) = \binom{n+5}{5} \cdot \frac{9n^5 + 30n^4 + 15n^3 - 6n^2}{288} \quad (22)$$

$$S_2(n+6, n) = \binom{n+6}{6} \cdot \frac{63n^6 + 315n^5 + 315n^4 - 91n^3 - 42n^2 + 16n}{4032} \quad (23)$$

Remark 4 Note that the formulas (18)-(22) also follow directly using the formulas (4)-(9).

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