AMENABILITY OF $A \oplus_T X$ AS AN EXTENSION OF BANACH ALGEBRA

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DOI: 10.20948/mathmontis-2020-49-3

Summary. Let $A, X, \mathcal{U}$ be Banach algebras and A be a Banach $\mathcal{U}$ -bimodule also X be a Banach $A - \mathcal{U}$ -module. In this paper we study the relation between module amenability, weak module amenability and module approximate amenability of Banach algebra $A \oplus_T X$ and that of Banach algebras A, X . Where $T: A \times A \rightarrow X$ is a bounded bi-linear mapping with specific conditions.

1 INTRODUCTION

The notation of amenability of Banach algebras was introduced by B.Johnson in [9]. A Banach algebra A is amenable if every bounded derivation from A into any dual Banach A -bimodule is inner, equivalently if $H(A, X^*) = \{0\}$ for any Banach A -bimodule X , where $H(A, X^*)$ is the first Hochschild co- homology group of A with coefficient in X^* . Also, a Banach algebra A is weakly amenable if $H(A, A^*) = \{0\}$. Bade, Curtis and Dales introduced the notion of weak amenability on Banach algebras in [5]. They considered this concept only for commutative Banach algebras. After a while, Johnson defined the weak amenability for arbitrary Banach algebras [8].

For a morphism $T: B \rightarrow A$ from a Banach algebra B to a commutative Banach algebra A . The notion of module amenability of Banach algebras was introduced by Amini in [1]. Amini and Ebrahimi Bagha in [3] studied the concept of weak module amenability. In [10] the notation of module approximate amenability and contractibility as modules over of another Banach algebra was introduced for the notion of Banach algebras.

M. Sangani-Monfared in [11] defined a product on $A \times B$ and obtained the Banach algebra $A \times_\theta B$ using a character $\theta \in \sigma(B)$, for Banach algebras in a fairly general setting.

Later, S.J. Bhatt and P.A. Dabhi in [6] defined a product on $A \times B$ and obtained a Banach algebra $A \times_T B$ for a morphism $T: B \rightarrow A$ from a Banach algebra B to a commutative Banach algebra A .

The first and the second authors generalized all these constructions, and defined the module Lau product $A \times_\alpha B$ for Banach algebras A and B such that A is a Banach B -bimodule. They studied the ideal amenability of $A \times_\alpha B$ in [4].

T.Yazdan panah in [12] studied the concept of expanded modular of Banach algebra denoted by $A \oplus_T X$. He showed that $A \oplus_T X$ is amenable if and only if A is amenable and $X = \{0\}$. In this paper, we define a new Banach algebra different from of all above Banach algebras, named $A \oplus_T X$ in section 2. Then, some required basic properties of the following part are studied. In section 3, as the main section of paper, we study the relationship between module amenability of $A \oplus_T X$ and module amenability of A and X . We show that If $T(A, 0) = X$ and $A^2 = A$, then the module amenability of A implies module amenability of $A \oplus_T X$. Furthermore, it's

2010 Mathematics Subject Classification: 46H20, 46H25.

Key words and Phrases: Banach module, module amenability, weak module amenability, module approximate amenability, module derivation, Expanded Banach algebra.

conversly obtained that the module amenability of $A \oplus_T X$ implies module amenability of A and moreover if $T(A, 0) = X$, then X also is module amenable. In sections 4 and 5 respectively we study the relationship between weak module amenability (based as definition in [1] and [2]) and module approximte amenability of $A \oplus_T X$ and weak module amenability and module approximte amenability of A, X .

2 DEFINITIONS AND BASIC PROPERTIES

Throughout this paper it's assumed that $\mathcal{U}$ be a Banach algebra, A be a Banach $\mathcal{U}$ -bimodule and X be a Banach A - $\mathcal{U}$ -bimodule. Module actions are assumed as follow too:

$$\begin{aligned} A \times \mathcal{U} &\rightarrow A; (a, \alpha) \mapsto a \circ \alpha, \mathcal{U} \times A \rightarrow A; (\alpha, a) \mapsto \alpha \cdot a. \\ X \times \mathcal{U} &\rightarrow X; (x, \alpha) \mapsto x \Delta \alpha, \mathcal{U} \times X \rightarrow X; (\alpha, x) \mapsto \alpha \nabla x. \\ X \times A &\rightarrow X; (x, a) \mapsto x \circ a, A \times X \rightarrow X; (a, x) \mapsto a \cdot x. \end{aligned}$$

Consider the bounded bilinear map $T : A \times A \rightarrow X$, which has the following properties:

$$\begin{aligned} a \cdot T(a_1 a_2, 0) &= T(a a_1, 0) \circ a_2, T(a_1 a_2, 0) = T(a_1, 0) T(a_2, 0), \\ T(\alpha \cdot a, \alpha \nabla x) &= \alpha \cdot T(a, x), T(a \circ \alpha, x \Delta \alpha) = T(a, x) \cdot \alpha, \\ \|T(a, 0)\| &= \|a\|, \text{ for all } a, a_1, a_2 \in A, x \in X, \alpha \in \mathcal{U}. \end{aligned}$$

Module extension $A \oplus X$, with the product

$$(a, x)(a_1, x_1) = (a a_1, a \cdot x_1 + x \circ a_1 + T(a a_1, 0))$$

and the norm $\|(a, x)\| = \|a\| + \|x\|$ is a Banach algebra denoted by $A \oplus_T X$.

Definition 2.1 *The bounded map $D: A \rightarrow X^*$ with $D(a + b) = D(a) + D(b)$, $D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in A$, and $D(\alpha \cdot a) = \alpha \cdot D(a)$, $D(a \cdot \alpha) = D(a) \cdot \alpha$ ($\alpha \in \mathcal{U}$, $a \in A$), is called module derivation.*

Note that X^* is also Banach module over A and $\mathcal{U}$ with compatible actions under the canonical actions of A and $\mathcal{U}$, $\alpha \cdot (a \cdot f) = (\alpha \cdot a) \cdot f$, ($a \in A$, $\alpha \in \mathcal{U}$, $f \in X^*$), and the same for right action. Here the canonical actions of A and $\mathcal{U}$ on X^* are defined by $(\alpha \cdot f)(x) = f(x \Delta \alpha)$, $(a \cdot f)(x) = f(x \circ a)$, ($\alpha \in \mathcal{U}$, $a \in A$, $f \in X^*$, $x \in X$) and it's the same for right actions. As in [1] we call A -module X which have a compatible $\mathcal{U}$ -action as above, a $A - \mathcal{U}$ modules, above assertion is to say that if X is an $A - \mathcal{U}$ -module, then so is X^* . Also we use the notation $Z_{\mathcal{U}}(A, X^*)$ for the set of all module derivations $D: A \rightarrow X^*$, and $N_{\mathcal{U}}(A, X^*)$ for those which are inner and $H_{\mathcal{U}}(A, X^*)$ for the quotient group.

Proposition 2.2 *$A \oplus_T X$ is a Banach $\mathcal{U}$ -bimodule.*

Proof. Consider the module actions as follow:

$$\mathcal{U} \times (A \oplus_T X) \rightarrow A \oplus_T X; \alpha \cdot (a, x) = (\alpha \cdot a, \alpha \nabla x), \text{ and } (A \oplus_T X) \times \mathcal{U} \rightarrow A \oplus_T X; (a, x) \cdot \alpha = (\alpha \circ a, \alpha \Delta x). \text{ It is easy to check the satifaction of the properties. } \blacksquare$$

Proposition 2.3 *If Y is an A - $\mathcal{U}$ -module, then $Y \oplus \{0\}$ is a Banach $A \oplus_T X - \mathcal{U}$ -bimodule.*

Proof. Assume that the module actions on Y , are as follows:

$$\mathcal{U} \times Y \rightarrow Y; (\alpha, y) \mapsto \alpha \Delta y, Y \times \mathcal{U} \rightarrow Y; (y, \alpha) \mapsto y \cdot \alpha. \text{ And } A \times Y \rightarrow Y; (a, y) \mapsto a \cdot y, Y \times A \rightarrow Y; (y, a) \mapsto y \cdot a. \text{ Define the module actions as: } (Y \oplus \{0\}) \times \mathcal{U} \rightarrow Y \oplus$$

$\{0\}; (y, 0) \cdot \alpha = (y \cdot \alpha, 0), \mathcal{U} \times (Y \oplus \{0\}) \rightarrow Y \oplus \{0\}; \alpha \cdot (y, 0) = (\alpha \Delta y, 0)$. And $(A \oplus_T X) \times (Y \oplus \{0\}) \rightarrow Y \oplus \{0\}; (a, x) \cdot (y, 0) = (a \cdot y, 0), (Y \oplus \{0\}) \times (A \oplus_T X) \rightarrow Y \oplus \{0\}; (y, 0) \circ (a, x) = (y \cdot a, 0)$. We only need to show that the actions are compatible.

$$\begin{aligned} 1) \alpha \cdot ((a, x) \cdot (y, 0)) &= \alpha \cdot (a \cdot y, 0) \\ &= (\alpha \Delta (a \cdot y), 0) = ((\alpha \cdot a) \cdot y, 0) \\ &= ((\alpha \cdot a, \alpha \nabla x) \cdot (y, 0) = (\alpha \cdot (a, x)) \cdot (y, 0) . \end{aligned}$$

$$\begin{aligned} 2) ((a, x) \cdot (y, 0)) \cdot \alpha &= (a \cdot y, 0) \cdot \alpha \\ &= ((a \cdot y) \cdot \alpha, 0) = (a \cdot (y \cdot \alpha), 0) \\ &= (a, x) \cdot (y \cdot \alpha, 0) = (a, x) \cdot ((y, 0) \cdot \alpha) . \end{aligned}$$

$$\begin{aligned} 3) (\alpha \cdot (y, 0)) \cdot (a, x) &= (\alpha \Delta y, 0) \cdot (a, x) \\ &= ((\alpha \Delta y) \cdot a, 0) = (\alpha \Delta (y \cdot a), 0) \\ &= \alpha \cdot (y \cdot a, 0) = \alpha \cdot ((y, 0) \circ (a, x)) \end{aligned}$$

■

Proposition 2.4 Let $M \oplus N$ be a Banach $A \oplus_T X - \mathcal{U}$ -bimodule, then M is a Banach $A - \mathcal{U}$ -bimodule.

Proof. Consider the map $Q_M : M \oplus N \rightarrow M; (m, n) \mapsto m$ and define the module actions as: $M \times \mathcal{U} \rightarrow M; (m, n) \mapsto m \cdot \alpha = Q_M((m, 0) \cdot \alpha), \mathcal{U} \times M \rightarrow M; (\alpha, m) \mapsto \alpha \circ m = Q_M(\alpha \cdot (m, 0))$ $M \times A \rightarrow M; (m, a) \mapsto m \cdot a = Q_M((m, 0) \circ (a, 0))$ and $A \times M \rightarrow A; (a, m) \mapsto a \cdot m = Q_M((a, 0) \cdot (m, 0))$ ■

Proposition 2.5 Let M be a Banach $A - \mathcal{U}$ -module and N be a Banach $X - \mathcal{U}$ -bimodule, $M \oplus N$ is a Banach $A \oplus_T X - \mathcal{U}$ -bimodule.

Proof. Given module actions on $M \oplus N$ as follows:

$$\begin{aligned} (M \oplus N) \times \mathcal{U} \rightarrow M \oplus N; (m, n) \cdot \alpha &= (m \cdot \alpha, n \nabla \alpha), \mathcal{U} \times (M \oplus N) \rightarrow M \oplus N; \alpha \cdot (m, n) = (\alpha \cdot m, \alpha \Delta m), \\ (M \oplus N) \times (A \oplus_T X) \rightarrow (M \oplus N); (m, n) \cdot (a, x) &= (m \cdot a, n \cdot T(a, 0)) \\ (A \oplus_T X) \times (M \oplus N) \rightarrow M \oplus N; (a, x) \cdot (m, n) &= (a \cdot m, T(a, 0) \odot n) . \end{aligned}$$

■

Proposition 2.6 For each $(f, g) \in M^* \oplus N^*, (a, x) \in A \oplus_T X, (m, n) \in M \oplus N$ we have $(f, g) \cdot (a, x) = (f \cdot a, g \cdot T(a, 0))$ and $(a, x) \cdot (f, g) = (a \cdot f, T(a, 0) \odot g)$.

Proof.

$$\begin{aligned} \langle (f, g) \cdot (a, x), (m, n) \rangle &= \langle (f, g), (a, x) \cdot (m, n) \rangle \\ &= \langle (f, g), (a \cdot m, T(a, 0) \odot n) \rangle \\ &= \langle f, a \cdot m \rangle + \langle g, T(a, 0) \odot n \rangle \\ &= \langle f \cdot a, m \rangle + \langle g \cdot T(a, 0), n \rangle \\ &= \langle (f \cdot a, g \cdot T(a, 0)), (m, n) \rangle \end{aligned}$$

■

Proposition 2.7 If N is a Banach $X - \mathcal{U}$ -bimodule, then N is a Banach $A - \mathcal{U}$ -bimodule.

Proof. The module actions are defined as follow:

$$A \times N \rightarrow N; a \cdot n = T(a, 0) \odot n \text{ and } N \times A \rightarrow N; n \cdot a = n \cdot T(a, 0) . \quad \blacksquare$$

3 MODULE AMENABILITY

Lemma 3.1 $D \in Z_{\mathcal{U}}(A \oplus_T X, M^* \oplus N^*)$ if and only if there are $D_1 \in Z_{\mathcal{U}}(A, M^*)$, $D_3 \in Z_{\mathcal{U}}(X, N^*)$, $R \in Z_{\mathcal{U}}(A, N^*)$ and linear map $D_2 : X \rightarrow M^*$ such that

- 1) $D(a, x) = (D_1(a) + D_2(x), R(a) + D_3(x))$,
- 2) $D_2(a \cdot x) = a \cdot D_2(x)$,
- 3) $D_2(x \circ a) = D_2(x) \cdot a$,
- 4) $R(bd) = R(b) \cdot T(d, 0) + T(b, 0) \cdot R(d) = R(b) \cdot d + b \cdot R(d)$,
- 5) $D_2(T(ab, 0)) = 0$,
- 6) $D_3(a \cdot x) = T(a, 0) \cdot D_3(x)$,
- 7) $D_3(x \circ a) = D_3(x) \cdot T(a, 0)$,
- 8) $D_3(T(ab, 0)) = 0$.

Proof. Suppose that $D \in Z_{\mathcal{U}}(A \oplus_T X, M^* \oplus N^*)$ then there are $d_1 : A \oplus \tau X \rightarrow M^*$, $d_2 : A \oplus_T X \rightarrow N^*$ such that $D = (d_1, d_2)$, Set

$$D_1 : A \rightarrow M^*; D_1(a) = d_1(a, 0),$$

$$D_2 : X \rightarrow N^*; D_2(x) = d_1(0, x),$$

$$D_3 : X \rightarrow N^*; D_3(x) = d_2(0, x), R : A \rightarrow N^*; R(a) = d_2(a, 0).$$

Now

$$\begin{aligned} D(a, x) &= (d_1, d_2)((a, 0) + (0, x)) = (d_1, d_2)(a, 0) + (d_1, d_2)(0, x) \\ &= (d_1(a, 0), d_2(a, 0)) + ((d_1(0, x), d_2(0, x))) \\ &= (d_1(a, 0) + d_1(0, x), d_2(a, 0) + d_2(0, x)) \\ &= (D_1(a) + D_2(x), R(a) + D_3(x)), \end{aligned} \quad (1)$$

Now

$$\begin{aligned} D((a, x)(m, x')) &= D(am, a \cdot x' + x \circ m + T(am, 0)) \\ &= (D_1(am) + D_2(a \cdot x') + D_2(x \circ m) + D_2(T(am, 0)), R(am) + D_3(a \cdot x') \\ &\quad + D_3(x \circ m) + D_3(T(am, 0))), \end{aligned} \quad (2)$$

since D is module derivation so

$$\begin{aligned} D((a, x)(m, x')) &= D(a, x) \cdot (m, x') + (a, x) \cdot D(m, x') \\ &= (D_1(a) + D_2(x), R(a) + D_3(x)) \cdot (m, x') \\ &\quad + (a, x) \cdot (D_1(m) + D_2(x), R(m) + D_3(x')) \\ &= ((D_1(a) \cdot m + D_2(x)) \cdot m + a \cdot D_2(x') + D_2(x) \cdot m, R(a) \cdot T(m, 0) \\ &\quad + T(a, 0) \cdot R(m) + D_3(x) \cdot T(m, 0) + T(a, 0) \cdot D_3(x')). \end{aligned} \quad (3)$$

In 3, 2 Take $x = x' = 0$ to get $D_1 \in Z_{\mathcal{U}}(A, M^*)$, (5), (4) and (8). Take $a = 0$ to get (3) and (6). Take $m = 0$ to get (2), (7). And in a similar way we can get other parameters. Conversely is in a same way. ■

Corollary 3.2 Let $X = \{0\}$ and D, D_1 and R be as in pervious lemma, then $D = \delta_{(f,g)}$ if and only if $D_1 = \delta_f$ and $g = \bar{\delta}_g$. Where $\bar{\delta}_g(a) = gT(a, 0) - T(a, 0) \cdot g$.

Proof. Since $X = \{0\}$ and $D(a, x) = (D_1(a) + D_2(x), R(a) + D(x))$ so $D(a, 0) = (D_1(a), R(a))$. If $D = \delta_{(f,g)}$ then

$$\begin{aligned} D(a, 0) &= \delta_{(f,g)}(a, 0) \\ &= (f, g) \cdot (a, 0) - (a, 0) \cdot (f, g) \\ &= (f \cdot a, g \cdot T(a, 0)) - (a \cdot f, T(a, 0) \cdot g) \\ &= (f \cdot a - a \cdot f, g \cdot T(a, 0) - T(a, 0) \cdot g) = (\delta_f(a), \bar{\delta}_g(a)). \end{aligned}$$

So $D_1 = \delta_f$ and $R = \bar{\delta}_g$. Conversely

$$\begin{aligned} D(a, 0) &= (D_1(a), R(a)) \\ &= (\delta_f(a), \bar{\delta}_g(a)) \\ &= (f \cdot a - a \cdot f, g \cdot T(a, 0) - T(a, 0) \cdot g) \\ &= (f, g) \cdot (a, 0) - (a, 0) \cdot (f, g) \\ &= \delta_{(f,g)}(a, 0). \end{aligned}$$

■

Theorem 3.3 *The module amenability of $A \oplus_T X$ implies module amenability of A . Moreover if $T(A, 0) = X$, then X is also module amenable.*

Proof. Assume that M, N are Banach $A - \mathfrak{U}$ -bimodule and Banach $X - \mathfrak{U}$ -bimodule respectively. Let $D_1: A \rightarrow M^*$ and $D_2: X \rightarrow N^*$ be module derivations. By Proposition 2.5, $M \oplus N$ is a Banach $A \oplus_T X - \mathfrak{U}$ -bimodule. Define $D' : A \oplus_T X \rightarrow M^* \oplus N^*$; $D'(a, x) = (D_1(a), D_2(T(a, 0)))$. Now

$$\begin{aligned} D'((a, x)(m, x')) &= D'(am, a \cdot x' + x \circ m + T(am, 0)) \\ &= (D_1(am), D_2(T(am, 0))) \\ &= (D_1(a) \cdot m + a \cdot D_1(m), D_2(T(a, 0)) \cdot T(m, 0) + T(a, 0) \cdot D_2(T(m, 0))) \\ &= (D_1(a), D_2(T(a, 0))) \cdot (m, x') + (a, x) \cdot (D_1(m), D_2(T(m, 0))) \\ &= D'(a, x) \cdot (m, x') + (a, x) \cdot D'(m, x'). \end{aligned}$$

Also

$$\begin{aligned} D'(\alpha \cdot (a, x)) &= D'(\alpha \cdot a, \alpha \nabla x) \\ &= ((D_1(\alpha \cdot a), D_2(T(\alpha \cdot a, 0)))) \\ &= ((\alpha \cdot D_1(a), \alpha \cdot D_2(T(a, 0)))) \\ &= (\alpha \cdot D'(a, x)). \end{aligned}$$

And

$$\begin{aligned} D'((a, x) + (m, x')) &= D'((a + m, x + x')) \\ &= (D_1(a + m), D_2(T(a + m, 0))) \\ &= (D_1(a) + D_1(m), D_2(T(a, 0)) + D_2(T(m, 0))) \\ &= (D_1(a), D_2(T(a, 0))) + (D_1(m), D_2(T(m, 0))) \\ &= D'(a, x) + D'(m, x'). \end{aligned}$$

So D' is a module derivation. Since $A \oplus_T X$ is module amenable, there exists $(f, g) \in M^* \oplus N^*$ such that $D' = \delta_{(f,g)}$. Thus

$$\begin{aligned} D'(a, x) &= \delta_{(f,g)}(a, x) \\ &= (f, g) \cdot (a, x) - (a, x) \cdot (f, g) \\ &= (f \cdot a + g \cdot T(a, 0)) - (a \cdot f, T(a, 0) \cdot g) \\ &= (f \cdot a - a \cdot f, g \cdot T(a, 0) - T(a, 0) \cdot g). \end{aligned}$$

Consequently $D_1(a) = f \cdot a - a \cdot f$ i.e. $D_1 = \delta_f$ and $D_2(T(a, 0)) = \delta_g(T(a, 0))$. Since $T(A, 0) = X$, $D_2(x) = \delta_g(x)$ for all $x \in X$. ■

Theorem 3.4 *The module amenability of A implies module amenability of $A \oplus_T \{0\}$.*

Proof. Let $M \oplus N$ be a Banach $A \oplus_T X - \mathfrak{U}$ -bimodule and $D: A \oplus_T X \rightarrow M^* \oplus N^*$ be a module derivation. By lemma 3.1, there are D_1, D_2, D_3 and R such that $D(a, x) = (D_1(a) + D_2(x), R(a) + D_3(x))$. Since here $X = \{0\}$ so $D(a, 0) = (D_1(a), R(a))$. Module amenability of A implies there exist $f \in M^*$ such that $D_1 = \delta_f$ and since N^* is an A - $\mathfrak{U}$ -bimodule and R is module derivation, there exist $g \in N^*$ such that $R = \delta_g$. Thus $D = \delta_{(f,g)}$. ■

Theorem 3.5 *If $T(A, 0) = X$ and $A^2 = A$, then the module amenability of A implies module amenability of $A \oplus_T X$.*

Proof. Let $M \oplus N$ be a Banach $A \oplus_T X - \mathfrak{U}$ -bimodule and $D: A \oplus_T X \rightarrow M^* \oplus N^*$ be a module derivation. By lemma 3.1, there are D_1, D_2, D_3 and R such that $D(a, x) = (D_1(a) + D_2(x), R(a) + D_3(x))$. Since here $T(A, 0) = X$ and $A^2 = A$ so $D(a, x) = (D_1(a), R(a))$. Module amenability of A implies there exist $f \in M^*$ such that $D_1 = \delta_f$ and since N^* is an $A - \mathfrak{U}$ -bimodule and R is module derivation, there exist $g \in N^*$ such that $R = \delta_g$. Thus $D = \delta_{(f,g)}$. ■

Example 3.6 *Let $\mathbb{N}$ be the set of positive integers. Consider $S = (\mathbb{N}, \vee)$ with the maximum operation $m \vee n = \max\{m, n\}$, then S is a amenable countable, abelian inverse semigroup with the identity 1. Clearly $E_S = S$. This semigroup is denoted by $\mathbb{N}_\vee$. $l^1(\mathbb{N}_\vee)$ is unital with unit δ_1 . Since $\mathbb{N}_\vee$ is amenable and $l^1(\mathbb{N}_\vee)$ is unital so $l^1(\mathbb{N}_\vee)$ is module amenable (as an $l^1(\mathbb{N}_\vee) - l^1(\mathbb{N}_\vee)$ -bimodule. Define $T: l^1(S) \times l^1(S) \rightarrow l^1(S)$; $T(\delta_x, \delta_y) = \begin{cases} \delta_x, & \delta_y = 0 \\ \delta_{x \vee y}, & \delta_y \neq 0. \end{cases}$ Then $l^1(\mathbb{N}_\vee) \oplus_T l^1(\mathbb{N}_\vee)$ is module amenable.*

4 WEAK MODULE AMENABILITY

The Banach algebra A is called weak module amenable (as an $\mathfrak{U}$ -bimodule), if $H_{\mathfrak{U}}(A, X) = \{0\}$, where X is a commutative $\mathfrak{U}$ -submodule of A^* ([2]).

Theorem 4.1 *The weak module amenability of $A \oplus_T X$ implies weak module amenability of A . In addition if $T(A, 0) = X$ then X is also weak module amenable.*

Proof. Assume that M, N are commutative $\mathfrak{U}$ -submodule of A^* and X^* , respectively. we can show that $M \oplus N$ is a commutative $\mathfrak{U}$ -submodule of $(A \oplus_T X)^*$. Let $D_1 \in Z_{\mathfrak{U}}(A, M)$ and $D_2 \in Z_{\mathfrak{U}}(X, N)$. Define $D: A \oplus_T X \rightarrow M \oplus N$; $D(a, x) = (D_1(a), D_2(T(a, 0)))$, it is easy to see that $D \in Z_{\mathfrak{U}}(A \oplus_T X, M \oplus N)$. Since $A \oplus_T X$ is weak module amenable there is $(f, g) \in M \oplus N$ such that $D = \delta_{(f,g)}$ and

$$\begin{aligned} (D_1(a), D_2(T(a, 0))) &= D(a, x) \\ &= \delta_{(f,g)}(a, x) \end{aligned}$$

$$\begin{aligned}
&= (f, g) \cdot (a, x) - (a, x) \cdot (f, g) \\
&= (f \cdot a - a \cdot f, g \cdot T(a, 0) - T(a, 0) \cdot g) \\
&= (\delta_f(a), \delta_g(T(a, 0)))
\end{aligned}$$

Hence A, X are weak module amenable. ■

Theorem 4.2 *The weak module amenability of A implies the weak module amenability of $A \oplus_T \{0\}$.*

Proof. Suppose that $M \oplus N$ is a commutative Banach $\mathfrak{U}$ -submodule of $(A \oplus \tau\{0\})^*$, and $D \in Z_{\mathfrak{U}}(A \oplus \tau\{0\}, M \oplus N)$. Then M and N are commutative $\mathfrak{U}$ -submodule of A^* . Since $D \in Z_{\mathfrak{U}}(A \oplus \tau\{0\}, M \oplus N)$, by lemma 3.1 there are $D_1 \in Z_{\mathfrak{U}}(A, M)$, and $R \in Z_{\mathfrak{U}}(A, N)$, such that $D(a, 0) = (D_1(a), R(a))$. Since A is weak module amenable so there are $m \in M$ and $n \in N$ such that $D_1 = \delta_m, R = \delta_n$, where $\delta_n(a) = a \cdot n - n \cdot a = T(a, 0) \odot n - n \cdot T(a, 0)$.

Now

$$\begin{aligned}
D(a, x) &= (D_1(a), R(a)) \\
&= (\delta_m(a), \delta_n(a)) \\
&= (a \cdot m - a \cdot m, T(a, 0) \odot n - n \cdot T(a, 0)) \\
&= (a, 0) \cdot (m, n) - (m, n) \cdot (a, 0) \\
&= \delta_{(m,n)}(a, 0).
\end{aligned}$$

■

Theorem 4.3 *If $T(A, 0) = X$ and $A^2 = A$, then the weak module amenability of A implies the weak module amenability of $A \oplus_T X$.*

Proof. The proof is as above theorem. ■

Example 4.4 *Let $S = \mathbb{N}_V$ be as in Example 3.6, since $l^1(S)$ is $l^1(S) - l^1(S)$ -module and $l^1(S)$ is weak module amenable. Let $T: l^1(S) \times l^1(S) \rightarrow l^1(S)$ have the properties as above theorems, then $l^1(S) \oplus_T l^1(S)$ is weak module amenable.*

5 MODULE APPROXIMATE AMENABILITY

Let A be as above, then A is module approximately amenable (as an $\mathfrak{U}$ -bimodule), if for any commutative Banach $A - \mathfrak{U}$ -bimodule X , each module derivation $D: A \rightarrow X^*$ is approximately inner.

A derivation $D: A \rightarrow X$ is said to be approximately inner if there exists a net $(x_i)_i \subseteq X$ such that $D(a) = \lim_i (a \cdot x_i - x_i \cdot a), a \in A. ([10])$.

Lemma 5.1 *Let D_1, R, D_3 and D_2 are such as in the Lemma 3.1, and $D(a, b) = (D_1(a) + D_2(b), R(a) + D_3(b))$. If $T(A, 0) = X$ and $A^2 = A$ then: D is approximately inner if and only if D_1 and R are approximately inner.*

Proof. Assume that M is a commutative $A - \mathfrak{U}$ -bimodule and also N is commutative $X - \mathfrak{U}$ -bimodule, then $M \oplus N$ is a commutative $A \oplus_T X - \mathfrak{U}$ -bimodule. Let D be approximately inner so there is $(f_i, g_i)_i \subseteq M^* \oplus N^*$ such that

$$\begin{aligned}
D(a, x) &= T(a', 0) \\
&= \lim_i ((a, x) \cdot (f_i, g_i) - (f_i, g_i) \cdot (a, x)) \\
&= \lim_i ((a \cdot f_i, T(a, 0) \cdot g_i) - (f_i \cdot a, g_i \cdot T(a, 0))) \\
&= \lim_i (a \cdot f_i - f_i \cdot a, T(a, 0) \cdot g_i - g_i \cdot T(a, 0)),
\end{aligned}$$

i.e. $D(a) = \lim_i (a \cdot f_i - f_i \cdot a)$ and $R(a) = \lim_i (T(a, 0) \cdot g_i - g_i \cdot T(a, 0))$.

Conversely, let $D_1(a) = \lim_{i \in I} (a \cdot f_i - f_i \cdot a)$ and $R(a) = \lim_{j \in J} (T(a, 0) \cdot g_j - g_j \cdot T(a, 0))$

Since the index sets $(I, \leq), (J, \leq)$ are ordered sets, so the set $\Lambda = I \times J = \{(i, j) : i \in I, j \in J\}$ is ordered as follows

$$(i, j) \leq (i', j') \Leftrightarrow (i \leq i', j \leq j').$$

For $\lambda = (i, j) \in \Lambda$ set $t_\lambda = (f_i, g_j)$. Let $\epsilon > 0$ be given. Since $D_1(a) = \lim_i (a \cdot f_i - f_i \cdot a)$ and $R(a) = \lim_j (T(a, 0) \cdot g_j - g_j \cdot T(a, 0))$ there are $i_0 \in I, j_0 \in J$ such that

$$1) \text{ For all } i \geq i_0, \|D_1(a) - (a \cdot f_i - f_i \cdot a)\| \leq \frac{\epsilon}{3}.$$

$$2) \text{ For all } j \geq j_0, \|R(a) - (T(a, 0) \cdot g_j - g_j \cdot T(a, 0))\| \leq \frac{\epsilon}{3}.$$

Now set $\lambda_0 = (i_0, j_0)$, then for all $\lambda \geq \lambda_0$, since $D(a, T(a', 0)) = (D_1(a), R(a))$, we have

$$\begin{aligned}
\|D(a, x) - ((a, x) \cdot t_\lambda - t_\lambda \cdot (a, x))\| &= \|D(a, x) - ((a, x) \cdot (f_i, g_j) - (f_i, g_j) \cdot (a, x))\| \\
&= \|D(a, x) - (a \cdot f_i - f_i \cdot a, T(a, 0) \cdot g_j - g_j \cdot T(a, 0))\| \\
&= \|(D_1(a), R(a)) - (a \cdot f_i - f_i \cdot a, T(a, 0) \cdot g_j - g_j \cdot T(a, 0))\| \\
&= \|((D_1(a) - (a \cdot f_i - f_i \cdot a)), R(a) - (T(a, 0) \cdot g_j - g_j \cdot T(a, 0)))\| \\
&\leq \|D_1(a) - (a \cdot f_i - f_i \cdot a)\| + \|R(a) - (T(a, 0) \cdot g_j - g_j \cdot T(a, 0))\| \\
&< \epsilon.
\end{aligned}$$

Hence $D(a, x) = \lim_\lambda ((a, x) \cdot t_\lambda - t_\lambda \cdot (a, x))$ where $x = T(a', 0)$ i.e. D is approximately inner. ■

Theorem 5.2 *If $A \oplus \tau X$ is module approximately amenable then A is module approximately amenable. Furthermore, if $T(A, 0) = X$ also X is module approximately amenable.*

Proof. In an argument as in the proof of Theorem 3.3 and the application, the usage of above lemma. ■

Theorem 5.3 *If $T(A, 0) = X$ and $A^2 = A$ then the module approximate amenability of A implies the module approximate amenability of $A \oplus_T X$.*

Proof. Let $M \oplus N$ be a commutative $A \oplus_T X - \mathcal{U}$ -bimodule and $D \in Z_{\mathcal{U}}(A \oplus \tau X, M^* \oplus N^*)$. There are $D_1 \in Z_{\mathcal{U}}(A, M^*), D_3 \in Z_{\mathcal{U}}(X, N^*), R \in Z_{\mathcal{U}}(A, N^*)$ and $D_2 : X \rightarrow N^*$ such that $D(a, x) = (D_1(a) + D_2(x), R(a) + D(x))$ and since $T(A, 0) = X$ and $A^2 = A$ we have $D(a, x) = (D_1(a), R(a))$. Since A, X are module approximate amenable, so D_1 and R are approximately inner. Thus by the above lemma, D is approximately inner. ■

Example 5.4 Let S be an amenable inverse semigroup such that the set of idempotents E_S be equal to S and $l^1(S)$ has approximately unit. Since S is amenable, $l^1(S)$ is module approximately amenable, [10]. Also $l^1(S)$ is $l^1(S) - l^1(S)$ -bimodule, thus $l^1(S) \oplus_T l^1(S)$ is module approximately amenable. Where $T: l^1(S) \times l^1(S) \rightarrow l^1(S)$ is defined by

$$T(\delta_x, \delta_y) = \begin{cases} \delta_x, & \delta_y = 0 \\ \delta_{xy}, & \delta_y \neq 0. \end{cases}$$

6 CONCLUSIONS

The module amenability of $A \oplus_T X$ implies module amenability of A and The module amenability of A implies module amenability of $A \oplus_T \{0\}$. Also If $T(A, 0) = X$ and $A^2 = A$, then the module amenability of A implies module amenability of $A \oplus_T X$. meanwhile, The weak module amenability of $A \oplus_T X$ implies weak module amenability of A . On the contrary, if $T(A, 0) = X$ and $A^2 = A$, then the weak module amenability of A implies the weak module amenability of $A \oplus_T X$.

Considering approximately, if $A \oplus \tau X$ is module approximately amenable then A is module approximately amenable. On the contrary, if $T(A, 0) = X$ and $A^2 = A$ then the module approximate amenability of A implies the module approximate amenability of $\oplus_T X$. For example, we have S be an amenable inverse semigroup such that the set of idempotents E_S be equal to S and $l^1(S)$ has approximately unit. Since S is amenable, $l^1(S)$ is module approximately amenable.

Acknowledgements: The authors would like to thank the referees for their careful reading and constructive comments which improved the paper.

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Received March 16, 2020