SOME NEW INTEGRAL INEQUALITIES THROUGH THE STEKLOV OPERATOR

B. BENAÏSSA¹, A. SENOUCI^{2*}

¹Faculty of Material Sciences, University of Tiaret-Algeria.

²Department of Mathematics, University of Tiaret-Algeria.

Laboratory of Informatics and Mathematics, University of Tiaret-Algeria.

*Corresponding author. E-mail: kamer295@yahoo.fr

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Summary. Hardy and Copson type inequalities have been studied by a large number of authors during the twentieth century and has motivated some important lines of study which are currently active. A large number of papers have been appeared involving Copson and Hardy inequalities (see [2-16] for more details).

In this paper some Hardy-Steklov and Copson-Steklov type integral inequalities were established. Namely the integral inequalities were proved there.

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_s f)^p(x) dx \leq \left(\frac{p}{|p-\alpha-1|} \right)^p \left(\int_0^b v(x) dx \right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} |K(x)|^q dx \right)^{\frac{p}{q}}, \quad (*)$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s f)^p(x) dx \leq \left(\frac{p}{|p-\alpha-1|} \right)^p \left(\int_0^b \phi(x) dx \right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} |J(x)|^q dx \right)^{\frac{p}{q}}. \quad (**)$$

Where $(\mathcal{F}_s f)$ is the Hardy-Steklov type operator and $(\mathcal{C}_s f)$ is the Copson-Steklov type operator (see the main results for more details).

Several Hardy-Steklov type, Hardy-type and Hardy integral inequalities were derived from (*). Similarly, some Copson-Steklov type and Copson type integral inequalities are deduced from (**).

1. INTRODUCTION

In 1928, G.H. Hardy proved the following integral inequalities [6]. Let f non-negative measurable function on $(0, \infty)$

$$(\mathcal{F}f)(x) = \begin{cases} \int_0^x f(t) dt & \text{for } \alpha < p-1, \\ \int_x^\infty f(t) dt & \text{for } \alpha > p-1, \end{cases}$$

then

$$\int_0^\infty x^{\alpha-p} (\mathcal{F}f)^p(x) dx \leq \left(\frac{p}{|p-\alpha-1|} \right)^p \int_0^\infty x^\alpha f^p(x) dx, \quad \text{for } p > 1. \quad (1)$$

In 1976, E.T. Copson proved the following integral inequalities (see [4], Theorem 1, Theorem 3). Let f, ϕ non-negative measurable functions on $(0, \infty)$

$$\Phi(x) = \int_0^x \phi(t) dt, \quad (\mathcal{C}f)(x) = \begin{cases} \int_0^x f(t) \phi(t) dt, & \text{for } c > 1, \\ \int_x^\infty f(t) \phi(t) dt, & \text{for } c < 1, \end{cases}$$

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then

$$\int_0^b (\mathcal{C}f)^p(x) \Phi^{-c}(x) \phi(x) dx \leq \left(\frac{p}{|c-1|} \right)^p \int_0^b f^p(x) \Phi^{p-c}(x) \phi(x) dx, \quad \text{for } p \geq 1. \quad (2)$$

Inequality (2) can be easily rewritten in the following form

$$(\mathcal{C}f)(x) = \begin{cases} \int_0^x f(t) \phi(t) dt, & \text{for } \alpha < p-1, \\ \int_x^\infty f(t) \phi(t) dt, & \text{for } \alpha > p-1, \end{cases}$$

then

$$\int_0^b (\mathcal{C}f)^p(x) \Phi^{\alpha-p}(x) \phi(x) dx \leq \left(\frac{p}{|p-1-\alpha|} \right)^p \int_0^b f^p(x) \Phi^\alpha(x) \phi(x) dx, \quad \text{for } p \geq 1. \quad (3)$$

The Hardy-Steklov operator is defined by

$$(Tf)(x) = g(x) \int_{r(x)}^{h(x)} f(t) dt, \quad f \geq 0,$$

where g is a positive measurable function and r, h are functions defined on an interval (a, b) such that $r(x) < h(x)$ for all $x \in (a, b)$.

Particular cases of this operator are Hardy operator $(\mathcal{F}f)(x) = \int_0^x f(t) dt$, the Hardy averaging operator $(F_\mu f)(x) = x^\mu \int_0^x f(t) dt$ and the Steklov operator $(Sf)(x) = \int_{x-1}^{x+1} f(t) dt$, which has been studied intensively (see [9] for example).

Let f, v, ϕ be non-negative measurable functions on $(0, \infty)$. Suppose that r and h are increasing differentiable functions on $[0, \infty)$, such that

$$\begin{cases} 0 < r(x) < h(x) < \infty & \text{for all } x \in (0, \infty), \\ r(0) = h(0) = 0 & \text{and } r(\infty) = h(\infty) = \infty. \end{cases} \quad (4)$$

The Hardy-Steklov and Copson-Steklov type operators are defined as follows,

$$(\mathcal{F}_s f)(x) = \int_{r(x)}^{h(x)} f(y) v(y) dy, \quad x > 0, \quad (5)$$

$$(\mathcal{C}_s f) = \int_{r(x)}^{h(x)} \frac{f(y) \phi(y)}{\Phi(y)} dy, \quad x > 0, \quad (6)$$

where

$$\Phi(x) = \int_0^x \phi(t)dt, \quad \text{for } x \in (0, \infty).$$

We adopt the usual convention: $\frac{0}{0} = \frac{\infty}{\infty} = 0$.

2. MAIN RESULTS

Let $0 < b \leq \infty$. Throughout the paper, we will assume that the integrals exist and are finite. The following lemma is needed in the proof of the main results (was proved in [1]).

Lemma 2.1. Let $1 < p \leq q < \infty$ and f, g, w be non-negative measurable functions on (a, b) such that $W(x) = \int_0^x w(t)dt$. If $m \in \mathbb{R}, m \neq 1$, then

$$\int_a^b \frac{w(x)}{W^m(x)} g^p(f(x))dx \leq \left(\int_a^b w(x)dx \right)^{1-\frac{p}{q}} \left(\int_a^b \frac{w(x)}{W^{\frac{mq}{p}}(x)} g^q(f(x))dx \right)^{\frac{p}{q}}. \quad (7)$$

Remark 2.1. Let $V(x) = \int_0^x v(t)dt$. By putting $m = p - \alpha$ in inequality (7), $w(x) = v(x)$, $W(x) = V(x)$ (respectively $w(x) = \phi(x)$, $W(x) = \Phi(x)$ and $f(x) = g(f(x))$), we obtain

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} f^p(x)dx \leq \left(\int_0^b v(x)dx \right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} f^q(x)dx \right)^{\frac{p}{q}}. \quad (8)$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} f^p(x)dx \leq \left(\int_0^b \phi(x)dx \right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} f^q(x)dx \right)^{\frac{p}{q}}. \quad (9)$$

The main results are presented in the following Theorem and Corollaries.

Theorem 2.1. Let f, v, ϕ be non-negative measurable functions on $(0, \infty)$, $1 < p \leq q < \infty$ and $r(x), h(x)$ satisfied the conditions (4). If $\alpha < p - 1$, then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_s f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1} \right)^p \left(\int_0^b v(x)dx \right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} |K(x)|^q dx \right)^{\frac{p}{q}}, \quad (10)$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1} \right)^p \left(\int_0^b \phi(x)dx \right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} |J(x)|^q dx \right)^{\frac{p}{q}}, \quad (11)$$

where

$$K(x) = \frac{V(x)}{v(x)} \left\{ [V(h(x))]' f(h(x)) - [V(r(x))]' f(r(x)) \right\},$$

$$J(x) = \frac{\Phi(x)}{\phi(x)} \left\{ \frac{[\Phi(h(x))]' }{\Phi(h(x))} f(h(x)) - \frac{[\Phi(r(x))]' }{\Phi(r(x))} f(r(x)) \right\}.$$

Proof. We consider the inequality (11), then

$$\begin{aligned} (\mathcal{C}_s f)'(x) &= h'(x) \frac{\phi(h(x))}{\Phi(h(x))} f(h(x)) - r'(x) \frac{\phi(r(x))}{\Phi(r(x))} f(r(x)) \\ &= \frac{\phi(x)J(x)}{\Phi(x)}, \end{aligned}$$

integrating by part in the left-hand side of (11), we get

$$\begin{aligned} \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s f)^p(x) dx &= \left[\frac{-(\mathcal{C}_s f)^p(x)}{(p-\alpha-1)\Phi^{p-\alpha-1}(x)} \right]_0^b + \frac{p}{p-\alpha-1} \\ &\times \int_0^b \frac{\phi(x)J(x)(\mathcal{C}_s f)^{p-1}(x)}{\Phi^{p-\alpha}(x)} dx. \end{aligned}$$

Since $\alpha < p-1$, we have

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s f)^p(x) dx \leq \frac{p}{p-\alpha-1} \int_0^b \frac{\phi(x)J(x)(\mathcal{C}_s f)^{p-1}(x)}{\Phi^{p-\alpha}(x)} dx.$$

The Hölder integral inequality for $\frac{1}{p} + \frac{1}{p'} = 1$, gives

$$\begin{aligned} \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s f)^p(x) dx &\leq \frac{p}{p-\alpha-1} \left(\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s f)^p(x) dx \right)^{\frac{1}{p'}} \\ &\times \left(\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} |J(x)|^p dx \right)^{\frac{1}{p}}, \end{aligned}$$

therefore

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s f)^p(x) dx \leq \left(\frac{p}{p-\alpha-1} \right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} |J(x)|^p dx.$$

Finally, by using inequality (9), we get (11).

The proof of inequality (10) is similar. So, the proof of Theorem is complete.

Now let $r(x) = 0$ in (5) and (6), thus

$$(\mathcal{F}_{s,1}f)(x) = \int_0^{h(x)} f(y)v(y)dy, \quad x > 0,$$

$$(\mathcal{C}_{s,1}f)(x) = \int_0^{h(x)} \frac{f(y)\phi(y)}{\Phi(y)} dy, \quad x > 0.$$

If we set $q = p$ in (10) and (11), we obtain the following corollary.

Corollary 2.1. Let f, v, ϕ be non-negative measurable functions on $(0, \infty)$, $p > 1$, $\alpha < p - 1$ and

$$\begin{cases} 0 < h(x) < \infty & \text{for all } x \in (0, \infty), \\ h(0) = 0 & \text{and } h(\infty) = \infty. \end{cases} \quad (12)$$

Then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_{s,1}f)^p(x) dx \leq \left(\frac{p}{p-\alpha-1} \right)^p \int_0^b \frac{v(x)}{V^{p-\alpha}(x)} |K_1(x)|^p dx, \quad (13)$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_{s,1}f)^p(x) dx \leq \left(\frac{p}{p-\alpha-1} \right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} |J_1(x)|^p dx, \quad (14)$$

where

$$\begin{aligned} K_1(x) &= \frac{V(x)[V(h(x))]'f(h(x))}{v(x)}, \\ J_1(x) &= \frac{\Phi(x)[\Phi(h(x))]'f(h(x))}{\phi(x)\Phi(h(x))}. \end{aligned}$$

Remark 2.2. If $h(x) = x$ in Corollary 2.1, we obtain the following weighted Hardy inequality and Copson-type inequality

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_{s,2}f)^p(x) dx \leq \left(\frac{p}{p-\alpha-1} \right)^p \int_0^b \frac{v(x)}{V^{-\alpha}(x)} f^p(x) dx, \quad (15)$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_{s,2}f)^p(x) dx \leq \left(\frac{p}{p-\alpha-1} \right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} f^p(x) dx, \quad (16)$$

where

$$\begin{aligned} (\mathcal{F}_{s,2}f)(x) &= \int_0^x f(y)v(y)dy, \quad x > 0, \\ (\mathcal{C}_{s,2}f)(x) &= \int_0^x \frac{f(y)\phi(y)}{\Phi(y)} dy, \quad x > 0. \end{aligned}$$

If we put $h(x) = \lambda x$ and $r(x) = \beta x$ and $q = p$ in Theorem 2.1, we get following corollary.

Corollary 2.2. Let f, v, ϕ be non-negative measurable functions on $(0, \infty)$, $0 < \beta < \lambda < \infty$, $p > 1$ and

$$(\mathcal{F}_{s,3}f)(x) = \int_{\beta x}^{\lambda x} f(y)v(y)dy, \quad x > 0,$$

$$(\mathcal{C}_{s,3}f)(x) = \int_{\beta x}^{\lambda x} \frac{f(y)\phi(y)}{\Phi(y)} dy, \quad x > 0.$$

If $\alpha < p - 1$, then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_{s,3}f)^p(x) dx \leq \left(\frac{p}{p-\alpha-1} \right)^p \int_0^b \frac{v(x)}{V^{p-\alpha}(x)} |K_2(x)|^p dx, \quad (17)$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_{s,3}f)^p(x) dx \leq \left(\frac{p}{p-\alpha-1} \right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} |J_2(x)|^p dx. \quad (18)$$

Where

$$K_3(x) = \frac{V(x)[\lambda v(\lambda x)f(\lambda x) - \beta v(\beta x)f(\beta x)]}{v(x)},$$

$$J_3(x) = \frac{\Phi(x)}{\phi(x)} \left\{ \frac{\lambda \phi(\lambda x)}{\Phi(\lambda x)} f(\lambda x) - \frac{\beta \phi(\beta x)}{\Phi(\beta x)} f(\beta x) \right\}.$$

Remark 2.3. One can prove the boundedness of the operator $\mathcal{F}_{s,3}$ from $L_p(0, \infty)$ to $L_p(0, \infty)$ by using the Minkowski integral inequality for $p > 1$, it means that $\|(\mathcal{F}_{s,3}f)(x)\|_{L_p(0,\infty)} \leq C_{(\lambda,\beta,p)} \|f(x)\|_{L_{p,v}(0,\infty)}$, where $L_p(0, \infty)$ is the classical Lebesgue space and $L_{p,v}(0, \infty)$ is the weighted Lebesgue space, with the following norm $\|f(x)\|_{L_{p,v}(0,\infty)} = \left(\int_0^\infty |f(x)v(x)|^p dx \right)^{\frac{1}{p}}$ and $C_{(\lambda,\beta,p)}$ is a positive constant depending only on λ, β and p .

Remark 2.4. For $\lambda = 1$ and $\beta = \frac{1}{2}$, we get a Pachpatte-type inequality.

Let

$$(\mathcal{F}_s^*f)(x) = \int_{r(x)}^\infty f(y)v(y)dy, \quad (\mathcal{C}_s^*f)(x) = \int_{r(x)}^\infty \frac{f(y)\phi(y)}{\Phi(y)} dy, \quad x > 0,$$

with

$$\begin{cases} 0 < r(x) < \infty & \text{for all } x \in (0, \infty), \\ r(0) = 0 & \text{and } r(\infty) = \infty. \end{cases} \quad (19)$$

By setting $h(x) = \infty$ and reasoning a manner analogous to the proof of Theorem 2.1, we get the following corollary.

Corollary 2.3 Let f, v, ϕ be non-negative measurable functions on $(0, \infty)$, $1 < p \leq q < \infty$. If $\alpha > p - 1$, then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_s^* f)^p(x) dx \leq \left(\frac{p}{\alpha - p + 1} \right)^p \left(\int_0^b v(x) dx \right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} |K^*(x)|^q dx \right)^{\frac{p}{q}}, \quad (20)$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s^* f)^p(x) dx \leq \left(\frac{p}{\alpha - p + 1} \right)^p \left(\int_0^b \phi(x) dx \right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} |J^*(x)|^q dx \right)^{\frac{p}{q}}, \quad (21)$$

where

$$K^*(x) = - \frac{V(x) [V(r(x))]' f(r(x))}{v(x)},$$

$$J^*(x) = - \frac{\Phi(x) [\Phi(r(x))]' f(r(x))}{\phi(x) \Phi(r(x))}.$$

Remark 2.5. The following particular case of Corollary 2.3 can be derived by taking $r(x) = x$ and $q = p$.

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\widetilde{\mathcal{F}}_s^* f)^p(x) dx \leq \left(\frac{p}{\alpha - p + 1} \right)^p \int_0^b \frac{v(x)}{V^{-\alpha}(x)} f^p(x) dx, \quad (22)$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\widetilde{\mathcal{C}}_s^* f)^p(x) dx \leq \left(\frac{p}{\alpha - p + 1} \right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} f^p(x) dx, \quad (23)$$

where

$$(\widetilde{\mathcal{F}}_s^* f)(x) = \int_x^\infty f(y) v(y) dy, \quad x > 0, \quad (\widetilde{\mathcal{C}}_s^* f)(x) = \int_x^\infty \frac{f(y) \phi(y)}{\Phi(y)} dy, \quad x > 0.$$

Remark 2.6 We note that if $v(x) = 1$ in the inequalities (15) and (22), we get the Hardy inequalities (1).

3. CONCLUSION

By using Hardy-Steklov and Copson-Steklov type operators and by introducing a second parameter of integrability q , some new integral inequalities were established and proved. These integral inequalities generalize certain classical inequalities like those of Hardy Copson and Pachpatte. As a perspective, we propose to extended these results to $\mathbb{R}^n$ or subsets of $\mathbb{R}^n$ for $n \geq 2$. Also it would of interest to try apply some of this integral inequalities in the study of deferent fields of mathematics (partial deferential equations, functional spaces, mathematical modeling, ...).

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