On the Moving Coordinate System and Euler-Savary Formula in Affine Cayley-Klein Planes

Afin Cayley-Klein Düzlemlerinde Hareketli Koordinat Sistemi ve Euler-Savary Formülü Üzerine

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Abstract

In this present paper, we will take three affine Cayley-Klein planes into consideration: A_ϵ , $\mathbb{P}_\epsilon$ and $\mathbb{P}'_\epsilon$. The plane $\mathbb{P}'_\epsilon$ is a fixed plane relative to two other moving affine Cayley-Klein (CK)-planes. We will describe one-parameter motions $A_\epsilon / \mathbb{P}_\epsilon$, $A_\epsilon / \mathbb{P}'_\epsilon$ and $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ and discuss the relationship between the motions $A_\epsilon / \mathbb{P}_\epsilon$, $A_\epsilon / \mathbb{P}'_\epsilon$ and $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ by evaluating their derivative formulae, velocity vectors and pole points. Also, we will observe moving coordinate system and after that, we will examine the canonical relative system for one-parameter planar motions in the affine CK-planes by using the notions of moving coordinate system. Moreover, Euler-Savary formula, which gives the relationship between the curvatures of trajectory curves, will be obtained with the help of canonical relative system for one-parameter motions in affine CK-planes by using the method given by H. R. Müller in 1956 [1].

Keywords: Cayley-Klein planes, one-parameter planar motion, moving coordinate system, kinematics, Euler-Savary Formula.

Öz

Bu çalışmada A_ϵ , $\mathbb{P}_\epsilon$ ve $\mathbb{P}'_\epsilon$ üç afin Cayley-Klein düzlemi gözönüne alınmıştır. $\mathbb{P}'_\epsilon$ düzlemi diğer iki hareketli afin Cayley-Klein (CK)-düzlemine göre sabittir. Çalışmada bir parametrelili $A_\epsilon / \mathbb{P}_\epsilon$, $A_\epsilon / \mathbb{P}'_\epsilon$ ve $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ hareketleri tarif edilecek; türev formülleri, hız vektörleri ve pol noktaları elde edilerek $A_\epsilon / \mathbb{P}_\epsilon$, $A_\epsilon / \mathbb{P}'_\epsilon$ ve $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ hareketleri arasındaki ilişki tartışılacaktır. Ayrıca afin (CK)-düzlemlerinde hareketli koordinat sistemi araştırılarak bu hareketli koordinat sisteminin kavramları ile kavramları bir parametrelili hareketler için kanonik izafe sistemi incelenecektir. Bu ifadeler ek olarak, kanonik izafe sistemi yardımıyla afin (CK)-düzlemlerinde bir parametrelili hareketler için yörünge eğrilerinin eğrilikleri arasındaki ilişkiyi veren Euler Savary formülü H. R. Müller tarafında 1956 yılında verilen metotla elde edilecektir [1].

Anahtar Kelimeler: Cayley-Klein düzlemleri, bir parametrelili düzlemsel hareket, hareketli koordinat sistemi, kinematik, Euler-Savary formülü

1. Introduction

Cayley-Klein (CK) geometries, were originated in the 19th century, are number of geometries including Euclidean, Galilean, Minkowskian and Bolyai-Lobachevskian, [2,3]. Following Cayley and Klein, Yaglom distinguished these geometries by choosing one of three ways of measuring length (parabolic, elliptic, or hyperbolic) between two points on a line and one of the three ways of measuring angles between two lines (parabolic, elliptic, or hyperbolic). This gives nine ways of measuring lengths and angles, [4].

Much recent research is conducted in CK-planes, [4-19]. There is a known (but not well-known) relationship between the plane geometries which have parabolic measure of distance: Euclidean, Galilean and Minkowskian (Lorentz) geometries. They are called *affine CK-plane geometries*, [4].

To observe one-parameter motion in plane geometries has a significant role in kinematics. In this aspect, many researchers have received considerable attention in the kinematic literature, [20-24]. In 1956, H. R. Müller defined one-parameter planar motion in the Euclidean plane $\mathbb{E}^2$ and studied the relationship between absolute, relative and sliding velocities (accelerations) [1]. Then, one-parameter planar motions and the above same notions are investigated in Lorentzian (Minkowskian) plane $\mathbb{L}^2$ and Galilean plane $\mathbb{G}^2$ by [22] and [23], respectively. Besides, in [24] the one-parameter motions in the affine CK-planes $\mathbb{P}_\epsilon$ are introduced by generalizing the concepts introduced by above scientists.

It is known that the moving coordinate systems are important because, no material body is at absolute rest. As we know, even galaxies are not stationary. In reality, we have the moving frames, major example being Earth itself. In the light of this truth; the researchers argued this notion by considering different plane geometries: Lorentzian and Galilean planes, [25,26].

Furthermore, the canonical relative system for one-parameter planar motions were studied in [1], [27] and [28] in the planes $\mathbb{E}^2$, $\mathbb{L}^2$ and $\mathbb{G}^2$ by using the notions of moving coordinate system, respectively. Three Lorentzian planes

moving with respect to one another and pole points are studied in [29].

Euler-Savary formula which gives the relationship between the curvature of trajectory curves, during one-parameter planar motions, was studied by [1]. This formula was studied in Lorentzian plane for the one-parameter Lorentzian motions by using two different ways: In 2002, I. Aytun studied the this formula for the one-parameter Lorentzian motions with using the Müller's Method [30]. In 2003, T. Ikawa gave this formula on Minkowski plane by taking a new aspect without using the Müller's Method [31]. Ikawa gave the relationship between the curvature of roulette and curvatures of these base curve and rolling curve, [31]. Euler-Savary formula is a well documented and an admitted formula in the literature and many scientists have contributed to the development of fundamental knowledge of Euler-Savary formula, [32-39].

In 1983, the kinematics in the isotropic plane was studied by O. Röschel. In [40], the fundamental properties of the point-paths are investigated, a formula analog to the well-known formula of Euler-Savary was developed and special motions: an isotropic elliptic motion and an isotropic four-bar-motion are studied.

Besides, in 1985, the motions $\sum/\sum_0$ in the isotropic plane was studied in [41]. Given C^2 -curve k in the moving frame $\sum$, Röschel found the enveloped curve k_0 in the fixed frame $\sum_0$ and considered the correspondence between the isotropic curvatures A and A_0 of k and k_0 . Then third-order properties of the point-paths are investigated.

In this present paper, we will consider three affine CK-planes into consideration: $A_\epsilon, \mathbb{P}_\epsilon$ and $\mathbb{P}'_\epsilon$. $\mathbb{P}'_\epsilon$ is a fixed plane relative to two other moving affine CK-planes. We will aim to examine the relationship between the motions $A_\epsilon/\mathbb{P}_\epsilon, A_\epsilon/\mathbb{P}'_\epsilon$ and $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$ by evaluating their derivative formulae, velocity vectors and pole points. We will introduce canonical relative system for one-parameter planar motions in the affine CK-planes by using the notions of moving coordinate system. Moreover, Euler-Savary formula is obtained with the help of canonical

relative system for one-parameter motions in affine CK-planes by using the Müller's Method. We will establish a simple but effective method by unifying moving coordinate system and Euler-Savary formula in Euclidean, Lorentzian and Galilean planes.

2. Preliminaries

In this section, we will investigate the basic notations of affine CK-planes and one-parameter planar motions in affine CK-planes, [24].

2.1. Basic notations

In this subsection, we examine the basic notations of affine CK-planes [4,8,24] which are denoted by $\mathbb{P}_\epsilon$.

Let us consider $\mathbb{R}^2$ with the bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle_\epsilon = x_1 y_1 + \epsilon x_2 y_2,$$

where ϵ may be 1, 0 or -1 and $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$. The matrix of this bilinear form can be given as below:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}.$$

For all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{P}_\epsilon$ we can write

$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T B \mathbf{y}$. For $\epsilon = 1$ we have Euclidean plane $\mathbb{E}^2$, for $\epsilon = 0$ we have Galilean plane $\mathbb{G}^2$ and for $\epsilon = -1$ we have Lorentzian plane $\mathbb{L}^2$. If $\langle \mathbf{x}, \mathbf{y} \rangle_\epsilon = 0$, then the vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{P}_\epsilon$ are orthogonal. Self-orthogonal vectors are called *isotropic*. The norm of the vector $\mathbf{x} = (x_1, x_2)$ in $\mathbb{P}_\epsilon$ is defined by

$$\|\mathbf{x}\|_\epsilon = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle_\epsilon|} = \sqrt{|x_1^2 + \epsilon x_2^2|}.$$

The distance between two points $A = (x_1, x_2)$ and $B = (y_1, y_2)$ is given by

$$\|\mathbf{AB}\| = \sqrt{|\langle \mathbf{AB}, \mathbf{AB} \rangle_\epsilon|} = d_{AB} = \sqrt{|(y_1 - x_1)^2 + \epsilon (y_2 - x_2)^2|}.$$

For $\epsilon = 1$ only the zero vector is isotropic, for $\epsilon = 0$ zero vector and vertical vectors are isotropic and for $\epsilon = -1$ zero vector and vectors parallel to $(\pm 1, 1)$ are isotropic, [8]. A circle is a locus of points equidistant from a given fixed point, namely the center of the circle. The unit circle in $\mathbb{P}_\epsilon$ is the set of points with $\|\mathbf{P}\| = 1$, for

all $P \in \mathbb{P}_\epsilon$. The equation of the unit circle in $\mathbb{P}_\epsilon$

is $\mathbf{x}^2 + \epsilon \mathbf{y}^2 = \pm 1$. The linear transformation

$J : \mathbb{P}_\epsilon \rightarrow \mathbb{P}_\epsilon$ with matrix, also denoted by J and can be seen as below:

$$J = \begin{bmatrix} 0 & -\epsilon \\ 1 & 0 \end{bmatrix}.$$

This linear transformation converts any vector $\mathbf{x}$ to an orthogonal vector $J\mathbf{x}$. If $\mathbf{x}$ is a nonisotropic vector and $\mathbf{y}$ is orthogonal to $\mathbf{x}$, then we can write $\mathbf{y} = kJ\mathbf{x}$ for some real number k , [8].

It is not difficult to verify directly from the definition of the matrix exponential as

$$e^{J\varphi} = \sum_{n=0}^{\infty} \frac{(J\varphi)^n}{n!}$$

that

$$e^{J\varphi} = \cos_\epsilon \varphi + J \sin_\epsilon \varphi = \begin{bmatrix} \cos_\epsilon \varphi & -\epsilon \sin_\epsilon \varphi \\ \sin_\epsilon \varphi & \cos_\epsilon \varphi \end{bmatrix}$$

where

$$\cos_\epsilon \varphi = \sum_{n=0}^{\infty} \frac{(-\epsilon)^n \varphi^{2n}}{(2n)!}, \quad \sin_\epsilon \varphi = \sum_{n=0}^{\infty} \frac{(-\epsilon)^n \varphi^{2n+1}}{(2n+1)!}.$$

For $\epsilon = 1$ these are usual cosine and sine functions, for $\epsilon = -1$ they are hyperbolic cosine and sine functions, and for $\epsilon = 0$ they are just

$\cos_0 \varphi = 1$ and $\sin_0 \varphi = \varphi$ for all φ . In all cases, we obtain

$$\cos_\epsilon^2 \varphi + \epsilon \sin_\epsilon^2 \varphi = 1$$

and

$$\partial_\varphi \cos_\epsilon \varphi = -\epsilon \sin_\epsilon \varphi, \quad \partial_\varphi \sin_\epsilon \varphi = \cos_\epsilon \varphi.$$

By writing corresponding entries of the matrix equation $e^{J(\varphi+\theta)} = e^{J\varphi} e^{J\theta}$, we can find the sum formulae [4] as follows:

$$\begin{aligned}\cos_{\epsilon}(\varphi + \theta) &= \cos_{\epsilon} \varphi \cos_{\epsilon} \theta - \epsilon \sin_{\epsilon} \varphi \sin_{\epsilon} \theta \\ \sin_{\epsilon}(\varphi + \theta) &= \sin_{\epsilon} \varphi \cos_{\epsilon} \theta + \cos_{\epsilon} \varphi \sin_{\epsilon} \theta.\end{aligned}$$

2.2. One-parameter planar motions in affine CK-planes

The main purpose of this subsection is to argue the one-parameter planar motions in affine CK-planes, [24].

Let $\mathbb{P}_{\epsilon}$ and $\mathbb{P}_{\epsilon}'$ be moving and fixed affine CK-planes and $\{O; \mathbf{c}_1, \mathbf{c}_2\}$ and $\{O'; \mathbf{c}'_1, \mathbf{c}'_2\}$ be their orthonormal coordinate systems, respectively. Let us take the vector

$$\mathbf{OO}' = \mathbf{u} = u_1 \mathbf{c}_1 + u_2 \mathbf{c}_2 \text{ for } u_1, u_2 \in \mathbb{R}. \quad (1)$$

Let us define a transformation as given below:

$$\mathbf{x}' = \mathbf{x} - \mathbf{u}, \quad (2)$$

where $\mathbf{x}, \mathbf{x}'$ are coordinate vectors with respect to the moving and fixed rectangular coordinate system of a point $X = (x_1, x_2) \in \mathbb{P}_{\epsilon}$, respectively. By the equation (2), *one-parameter planar motions in affine CK-planes* are defined. These motions denoted by $\mathbb{P}_{\epsilon} / \mathbb{P}_{\epsilon}'$, [24].

Moreover, φ , the angle between the vectors $\mathbf{c}_1$ and $\mathbf{c}'_1$, is the *rotation angle* of the motions $\mathbb{P}_{\epsilon} / \mathbb{P}_{\epsilon}'$ and $\mathbf{x}, \mathbf{x}', \mathbf{u}$ are continuously differentiable functions of the time parameter $t \in I \subset \mathbb{R}$. For $t = 0$, the coordinate systems are coincident. By taking $\varphi = \varphi(t)$, we can write

$$\begin{cases} \mathbf{c}_1 = \cos_{\epsilon} \varphi \mathbf{c}'_1 + \sin_{\epsilon} \varphi \mathbf{c}'_2 \\ \mathbf{c}_2 = -\epsilon \sin_{\epsilon} \varphi \mathbf{c}'_1 + \cos_{\epsilon} \varphi \mathbf{c}'_2 \end{cases} \quad (3)$$

We assume that $\dot{\varphi}(t) = d\varphi/dt \neq 0$. In this case $\dot{\varphi}(t)$ is called the *angular velocity* of the motions $\mathbb{P}_{\epsilon} / \mathbb{P}_{\epsilon}'$. By differentiating the equations

(1) and (3) with respect to the parameter t , the *derivative formulae* of the motions $\mathbb{P}_{\epsilon} / \mathbb{P}_{\epsilon}'$ are obtained as follows:

$$\begin{cases} \dot{\mathbf{c}}_1 = \dot{\varphi} \mathbf{c}_2, \\ \dot{\mathbf{c}}_2 = -\epsilon \dot{\varphi} \mathbf{c}_1, \\ \dot{\mathbf{u}} = (\dot{u}_1 - \epsilon \dot{\varphi} u_2) \mathbf{c}_1 + (\dot{u}_2 + \dot{\varphi} u_1) \mathbf{c}_2. \end{cases} \quad (4)$$

By using these derivative formulae, we will determine velocities of a point

$$X = (x_1, x_2) \in \mathbb{P}_{\epsilon} \text{ where}$$

$$X = \mathbf{x} = x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2.$$

The velocity of the point X with respect to $\mathbb{P}_{\epsilon}$ is called the *relative velocity vector* denoted by

$$\mathbf{V}_r = \frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} \text{ and it is described by}$$

$$\mathbf{V}_r = \dot{x}_1 \mathbf{c}_1 + \dot{x}_2 \mathbf{c}_2. \quad (5)$$

Besides, the *absolute velocity* of the X with respect to $\mathbb{P}_{\epsilon}'$ is obtained by differentiating the equation (2) with respect to t and using

derivative formulae. It is denoted by $\mathbf{V}_a = \frac{d\mathbf{x}'}{dt}$

and obtained as follows:

$$\begin{aligned} \mathbf{V}_a &= \{-\dot{u}_1 + \epsilon \dot{\varphi}(u_2 - x_2)\} \mathbf{c}_1 \\ &\quad + \{-\dot{u}_2 + \dot{\varphi}(-u_1 + x_1)\} \mathbf{c}_2 + \mathbf{V}_r. \end{aligned} \quad (6)$$

By using equation (6), we get the *sliding velocity vector* as described below:

$$\begin{aligned} \mathbf{V}_f &= \{-\dot{u}_1 + \epsilon \dot{\varphi}(u_2 - x_2)\} \mathbf{c}_1 \\ &\quad + \{-\dot{u}_2 + \dot{\varphi}(-u_1 + x_1)\} \mathbf{c}_2. \end{aligned} \quad (7)$$

From equations (5), (6) and (7) the following theorem can be given.

Theorem 2.1.

Let X be a moving point on the plane $\mathbb{P}_{\epsilon}$ and $\mathbf{V}_r, \mathbf{V}_a$ and $\mathbf{V}_f$ be the relative, absolute and sliding velocity vectors of X under the one-parameter planar CK-motions $\mathbb{P}_{\epsilon} / \mathbb{P}_{\epsilon}'$,

respectively. Then, the relationship between the velocities as indicated below:

$$\mathbf{V}_a = \mathbf{V}_f + \mathbf{V}_r.$$

Now, we will investigate the points that do not move during the motions $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$. At this point, the sliding velocity vector $\mathbf{V}_f$ is equal to zero for

every $t \in [t_0, t_1]$. These points are called the *pole points* or the *instantaneous rotation pole centers*. If we use the equation (7) for a pole point $P = (p_1, p_2) \in \mathbb{P}_\epsilon$ of the motions $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$, we have

$$\begin{cases} -\dot{u}_2 + \dot{\phi}(-u_1 + x_1) = 0 \\ -\dot{u}_1 + \epsilon \dot{\phi}(u_2 - x_2) = 0. \end{cases} \quad (8)$$

So, we obtain the pole point from the solution of the system (8) as follows:

$$\begin{cases} p_1(t) = x_1(t) = u_1(t) + \frac{\dot{u}_2(t)}{\dot{\phi}(t)} \\ \epsilon p_2(t) = \epsilon x_2(t) = \epsilon u_2(t) - \frac{\dot{u}_1(t)}{\dot{\phi}(t)}. \end{cases} \quad (9)$$

Therefore, the point P is fixed in the plane $\mathbb{P}_\epsilon$.

Let us rearrange the sliding velocity vector (7) by using the equation (9):

$$\mathbf{V}_f = \{-\epsilon(x_2 - p_2)\mathbf{c}_1 + (x_1 - p_1)\mathbf{c}_2\}\dot{\phi}. \quad (10)$$

With reference to the above equation, we can give the following corollaries:

Corollary 2.1.

During the one-parameter planar motions $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ in affine CK-planes, the pole ray $\mathbf{PX}$ and the sliding velocity vector $\mathbf{V}_f$ are perpendicular vectors in the sense of affine CK-geometry, i.e., $\langle \mathbf{PX}, \mathbf{V}_f \rangle_\epsilon = 0$. Then, the focus of the point X of the motions $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ is an orbit that its normal pass through the rotation pole P .

Corollary 2.2.

Under the motions $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$, the affine CK-norm of the sliding velocity vector $\mathbf{V}_f$ is written below:

$$\|\mathbf{V}_f\|_\epsilon = \|\mathbf{PX}\|_\epsilon |\dot{\phi}|.$$

3. Moving Coordinate System and Pole Points in Affine CK-Planes

In this first original section, we will introduce the one-parameter motions $A_\epsilon / \mathbb{P}_\epsilon$ and $A_\epsilon / \mathbb{P}'_\epsilon$ in affine CK-planes. Let A_ϵ and $\mathbb{P}_\epsilon$ be moving and $\mathbb{P}'_\epsilon$ be fixed affine CK-planes and $\{B; \mathbf{a}_1, \mathbf{a}_2\}, \{O; \mathbf{c}_1, \mathbf{c}_2\}$ and $\{O'; \mathbf{c}'_1, \mathbf{c}'_2\}$ be their coordinate systems, respectively.

Let us take the vectors $\mathbf{BO}$ and $\mathbf{BO}'$ as follows:

$$\begin{cases} \mathbf{BO} = \mathbf{b} = b_1\mathbf{a}_1 + b_2\mathbf{a}_2 \\ \mathbf{BO}' = \mathbf{b}' = b'_1\mathbf{a}_1 + b'_2\mathbf{a}_2. \end{cases}$$

We assume that φ , the angle between the vectors $\mathbf{a}_1$ and $\mathbf{c}_1$, is the rotation angle of one-parameter planar motions $A_\epsilon / \mathbb{P}_\epsilon$. Similarly, φ' , the angle between the vectors $\mathbf{a}_1$ and $\mathbf{c}'_1$, is the rotation angle of one-parameter planar motions $A_\epsilon / \mathbb{P}'_\epsilon$.

By taking $\varphi = \varphi(t)$ and $\varphi' = \varphi'(t)$ into account to avoid the cases of pure translation, we can define the derivative formulae and velocities of these motions. For

$$\dot{\varphi}(t) = d\varphi / dt \neq 0$$

and

$$\dot{\varphi}'(t) = d\varphi' / dt \neq 0,$$

we can write the above equations by using the equation (3):

$$\begin{cases} \mathbf{a}_1 = \cos_\epsilon \varphi \mathbf{c}_1 + \sin_\epsilon \varphi \mathbf{c}_2 \\ \mathbf{a}_2 = -\epsilon \sin_\epsilon \varphi \mathbf{c}_1 + \cos_\epsilon \varphi \mathbf{c}_2, \end{cases} \quad (11)$$

and

$$\begin{cases} \mathbf{a}_1 = \cos_\epsilon \dot{\varphi} \mathbf{c}_1' + \sin_\epsilon \dot{\varphi} \mathbf{c}_2' \\ \mathbf{a}_2 = -\epsilon \sin_\epsilon \dot{\varphi} \mathbf{c}_1' + \cos_\epsilon \dot{\varphi} \mathbf{c}_2', \end{cases} \quad (12)$$

for $A_\epsilon / \mathbb{P}_\epsilon$ and $A_\epsilon / \mathbb{P}_\epsilon'$, respectively.

Additionally, $\dot{\varphi}(t)$ and $\dot{\varphi}'(t)$ are called the angular velocities of the motions $A_\epsilon / \mathbb{P}_\epsilon$ and $A_\epsilon / \mathbb{P}_\epsilon'$, respectively. Assume that " $d\dots$ " denotes the differential with respect to $\mathbb{P}_\epsilon$ and " $d'\dots$ " denotes the differential with respect to $\mathbb{P}_\epsilon'$. The derivative formulae of the motions $A_\epsilon / \mathbb{P}_\epsilon$ and $A_\epsilon / \mathbb{P}_\epsilon'$ (taking $d'b = d'b'$) can be calculated from the equation (11) and (12) as follows:

$$\begin{cases} d\mathbf{a}_1 = d\varphi \mathbf{a}_2 \\ d\mathbf{a}_2 = -\epsilon d\varphi \mathbf{a}_1 \\ d\mathbf{b} = (db_1 - \epsilon d\varphi b_2) \mathbf{a}_1 + (db_2 + d\varphi b_1) \mathbf{a}_2 \end{cases} \quad (13)$$

and

$$\begin{cases} d'\mathbf{a}_1 = d\varphi' \mathbf{a}_2 \\ d'\mathbf{a}_2 = -\epsilon d\varphi' \mathbf{a}_1 \\ d'\mathbf{b} = (db_1' - \epsilon d\varphi' b_2') \mathbf{a}_1 + (db_2' + d\varphi' b_1') \mathbf{a}_2. \end{cases} \quad (14)$$

For the sake of shortness, we use the following equalities:

$$\begin{cases} d\varphi = \tau \\ db_1 - \epsilon d\varphi b_2 = \sigma_1 \\ db_2 + d\varphi b_1 = \sigma_2 \end{cases}, \quad \begin{cases} d\varphi' = \tau' \\ db_1' - \epsilon d\varphi' b_2' = \sigma_1' \\ db_2' + d\varphi' b_1' = \sigma_2' \end{cases}. \quad (15)$$

Definition 3.1.

σ_1, σ_2, τ and $\sigma_1', \sigma_2', \tau'$ are called CK-Pfaffian forms of the one-parameter CK-motions $A_\epsilon / \mathbb{P}_\epsilon$ and $A_\epsilon / \mathbb{P}_\epsilon'$ with respect to t , respectively.

With reference to the above definition, the derivative formulae of the motions $A_\epsilon / \mathbb{P}_\epsilon$ and $A_\epsilon / \mathbb{P}_\epsilon'$ can be rearranged as follows:

$$\begin{cases} d\mathbf{a}_1 = \tau \mathbf{a}_2 \\ d\mathbf{a}_2 = -\epsilon \tau \mathbf{a}_1 \\ d\mathbf{b} = \sigma_1 \mathbf{a}_1 + \sigma_2 \mathbf{a}_2 \end{cases}, \quad \begin{cases} d'\mathbf{a}_1 = \tau' \mathbf{a}_2 \\ d'\mathbf{a}_2 = -\epsilon \tau' \mathbf{a}_1 \\ d'\mathbf{b} = \sigma_1' \mathbf{a}_1 + \sigma_2' \mathbf{a}_2. \end{cases} \quad (16)$$

Let us consider a point X with the coordinates of (x_1, x_2) in moving plane A_ϵ . Since $\mathbf{B}\mathbf{X} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2$ is a vector on the moving system of A_ϵ , we have

$$\begin{cases} \mathbf{x} = \mathbf{O}\mathbf{X} = \mathbf{O}\mathbf{B} + \mathbf{B}\mathbf{X} = -\mathbf{b} + x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 \\ \mathbf{x}' = \mathbf{O}'\mathbf{X} = \mathbf{O}'\mathbf{B} + \mathbf{B}\mathbf{X} = -\mathbf{b}' + x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2, \end{cases}$$

where $\mathbf{x}$ and $\mathbf{x}'$ are coordinate vectors of the point X with respect to $\mathbb{P}_\epsilon$ and $\mathbb{P}_\epsilon'$, respectively.

The differential of X with respect to $\mathbb{P}_\epsilon$ is

$$d\mathbf{x} = (dx_1 - \sigma_1 - \epsilon \tau x_2) \mathbf{a}_1 + (dx_2 - \sigma_2 + \tau x_1) \mathbf{a}_2. \quad (17)$$

Hence, the *relative velocity vector* of X with respect to $\mathbb{P}_\epsilon$ is as follows:

$$\mathbf{V}_r = \frac{d\mathbf{x}}{dt}.$$

Similarly, differential of X with respect to $\mathbb{P}_\epsilon'$ is

$$d'\mathbf{x} = d'\mathbf{x} = (dx_1 - \sigma_1' - \epsilon \tau' x_2) \mathbf{a}_1 + (dx_2 - \sigma_2' + \tau' x_1) \mathbf{a}_2. \quad (18)$$

Thus, the *absolute velocity vector* of X with respect to $\mathbb{P}_\epsilon'$ is

$$\mathbf{V}_a = \frac{d'\mathbf{x}}{dt}.$$

If $\mathbf{V}_r = 0$ and $\mathbf{V}_a = 0$ then the point X is fixed in the planes $\mathbb{P}_\epsilon$ and $\mathbb{P}'_\epsilon$, respectively. So, the conditions become

$$dx_1 = \sigma_1 + \epsilon \tau x_2, \quad dx_2 = \sigma_2 - \tau x_1 \quad (19)$$

and

$$dx_1 = \sigma'_1 + \epsilon \tau' x_2, \quad dx_2 = \sigma'_2 - \tau' x_1, \quad (20)$$

respectively. By using the equations (19) and (20) and considering that the sliding velocity vector of the point X is $\mathbf{V}_f = \frac{d_f \mathbf{x}}{dt}$, we have

$$d_f \mathbf{x} = [(\sigma_1 - \sigma'_1) - \epsilon(\tau' - \tau)x_2] \mathbf{a}_1 + [(\sigma_2 - \sigma'_2) + (\tau' - \tau)x_1] \mathbf{a}_2. \quad (21)$$

In this manner, from (17), (18) and (21) we can give the following theorem.

Theorem 3.1.

Let X be a fixed point on the plane $\mathbb{P}_\epsilon$ under the one-parameter planar CK-motions $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$. Then, there is a relation between the differentials as noted below:

$$d' \mathbf{x} = d_f \mathbf{x} + d\mathbf{x}. \quad (22)$$

The above equation enables us to write the relationship between the velocities: $\mathbf{V}_a = \mathbf{V}_f + \mathbf{V}_r$. Hence, the above theorem is implemented.

Now, by considering the planes $\mathbb{P}_\epsilon$ and $\mathbb{P}'_\epsilon$ are fixed, we give the following theorem:

Theorem 3.2.

Let $\tilde{\mathbf{V}}_a, \tilde{\mathbf{V}}_f$ and $\tilde{\mathbf{V}}_r$ be absolute, relative and sliding velocity vectors of the motions $A_\epsilon / \mathbb{P}_\epsilon$, respectively. Similarly, let $\tilde{\mathbf{V}}'_a, \tilde{\mathbf{V}}'_f$ and $\tilde{\mathbf{V}}'_r$ be absolute, relative and sliding velocity vectors of the motions $A_\epsilon / \mathbb{P}'_\epsilon$, respectively. Then, the sliding velocity vector of the motions $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ can be given as below:

$$\begin{aligned} \mathbf{V}_f &= \tilde{\mathbf{V}}'_f - \tilde{\mathbf{V}}_f \\ &= [(\sigma_1 - \sigma'_1) - \epsilon(\tau' - \tau)x_2] \mathbf{a}_1 \\ &\quad + [(\sigma_2 - \sigma'_2) + (\tau' - \tau)x_1] \mathbf{a}_2. \end{aligned} \quad (23)$$

Proof 3.1.

By taking into consideration the conditions (19) and (20), we have the sliding velocity vectors of the motions $A_\epsilon / \mathbb{P}_\epsilon$ and $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$, respectively:

$$\begin{cases} \tilde{\mathbf{V}}_f = (-\sigma_1 - \epsilon \tau x_2) \mathbf{a}_1 + (-\sigma_2 + \tau x_1) \mathbf{a}_2 \\ \tilde{\mathbf{V}}'_f = (-\sigma'_1 - \epsilon \tau' x_2) \mathbf{a}_1 + (-\sigma'_2 + \tau' x_1) \mathbf{a}_2 \end{cases}$$

Accordingly, the sliding velocity vector of the motions $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ can be calculated as follows:

$$\begin{aligned} \mathbf{V}_f &= \tilde{\mathbf{V}}'_f - \tilde{\mathbf{V}}_f \\ &= [(\sigma_1 - \sigma'_1) - \epsilon(\tau' - \tau)x_2] \mathbf{a}_1 \\ &\quad + [(\sigma_2 - \sigma'_2) + (\tau' - \tau)x_1] \mathbf{a}_2. \end{aligned}$$

Finally, It is quite obvious that we get the equation (21) again.

Corollary 3.1.

The motions $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ is characterized by the composition of the inverse motions $\mathbb{P}_\epsilon / A_\epsilon$ and the motions $A_\epsilon / \mathbb{P}'_\epsilon$ as follows:

$$\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon = (\mathbb{P}_\epsilon / A_\epsilon) \circ (A_\epsilon / \mathbb{P}'_\epsilon). \quad (24)$$

4. Moving Planes with Respect to the Another and Rotation Poles

In this original section, we will aim to find out the rotation poles during the motions $A_\epsilon / \mathbb{P}_\epsilon$, $A_\epsilon / \mathbb{P}'_\epsilon$ and $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ with the different perspective to be a moving or fixed plane. It is indicated that in the one-parameter planar

motions in affine CK-planes, the rotation pole is characterized by vanishing sliding velocity. In the consideration of this statement, the rotation poles can be found as follows:

The pole point $Q = (q_1, q_2)$ of the motions $A_\epsilon / \mathbb{P}_\epsilon$ is calculated with $\tilde{\mathbf{V}}_f = 0$ as below:

$$\begin{cases} q_1 = \frac{\sigma_2}{\tau}, \\ \epsilon q_2 = -\frac{\sigma_1}{\tau}. \end{cases} \quad (25)$$

The pole point $Q' = (q'_1, q'_2)$ of the motions $A_\epsilon / \mathbb{P}'_\epsilon$ is computed with $\tilde{\mathbf{V}}'_f = 0$ and given as follows:

$$\begin{cases} q'_1 = \frac{\sigma'_2}{\tau'}, \\ \epsilon q'_2 = -\frac{\sigma'_1}{\tau'}. \end{cases} \quad (26)$$

The pole point $P = (p_1, p_2)$ of the motions $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ characterised by $\mathbf{d}_f \mathbf{x} = 0$. So, the pole point P of the one-parameter planar motions $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ is obtained as follows:

$$\begin{cases} p_1 = \frac{\sigma'_2 - \sigma_2}{\tau' - \tau}, \\ \epsilon p_2 = -\frac{\sigma'_1 - \sigma_1}{\tau' - \tau}, \end{cases} \quad (27)$$

where $\mathbf{BP} = p_1 \mathbf{a}_1 + p_2 \mathbf{a}_2$.

Theorem 4.1.

If three affine CK-planes generate one-parameter planar CK-motions pairwise, there exist three relative rotation poles at every moment t .

Theorem 4.2.

Let P, Q and Q' be the pole points of the affine CK-motions $A_\epsilon / \mathbb{P}_\epsilon$, $A_\epsilon / \mathbb{P}'_\epsilon$ and $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$, respectively. Then P, Q and Q' are collinear.

Proof 4.1.

The slopes of $[PQ], [PQ']$ and $[QQ']$ are all equal to:

$$\frac{\sigma'_1 \tau' - \sigma_1 \tau}{\sigma'_2 \tau - \sigma_2 \tau'}.$$

This completes the proof.

Definition 4.1.

The straight line, indicated the above theorem, is called *the affine CK-pole line* of the one-parameter affine CK-motions $A_\epsilon / \mathbb{P}_\epsilon$, $A_\epsilon / \mathbb{P}'_\epsilon$ and $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$.

Corollary 4.1.

Generally, if there are n – affine CK-planes which form one-parameter planar affine CK-motions pairwise, then we mention about n – member kinematic chain. If the each motions are connected time parameter t (real), then there exist $\binom{n}{2}$ relative rotation poles at every moment t .

5. Euler-Savary Formula in Affine CK-Planes

In this original section, we will study Euler-Savary formula in affine CK-planes. We choose the relative system $\{B; \mathbf{a}_1, \mathbf{a}_2\}$ satisfying the following conditions:

- i) The initial point B of the system is the instantaneous rotation pole P (i.e. $B = P$)
- ii) The axis $\{B, \mathbf{a}_1\}$ coincides with the common tangent of the pole curves (P) and (P') .

By considering the condition i), we have $p_1 = p_2 = 0$, because of the fact that P is an

initial point. Hence, from the equations (27), we obtain the following equalities:

$$\sigma_1' = \sigma_1, \quad \sigma_2' = \sigma_2. \quad (28)$$

So, from the equations (16), we can give the pole tangent as follows:

$$\mathbf{db} = \mathbf{dp} = \sigma_1 \mathbf{a}_1 + \sigma_2 \mathbf{a}_2 = \mathbf{d}'\mathbf{p} = \mathbf{db}'. \quad (29)$$

The equation (29) means that, the relative and the absolute velocities are equal to each other (i.e. $\mathbf{V}_f = 0$). Thus, the scalar arc elements of the pole curves (P) and (P') can be given as below:

$$ds' = \|\mathbf{V}_a\|_e dt = \|\mathbf{V}_r\|_e dt = ds. \quad (30)$$

This means that the moving pole curve (P) and fixed pole curve (P') roll on each other without sliding. By considering the condition ii) yields us that $\sigma_2 = \sigma_2' = 0$. If we take $\sigma_1 = \sigma_1' = \sigma$, then the derivative formulae of the canonical relative system $\{P, \mathbf{a}_1, \mathbf{a}_2\}$ become

$$\begin{cases} d\mathbf{a}_1 = \tau \mathbf{a}_2 \\ d\mathbf{a}_2 = -\epsilon \tau \mathbf{a}_1 \\ d\mathbf{p} = \sigma \mathbf{a}_1 \end{cases}, \quad \begin{cases} d'\mathbf{a}_1 = \tau' \mathbf{a}_2 \\ d'\mathbf{a}_2 = -\epsilon \tau' \mathbf{a}_1 \\ d'\mathbf{p} = \sigma \mathbf{a}_1 \end{cases}. \quad (31)$$

The differential forms σ , τ and τ' of the equations (31) have specific meanings: $ds = \sigma$ is the scalar arc element of the pole curves (P) and (P') , τ and τ' are the central cotangent angle, that is, two neighboring tangents angle of (P) and (P') , respectively. Thus, the curvature of the moving pole curve (P) and (P') , at the point P are $\frac{d\phi}{ds} = \frac{\tau}{\sigma}$ and $\frac{d\phi'}{ds} = \frac{\tau'}{\sigma}$, respectively. Hence, the curvature radii of the pole curves (P) and (P') can be written as below:

$$r = \frac{\sigma}{\tau} \text{ and } r' = \frac{\sigma'}{\tau'}, \quad (32)$$

respectively. Moving plane $\mathbb{P}_e$ rotates the infinitesimal instantaneous angle $d\phi = \tau' - \tau$ around the rotation pole (P) within the time scale t with respect to fixed plane $\mathbb{P}_e'$. Hence, ω is the angular velocity of moving plane $\mathbb{P}_e$ with respect to the fixed plane $\mathbb{P}_e'$ is given below:

$$\frac{\tau' - \tau}{dt} = \frac{d\phi}{dt} = \dot{\phi} = \omega. \quad (33)$$

Also, we denote the angular acceleration of moving plane $\mathbb{P}_e$ with respect to the fixed plane $\mathbb{P}_e'$ by $\dot{\omega}$, where $\ddot{\phi} = \dot{\omega}$. From the equations (32) and (33), it is seen that

$$\frac{\tau' - \tau}{dt} = \frac{d\phi}{dt} = \frac{1}{r'} - \frac{1}{r}. \quad (34)$$

Let us assume that the direction of unit tangent vector is $\mathbf{a}_1$ and $ds/dt > 0$. Due to the fact that, the curvature center of the moving pole curve (P) stays in the same side of the directed pole curve $(P; \mathbf{a}_1)$, it is written that $r > 0$. Similarly $r' > 0$.

Let us rearrange the equations (17) and (18) with respect to the planes $\mathbb{P}_e$ and $\mathbb{P}_e'$ by taking a point $X = (x_1, x_2)$ in the plane A_e , respectively. These differentials can be calculated as below:

$$d\mathbf{x} = (dx_1 - \sigma - \epsilon \tau x_2) \mathbf{a}_1 + (dx_2 + \tau x_1) \mathbf{a}_2$$

and

$$d'\mathbf{x} = (dx_1 - \sigma - \epsilon \tau' x_2) \mathbf{a}_1 + (dx_2 + \tau' x_1) \mathbf{a}_2.$$

On the condition that $d\mathbf{x} = 0$, then the following conditions occur and X is a fixed point $\mathbb{P}_e$:

$$\begin{cases} dx_1 = \sigma + \epsilon \tau x_2 \\ dx_2 = -\tau x_1. \end{cases} \quad (35)$$

In a similar way, if $d'\mathbf{x} = 0$, then the following conditions exist and X is a fixed point $\mathbb{P}'_\epsilon$:

$$\begin{cases} dx_1 = \sigma + \epsilon \tau' x_2 \\ dx_2 = -\tau' x_1. \end{cases} \quad (36)$$

Thus, the sliding velocity of the motion can be given as follows:

$$d_f \mathbf{x} = d'\mathbf{x} - d\mathbf{x} = (\tau' - \tau)(\epsilon x_2 \mathbf{a}_1 - x_1 \mathbf{a}_2).$$

Now, we investigate the curvature centers of trajectory curves which are drawn in the fixed plane by the points of moving plane during the motion $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$. Let $M' = (m'_1, m'_2)$ represents the curvature center of trajectory curves which are drawn in $\mathbb{P}'_\epsilon$ by the point $X = (x_1, x_2)$ in $\mathbb{P}_\epsilon$ with respect to the canonical relative system at every time t . The points X and M' and the instantaneous rotation pole P lay on an instantaneous trajectory normal related to X at every time t . Therefore, the vectors

$$\mathbf{PX} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2$$

and

$$\mathbf{PM}' = m'_1 \mathbf{a}_1 + m'_2 \mathbf{a}_2$$

have the same direction which passes the rotation pole P . Accordingly, we can write

$$x_1 : x_2 = m'_1 : m'_2$$

or

$$x_1 m'_2 - m'_1 x_2 = 0. \quad (37)$$

If we differentiate the equation (37), we have

$$dx_1 m'_2 + x_1 dm'_2 - dm'_1 x_2 - m'_1 dx_2 = 0. \quad (38)$$

By using the new form of the equations (35) and (36) for the points X and M' in the equation (38), we obtain the following equation as below:

$$\sigma(m'_2 - x_2) - (\tau' - \tau)(x_1 m'_1 + \epsilon x_2 m'_2). \quad (39)$$

If we substitute the polar coordinates

$$\begin{cases} x_1 = a \cos_\epsilon \alpha \\ x_2 = a \sin_\epsilon \alpha \end{cases}, \quad \begin{cases} m'_1 = a' \cos_\epsilon \alpha \\ m'_2 = a' \sin_\epsilon \alpha \end{cases}$$

in the equation (39), we get

$$\sigma \sin_\epsilon \alpha (a' - a) - (\tau - \tau') a a' = 0,$$

where a and a' are the distance between the points X and M' and rotation pole P , respectively. Besides, α is the angle between the pole ray ($\mathbf{PX}$ and $\mathbf{PM}'$) and the common tangent of pole curves. Finally, by taking into account the equation (33), we obtain the last form of the above equation as follows:

$$\left(\frac{1}{a} - \frac{1}{a'}\right) \sin_\epsilon \alpha = \frac{1}{r'} - \frac{1}{r} = \frac{d\phi}{ds}. \quad (40)$$

Consequently, the equation (40) is called *Euler-Savary formula for one-parameter motions in affine CK-planes*. Hence, the following theorem can be given:

Theorem 5.1.

Let $\mathbb{P}_\epsilon$ and $\mathbb{P}'_\epsilon$ be moving and fixed affine CK-planes, respectively. A point X in moving CK-plane $\mathbb{P}_\epsilon$ draws a trajectory whose curvature center is at the point M' in fixed plane $\mathbb{P}'_\epsilon$ during the one-parameter planar CK-motion $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$. In the reverse motion $\mathbb{P}'_\epsilon / \mathbb{P}_\epsilon$ a point M' in $\mathbb{P}'_\epsilon$ whose curvature center is at the point X in $\mathbb{P}_\epsilon$. The relationship between the points X and M' is given by Euler-Savary formula by the equation (40).

6. Discussions and Conclusions

In this paper, the generalization of moving coordinate system and Euler-Savary formula have been successfully applied in affine Cayley-Klein planes (CK-planes) by using one-parameter planar motions [24]. We have considered three affine CK-planes: A_ϵ , $\mathbb{P}_\epsilon$ and $\mathbb{P}'_\epsilon$. The plane $\mathbb{P}'_\epsilon$ is a fixed plane relative to two other moving affine CK-planes. We have examined the relationship between the motions $A_\epsilon / \mathbb{P}_\epsilon$, $A_\epsilon / \mathbb{P}'_\epsilon$ and $\mathbb{P}_\epsilon / \mathbb{P}'_\epsilon$ by evaluating their

derivative formulae, velocity vectors and pole points. We have introduced canonical relative system for one-parameter planar motions in the affine CK-planes by using the notions of moving coordinate system. Furthermore, we have obtained Euler-Savary formula with the aid of canonical relative system by using the H. R. Müller's Method. We have established a simple but effective method by unifying moving coordinate system and Euler-Savary formula in Euclidean, Lorentzian and Galilean planes.

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