SOME ASPECTS OF NEYMAN TRIANGLES AND DELANNOY ARRAYS

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Summary. This note considers some number theoretic properties of the orthonormal Neyman polynomials which are related to Delannoy numbers and certain complex Delannoy numbers.

1 INTRODUCTION

Rayner and Best point out that “the concept of smooth goodness of fitness tests was introduced in Neyman (1937)” [22]. Goodness of fit concepts in general usually go back to Karl Pearson [20]. Rayner [21] further pointed out that Jerzy Neyman’s smooth alternative of order k to the uniform distribution on $(0,1)$ has probability density for

$$h(y, \theta) = \exp \left\{ \sum_{i=1}^k \theta_i \pi_i(y) - K(\theta) \right\}, 0 < y < 1, k = 1, 2, \dots \quad (1.1)$$

where $K(\theta)$ is a normalising constant and the $\pi_i(y)$ are orthonormal polynomials (Freeman) related to the Legendre polynomials.

It is the purpose of this note to consider some number theoretic properties of the $\pi_i(y)$ polynomials ($i = 0, 1, 2, 3, 4$ in Rayner) which, for convenience, we label as Neyman polynomials. In Deveci and Shannon [9] complex-type k -Fibonacci numbers are defined and the relationships between the k -step Fibonacci numbers and the complex-type k -Fibonacci numbers are provided together with miscellaneous properties of the complex-type k -Fibonacci numbers. In addition, they studied the complex-type k -Fibonacci sequence modulo m . Finally, they obtained the period of the complex-type 2-Fibonacci sequences in the Dihedral group D_{2n} , ($n \geq 2$).

In this paper, we define the complex-type Delannoy numbers and then give the relationships between the Delannoy numbers and the complex-type Delannoy numbers. Furthermore, we study the complex-type Delannoy sequence modulo m .

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2 NEYMAN POLYNOMIALS

Rayner elsewhere lists the first five such polynomials and we add some more in order to build up a picture of patterns. To help with this we have slightly modified some aspects of his notation as in Bera and Ghosh [3]:

$$\begin{aligned}
\pi_0(y) &= \sqrt{1}(1) \\
\pi_1(y) &= \sqrt{3}(2y - 1) \\
\pi_2(y) &= \sqrt{5}(6y^2 - 6y + 1) \\
\pi_3(y) &= \sqrt{7}(20y^3 - 30y^2 + 12y - 1) \\
\pi_4(y) &= \sqrt{9}(70y^4 - 140y^3 + 90y^2 - 20y + 1) \\
\pi_5(y) &= \sqrt{11}(252y^5 - 630y^4 + 560y^3 - 210y^2 + 30y - 1) \\
\pi_6(y) &= \sqrt{13}(924y^6 - 2772y^5 + 3150y^4 - 1680y^3 + 420y^2 - 42y + 1).
\end{aligned}$$

Blinov and Lemeshko [4] have set out corresponding Legendre polynomials as, in effect,

$$\begin{aligned}
p_0(y) &= \sqrt{1}(1) \\
p_1(y) &= \sqrt{3}(2y) \\
p_2(y) &= \sqrt{5}(6y^2 - 0.5) \\
p_3(y) &= \sqrt{7}(20y^3 - 3y) \\
p_4(y) &= \sqrt{9}(70y^4 - 15y^2 + 0.375).
\end{aligned}$$

3 NEYMAN TRIANGLE

We assemble the absolute values of the polynomial coefficients into a triangle, as the row sums are all unity if we include the signed values of the coefficients. The row sums are in the right-most column, and the pertinent OIES references [23] are in the bottom row.

1							1
2	1						3
6	6	1					13
20	30	12	1				63
70	140	90	20	1			321
252	630	560	210	30	1		1683
924	2772	3150	1680	420	42	1	8989
A000984	A002457	A002544	A007744	A106440	A013613	---	A001850

Table 1: Neyman triangle

The leading diagonals in this table generate the sequence $\{1, 2, 7, 26, 101, 404, 1645, \dots\}$ which does not seem to be in OEIS, but the anti-diagonals can related to OEIS sequences in Table 2(a).

1	1	1	1	1	1	1	1	A000012
2	6	12	20	30	42	56	72	A002378
6	30	90	210	420	756	1260	1980	A033487
20	140	560	1680	4200	9240	18480	34320	A105939
70	630	3150	11550	34650	90090	210210	450450	70xA000581

Table 2(a): Anti-diagonals in Neyman triangle

The patterns are clearer when we express the Neyman anti-diagonals as multiples of the first element in each row, as in Table 2 (b). The leading diagonal here yields a known sequence (A005809) as do the anti-diagonals (A001519), the odd Fibonacci numbers as a bisection of the Fibonacci sequence, but we shall not pursue these here.

1 X	1	1	1	1	1	1	1	1	A000012
2 X	1	3	6	10	15	21	28	36	A000217
6 X	1	5	15	35	70	126	210	330	A000332
20 X	1	7	28	84	210	462	924	1716	A000579
70 X	1	9	45	165	495	1287	3003	6435	A000581
A0.....	00012	05408	0384	000447	53134	02299	53135	53136	

Table 2(b): Anti-diagonals in Neyman triangle

The leading diagonals in Table 2(a) generate the sequence $\{1, 3, 13, 63, 321, 1683, 8989, \dots\}$ [A001850] the elements of which are the Central Delannoy numbers [2], so called because they constitute the central anti-diagonal in the infinite square Delannoy array [A008288] in Table 3. The leading anti-diagonal here is A005809.

$n \downarrow$ $m \rightarrow$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	3	5	7	9	11	13	15
2	1	5	13	25	41	61	85	113
3	1	7	25	63	129	231	377	575
4	1	9	41	129	321	681	1289	2241
5	1	11	61	231	681	1683	3653	7183
6	1	13	85	377	1289	3653	8989	19825
7	1	15	113	575	2241	7183	19825	48639

Table 3: Square Delannoy array

The leading diagonals in this array generate the Pell numbers $\{1, 2, 5, 12, 29, \dots\}$, and, in the sense of this paper, Alladi and Hoggatt [1] further related these numbers to Tribonacci triangles. When this array is turned clockwise through 45° we have the Pell triangle.

We also see regular intersections (as common elements) among the row and column sequences, which is a topic worth exploring as in Stein [24] who found it necessary to examine the intersection of Fibonacci sequences in order to answer the question of whether every member of a variety is a quasigroup given that every finite member is [25].

The Central Delannoy numbers $\{a_n\}, n \geq 0$, can be expressed as

$$a_n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \quad (3.1)$$

and

$$a_n = \frac{\pi_n(2)}{\sqrt{n}} \quad (3.2)$$

in terms of the Neyman numbers, which would appear to be new. This suggests we consider in turn

$$\frac{\pi_n(3)}{\sqrt{n}} = \{1, 5, 37, 305, 2641, 23525, \dots\}$$

which is A006442, the expansion of $(x^2 - 10x + 1)^{-\frac{1}{2}}$, which is also related to the Delannoy numbers. Likewise A084768 is

$$\frac{\pi_n(4)}{\sqrt{n}} = \{1, 7, 73, 847, 10321, 129367, 1651609, \dots\}$$

and so on.

4 THE COMPLEX-TYPE DELANNOY NUMBERS

Now we define a new sequence that we call the complex-type Delannoy sequence $\{D^i(m, n)\}$ as follows:

$$D^i(m, n) = \begin{cases} 1 & \text{if } m = 0 \text{ or } n = 0, \\ i \cdot D^i(m-1, n) + i \cdot D^i(m, n-1) - D^i(m-1, n-1) & \text{otherwise.} \end{cases} \quad (1)$$

Note that when $m = n = a$, the complex-type Delannoy sequence $\{D^i(m, n)\}$ is reduced to the central complex-type sequence $\{D^i(a, a)\}$.

A table for the values of the complex-type Delannoy numbers is given by below:

$n \downarrow$ $m \rightarrow$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	$2i-1$	-3	$-2i-1$	1	$2i-1$	-3	$-2i-1$
2	1	-3	$-8i+1$	13	$16i+1$	-19	$-24i+1$	29
3	1	$-2i-1$	13	$34i-1$	-63	$-98i-1$	141	$194i-1$
4	1	1	$16i+1$	-63	$-160i+1$	321	$560i+1$	-895
5	1	$2i-1$	-19	$-98i-1$	321	$802i-1$	-1683	$-3138i-1$
6	1	-3	$-24i+1$	141	$560i+1$	-1683	$-4168i+1$	8989
7	1	$-2i-1$	29	$194i-1$	-895	$-3138i-1$	8989	$22146i-1$

Table 4: Square complex-type Delannoy numbers

From the definitions of the Delannoy numbers and the complex-type Delannoy numbers, we derive the following relations:

i. For $m, n \geq 1$

$$D^i(m, n) = \begin{cases} 2(i)^n \cdot D(m-1, n-1) - D^i(m-1, n-1), & n \equiv 1 \pmod{4}, \\ 2(i)^{n+1} \cdot D(m-1, n-1) - D^i(m-1, n-1), & n \equiv 2 \pmod{4}, \\ 2(i)^{n+2} \cdot D(m-1, n-1) - D^i(m-1, n-1), & n \equiv 3 \pmod{4}, \\ 2(i)^{n+3} \cdot D(m-1, n-1) - D^i(m-1, n-1), & n \equiv 0 \pmod{4}. \end{cases}$$

ii. For $m, n \geq 0$, $D^i(m, n) = D^i(n, m)$.

iii. For $m, n \geq 0$, $D^i(n+1, n) = D^i(n, n+1) = (-1)^n \cdot D(n, n)$.

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence.

The research on the conformity of a single term, $a_n \pmod{p}$, has a long history forming most known Pascal's oldest fractal problem, which was originally created by the parities of binomial coefficients $\binom{n}{k}$; see for example, [5,6,7,8,10,12,14,16,17,18,29,30]. We now extend the concept to the complex-type Delannoy numbers.

Consider the sequence

$$\{D^i(m, n)\} = \{D^i(0, n), D^i(1, n), D^i(2, n), \dots\}$$

where n is a fixed positive integer and $m = 0, 1, 2, \dots$

If we reduce the sequence $\{D^i(m, n)\}$ modulo α , taking least nonnegative residues, then we can get the repeating sequence, denoted by

$$\{D^i(m, n)(\alpha)\} = \{D^i(0, n)(\alpha), D^i(1, n)(\alpha), D^i(2, n)(\alpha), \dots\}$$

where $D^i(u, n)(\alpha)$ is used to mean the u th element of the sequence $\{D^i(m, n)(\alpha)\}$ modulo α for the positive integer constant n .

We note here that the sequence $\{D^i(m, n)(\alpha)\}$ has the same recurrence relation as in (1).

Theorem 4.1. The sequence $\{D^i(m, n)(\alpha)\}$ is periodic.

Proof. It is clear that sequence $\{D^i(m, 1)(\alpha)\}$ is a constant sequence. Since the sequence $\{D^i(m, 1)(\alpha)\}$ is a constant sequence; that is, since it consists only the repetitions of a constant subsequence, we can say that the sequence $\{D^i(m, 2)(\alpha)\}$ is also a periodic sequence, using the recurrence relation in the sequence $\{D^i(m, n)(\alpha)\}$. Similarly, since the

sequences $\{D^i(m,1)(\alpha)\}$ and $\{D^i(m,2)(\alpha)\}$ are periodic; that is, they consist only the repetitions of constant sub-sequences, the sequence $\{D^i(m,n)(\alpha)\}$ is also periodic. By a similar idea, we get the repeating sequences

$$\{D^i(m,1)(\alpha)\}, \{D^i(m,2)(\alpha)\}, \dots, \{D^i(m,n-1)(\alpha)\}$$

are periodic; that is, they consist only the repetitions of constant sub-sequences, using the recurrence relation in the sequence $\{D^i(m,n)(\alpha)\}$. Thus, this implies that the sequence $\{D^i(m,n)(\alpha)\}$ is periodic. $\square$

Example 2.1. We have

$$\{D^i(m,3)(3)\} = \left\{ \begin{array}{l} 1, i-1, 1, i-1, 0, i-1, 0, 2i-1, 0, 2i-1, 1, i-1, \\ 1, i-1, 1, i-1, 0, i-1, 0, 2i-1, 0, 2i-1, 1, i-1, \dots \end{array} \right\}$$

and its terms repeat so we get $L(D^i(m,3)(3))=12$, where the period of the sequence $\{D^i(m,n)(\alpha)\}$ is denoted by $L(D^i(m,n)(\alpha))$.

Conjecture 4.1. Let p be prime, let n be a fixed positive integer and $m=0,1,2,\dots$. If u is the smallest positive integer such that $L(D^i(m,n)(p^{u+1})) \neq L(D^i(m,n)(p^u))$, then $L(D^i(m,n)(p^v)) = p^{v-u} \cdot L(D^i(m,n)(p^u))$.

Theorem 4.2. Let α_1 and α_2 be positive integers with $\alpha_1, \alpha_2 \geq 2$, then

$$L(D^i(m,n)(lcm(\alpha_1, \alpha_2))) = lcm[L(D^i(m,n)(\alpha_1)), L(D^i(m,n)(\alpha_2))].$$

Proof. Let $lcm(\alpha_1, \alpha_2) = \alpha$. Then,

$$\begin{aligned} D^i(m,n)[L(D^i(m,n)(\alpha))] &\equiv D^i(m,n)[L(D^i(m,n)(\alpha))+1] \\ &\equiv \dots \equiv D^i(m,n)[L(D^i(m,n)(\alpha))+n-1] \equiv 0 \pmod{\alpha} \end{aligned}$$

and

$$\begin{aligned} D^i(m,n)[L(D^i(m,n)(\alpha_k))] &\equiv D^i(m,n)[L(D^i(m,n)(\alpha_k))+1] \\ &\equiv \dots \equiv D^i(m,n)[L(D^i(m,n)(\alpha_k))+n-1] \equiv 0 \pmod{\alpha_k} \end{aligned}$$

for $k=1,2$. Using the least common multiple operation this implies that

$$\begin{aligned} D^i(m,n)[L(D^i(m,n)(\alpha))] &\equiv D^i(m,n)[L(D^i(m,n)(\alpha))+1] \\ &\equiv \dots \equiv D^i(m,n)[L(D^i(m,n)(\alpha))+n-1] \equiv 0 \pmod{\alpha_k} \end{aligned}$$

for $k = 1, 2$. So we have $L(D^i(m, n)(\alpha_1)) \mid L(D^i(m, n)(\alpha))$ and $L(D^i(m, n)(\alpha_2)) \mid L(D^i(m, n)(\alpha))$, which means that $\text{lcm}[L(D^i(m, n)(\alpha_1)), L(D^i(m, n)(\alpha_2))]$ divides $L(D^i(m, n)(\text{lcm}(\alpha_1, \alpha_2)))$. We also know that

$$\begin{aligned} D^i(m, n)[\text{lcm}(L(D^i(m, n)(\alpha_1)), L(D^i(m, n)(\alpha_2)))] &\equiv D^i(m, n)[\text{lcm}(L(D^i(m, n)(\alpha_1)), L(D^i(m, n)(\alpha_2)))+1] \\ &\equiv \dots \equiv D^i(m, n)[\text{lcm}(L(D^i(m, n)(\alpha_1)), L(D^i(m, n)(\alpha_2)))+n-1] \equiv 0 \pmod{\alpha_k}. \end{aligned}$$

Then,

$$\begin{aligned} D^i(m, n)[\text{lcm}(L(D^i(m, n)(\alpha_1)), L(D^i(m, n)(\alpha_2)))] &\equiv D^i(m, n)[\text{lcm}(L(D^i(m, n)(\alpha_1)), L(D^i(m, n)(\alpha_2)))+1] \\ &\equiv \dots \equiv D^i(m, n)[\text{lcm}(L(D^i(m, n)(\alpha_1)), L(D^i(m, n)(\alpha_2)))+n-1] \equiv 0 \pmod{\alpha}. \end{aligned}$$

and it follows that $L(D^i(m, n)(\text{lcm}(\alpha_1, \alpha_2)))$ divides $\text{lcm}[L(D^i(m, n)(\alpha_1)), L(D^i(m, n)(\alpha_2))]$. Therefore, we have the following conclusions. $\square$

Corollary 4.1. Let v and u be positive integers. If $n = 2^v$, then $L(D^i(m, n)(2^u)) = 2^{u-v-1}$ for $u + 2 \geq v$.

Corollary 4.2. Let n be a positive integer and u a positive integer such that $u \geq 2$. Then $L(D^i(m, n)(2^u)) = 2^{u-1}$.

5 CONCLUDING COMMENTS

Lavers' Lemma 5 [15] suggests a way to generalize (3.1) to produce corresponding pyramids, and Horadam [13] and Subba Rao [26,27,28] contain further ideas on the study of intersections of sequences.

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