STUDY ON LA-RING BY THEIR INTUITIONISTIC FUZZY IDEALS

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Summary. In this paper, we extend the characterizations of Kuroki [17], by initiating the concept of intuitionistic fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals in a class of non-associative and non-commutative rings (LA-ring). We characterize regular (intra-regular, both regular and intra-regular) LA-rings in terms such ideals.

1 INTRODUCTION

In ternary operations, the commutative law is given by $abc = cba$. Kazim and Naseerudin [7], have generalized this notion by introducing the parenthesis on the left side of this equation to get a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is called the left invertive law. A groupoid S is called a left almost semigroup (abbreviated as LA-semi-group) if it satisfies the left invertive law. An LA-semi-group is a midway structure between a commutative semigroup and a groupoid.

A groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). An LA-semi-group is medial, but in general an LA-semi-group needs not to be paramedial. Every LA-semi-group with left identity is paramedial and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Kamran [16], extended the notion of LA-semi-group to the left almost group (LA-group). An LA-semi-group G is called a left almost group, if there exists a left identity $e \in G$ such that $ea = a$ for all $a \in G$ and for every $a \in G$ there exists $b \in G$ such that $ba = e$.

Shah et al. [22], by a left almost ring, mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semi-group, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all $a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Although the structure is non-associative and non-commutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

A non-empty subset A of R is called an LA-subring of R if $a - b \in A$ and $ab \in A$ for all

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$a, b \in A$. A is called a left (resp. right) ideal of R if $(A, +)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R .

A non-empty subset A of R is called an interior ideal of R if $(A, +)$ is an LA-group and $(RA)R \subseteq A$. A non-empty subset A of R is called a quasi-ideal of R if $(A, +)$ is an LA-group and $AR \cap RA \subseteq A$. An LA-subring A of R is called a bi-ideal of R if $(AR)A \subseteq A$. A non-empty subset A of R is called a generalized bi-ideal of R if $(A, +)$ is an LA-group and $(AR)A \subseteq A$.

We will define the concept of intuitionistic fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an LA-ring R . We will establish a study by discussing the different properties of such ideals. We will characterize regular (resp. intra-regular, both regular and intra-regular) LA-rings by the properties of intuitionistic fuzzy (left, right, quasi-, bi-, generalized bi-) ideals such ideals.

2 INTUITIONISTIC FUZZY IDEALS IN LA-RINGS

After, the introduction of fuzzy set by Zadeh [24], several researchers explored on the generalization of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [1], as a generalization of the notion of fuzzy set. Liu [18], introduced the concept of fuzzy subrings and fuzzy ideals of a ring. Many authors have explored the theory of fuzzy rings (for example [3, 9, 11-15, 18, 19-20, 23]). Gupta and Kantroo [4], gave the idea of intrinsic product of fuzzy subsets of a ring. Kuroki [17], characterized regular (intra-regular, both regular and intra-regular) rings in terms of fuzzy left (right, quasi, bi-) ideals.

An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where the functions $\mu_A : X \rightarrow [0, 1]$ and $\gamma_A : X \rightarrow [0, 1]$ denote the degree of membership and the degree of non-membership, respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$ [1].

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ in X can be identified to be an ordered pair (μ_A, γ_A) in $I^X \times I^X$, where I^X is the set of all functions from X to $[0, 1]$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

Banerjee and Basnet [2] and Hur et al. [6], initiated the notion of intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring. Subsequently many authors studied the intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring by describing the different properties (see [5]). Shah et al. [21, 22] initiated the concept of intuitionistic fuzzy normal subrings over a non-associative ring and also characterized the non-associative rings by their intuitionistic fuzzy bi-ideals in [8]. Kausar [10] explored the notion of direct product of finite intuitionistic anti fuzzy normal subrings over non-associative rings.

We initiate the notion of intuitionistic fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an LA-ring R .

An intuitionistic fuzzy set (IFS) $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy LA-subring of R if

- (1) $\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\}$,
- (2) $\gamma_A(x - y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,
- (3) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$,
- (4) $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$, for all $x, y \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy left ideal of R if

- (1) $\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\}$,
- (2) $\gamma_A(x - y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,
- (3) $\mu_A(xy) \geq \mu_A(y)$,
- (4) $\gamma_A(xy) \leq \gamma_A(y)$, for all $x, y \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy right ideal of R if

- (1) $\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\}$,
- (2) $\gamma_A(x - y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,
- (3) $\mu_A(xy) \geq \mu_A(x)$,
- (4) $\gamma_A(xy) \leq \gamma_A(x)$, for all $x, y \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of R is called an intuitionistic fuzzy ideal of an LA-ring R if it is both an intuitionistic fuzzy left ideal and an intuitionistic fuzzy right ideal of R .

Let A be a non-empty subset of an LA-ring R . Then the intuitionistic characteristic of A is denoted by $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ and defined by

$$\mu_{\chi_A}(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \text{ and } \gamma_{\chi_A}(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$

We note that an LA-ring R can be considered an intuitionistic fuzzy set of itself and we write $R = I_R$, i.e., $R(x) = (\mu_R, \gamma_R) = (1, 0)$ for all $x \in R$.

Let A and B be two intuitionistic fuzzy sets of an LA-ring R . Then

- (1) $A \subseteq B \Leftrightarrow \mu_A \subseteq \mu_B \text{ and } \gamma_A \supseteq \gamma_B$,
- (2) $A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$,
- (3) $A^c = (\gamma_A, \mu_A)$,

$$(4) A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B) = (\mu_{A \wedge B}, \gamma_{A \vee B}),$$

$$(5) A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B) = (\mu_{A \vee B}, \gamma_{A \wedge B}),$$

$$(6) 0 \approx (0,1), 1 \approx (1,0).$$

The product of $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ is denoted by $A \circ B = (\mu_A \circ \mu_B, \gamma_A \circ \gamma_B)$ and defined as:

$$(\mu_A \circ \mu_B)(x) = \begin{cases} \bigvee_{x=\sum_{i=1}^n a_i b_i} \{\bigwedge_{i=1}^n \{\mu_A(a_i) \wedge \mu_B(b_i)\}\} & \text{if } x = \sum_{i=1}^n a_i b_i, a_i, b_i \in R \\ 0 & \text{if } x \neq \sum_{i=1}^n a_i b_i \end{cases}$$

$$\text{and } (\gamma_A \circ \gamma_B)(x) = \begin{cases} \bigwedge_{x=\sum_{i=1}^n a_i b_i} \{\bigvee_{i=1}^n \{\gamma_A(a_i) \vee \gamma_B(b_i)\}\} & \text{if } x = \sum_{i=1}^n a_i b_i, a_i, b_i \in R \\ 1 & \text{if } x \neq \sum_{i=1}^n a_i b_i \end{cases}$$

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy interior ideal of R if

- (1) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$,
- (2) $\gamma_A(x-y) \leq \gamma_A(x) \vee \gamma_A(y)$,
- (3) $\mu_A((xy)z) \geq \mu_A(y)$,
- (4) $\gamma_A((xy)z) \leq \gamma_A(y)$, for all $x, y, z \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy quasi-ideal of R if

- (1) $(\mu_A \circ R) \cap (R \circ \mu_A) \subseteq \mu_A$,
- (2) $(\gamma_A \circ R) \cup (R \circ \gamma_A) \supseteq \gamma_A$,
- (3) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$,
- (4) $\gamma_A(x-y) \leq \gamma_A(x) \vee \gamma_A(y)$, for all $x, y \in R$.

An Intuitionistic fuzzy LA-subring $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy bi-ideal of R if

- (1) $\mu_A((xy)z) \geq \mu_A(x) \wedge \mu_A(z)$,
- (2) $\gamma_A((xy)z) \leq \gamma_A(x) \vee \gamma_A(z)$, for all $x, y, z \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy generalized bi-ideal of R if

- (1) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$,

- (2) $\gamma_A(x - y) \leq \gamma_A(x) \vee \gamma_A(y)$,
- (3) $\mu_A((xy)z) \geq \mu_A(x) \wedge \mu_A(z)$,
- (4) $\gamma_A((xy)z) \leq \gamma_A(x) \vee \gamma_A(z)$, for all $x, y, z \in R$.

An intuitionistic fuzzy ideal $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy idempotent if $\mu_A \circ \mu_A = \mu_A$ and $\gamma_A \circ \gamma_A = \gamma_A$.

Now we give some imperative properties of such ideals of an LA-ring R , which will be very helpful in later sections.

Lemma 2.1: Let R be an LA-ring. Then the following properties hold:

- (1) $(A \circ B) \circ C = (C \circ B) \circ A$,
- (2) $(A \circ B) \circ (C \circ D) = (A \circ C) \circ (B \circ D)$,
- (3) $A \circ (B \circ C) = B \circ (A \circ C)$,
- (4) $(A \circ B) \circ (C \circ D) = (D \circ B) \circ (C \circ A)$,
- (5) $(A \circ B) \circ (C \circ D) = (D \circ C) \circ (B \circ A)$, for all intuitionistic fuzzy sets A, B, C and D of R .

Proof: Obvious.

Theorem 2.2: Let A and B be two non-empty subsets of an LA-ring R . then the following properties hold:

- (1) If $A \subseteq B$ then $\chi_A \subseteq \chi_B$.
- (2) $\chi_A \circ \chi_B = \chi_{AB}$.

$$(4) \quad \chi_A \cap \chi_B = \chi_{A \cap B}.$$

Proof: (1) Suppose that $A \subseteq B$ and $a \in R$. If $a \in A$, this implies that $a \in B$. Thus $\mu_{\chi_A}(a) = 1 = \mu_{\chi_B}(a)$ and $\gamma_{\chi_A}(a) = 0 = \gamma_{\chi_B}(a)$, i.e., $\chi_A \subseteq \chi_B$.

If $a \notin A$, and $a \notin B$. Thus $\mu_{\chi_A}(a) = 0 = \mu_{\chi_B}(a)$ and $\gamma_{\chi_A}(a) = 1 = \gamma_{\chi_B}(a)$, i.e., $\chi_A \subseteq \chi_B$.

If $\alpha \notin A$ and $\alpha \in B$. Thus $\mu_{\chi_A}(\alpha) = 0$ and $\mu_{\chi_B}(\alpha) = 1$ and $\gamma_{\chi_A}(\alpha) = 1$ and $\gamma_{\chi_B}(\alpha) = 0$, i.e.,

(2) Let $x \in R$ and $x \in AB$. This means that $x = ab$ for some $a \in A$ and $b \in B$.

Now

$$\begin{aligned} (\mu_{\chi_A} \circ \mu_{\chi_B})(x) &= \vee_{x = \sum_{i=1}^n a_i b_i} \{ \wedge_{i=1}^n \{ \mu_{\chi_A}(a_i) \wedge \mu_{\chi_B}(b_i) \} \} \\ &\geq \mu_{\chi_A}(a) \wedge \mu_{\chi_B}(b) = 1 \wedge 1 = 1 = \mu_{\chi_{AB}}(x) \\ \text{and } (\gamma_{\chi_A} \circ \gamma_{\chi_B})(x) &= \wedge_{x = \sum_{i=1}^n a_i b_i} \{ \vee_{i=1}^n \{ \gamma_{\chi_A}(a_i) \vee \gamma_{\chi_B}(b_i) \} \} \\ &\leq \gamma_{\chi_A}(a) \vee \gamma_{\chi_B}(b) = 0 \vee 0 = 0 = \gamma_{\chi_{AB}}(x). \end{aligned}$$

If $x \notin AB$, i.e., $x \neq ab$ for all $a \in A$ and $b \in B$. Then there are two cases.

(i) If $x = uv$ for some $u, v \in R$, then

$$\begin{aligned} (\mu_{\chi_A} \circ \mu_{\chi_B})(x) &= \bigvee_{x = \sum_{i=1}^n a_i b_i} \{ \bigwedge_{i=1}^n \{ \mu_{\chi_A}(a_i) \wedge \mu_{\chi_B}(b_i) \} \} \\ &= 0 \wedge 0 = 0 = \mu_{\chi_{AB}}(x) \\ \text{and } (\gamma_{\chi_A} \circ \gamma_{\chi_B})(x) &= \bigwedge_{x = \sum_{i=1}^n a_i b_i} \{ \bigvee_{i=1}^n \{ \gamma_{\chi_A}(a_i) \vee \gamma_{\chi_B}(b_i) \} \} \\ &= 1 \vee 1 = 1 = \gamma_{\chi_{AB}}(x). \end{aligned}$$

(ii) If $x \neq uv$ for all $u, v \in R$, then obviously $(\chi_A \circ \chi_B)(x) = 0 = \chi_{AB}(x)$. Hence $\chi_A \circ \chi_B = \chi_{AB}$.

Similarly, we can prove (3) and (4).

Theorem 2.3: Let A be a non-empty subset of an LA-ring R . then the following properties hold.

- (1) A is an LA-subring of R if and only if χ_A is an intuitionistic fuzzy LA-subring of R .
- (2) A is a left (resp. right, two-sided) ideal of R if and only if χ_A is an intuitionistic fuzzy left (resp. right, two-sided) ideal of R .

Proof: (1) Let A be an LA-subring of R and $a, b \in R$. If $a, b \in A$, then by definition $\mu_A(a) = 1 = \mu_A(b)$ and $\gamma_A(a) = 0 = \gamma_A(b)$. Since $a - b$ and $ab \in A$, A being an LA-subring of R , this implies that $\mu_A(a - b) = 1 = \mu_A(ab)$ and $\gamma_A(a - b) = 0 = \gamma_A(ab)$. Thus

$$\begin{aligned} \mu_A(a - b) &\geq \mu_A(a) \wedge \mu_A(b), \mu_A(ab) \geq \mu_A(a) \wedge \mu_A(b) \\ \text{and } \gamma_A(a - b) &\leq \gamma_A(a) \vee \gamma_A(b), \gamma_A(ab) \leq \gamma_A(a) \vee \gamma_A(b). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mu_A(a - b) &\geq \mu_A(a) \wedge \mu_A(b), \mu_A(ab) \geq \mu_A(a) \wedge \mu_A(b) \\ \text{and } \gamma_A(a - b) &\leq \gamma_A(a) \vee \gamma_A(b), \gamma_A(ab) \leq \gamma_A(a) \vee \gamma_A(b). \end{aligned}$$

when $a, b \notin A$. Hence χ_A is an intuitionistic fuzzy LA-subring of R .

Conversely, suppose that χ_A is an intuitionistic fuzzy LA-subring of R and let $a, b \in A$. This means that $\mu_A(a) = 1 = \mu_A(b)$ and $\gamma_A(a) = 0 = \gamma_A(b)$. Since

$$\begin{aligned} \mu_A(a - b) &\geq \mu_A(a) \wedge \mu_A(b) = 1 \wedge 1 = 1, \\ \mu_A(ab) &\geq \mu_A(a) \wedge \mu_A(b) = 1 \wedge 1 = 1, \\ \gamma_A(a - b) &\leq \gamma_A(a) \vee \gamma_A(b) = 0 \vee 0 = 0, \\ \gamma_A(ab) &\leq \gamma_A(a) \vee \gamma_A(b) = 0 \vee 0 = 0, \end{aligned}$$

χ_A being an intuitionistic fuzzy LA-subring of R . Thus $\mu_A(a - b) = 1 = \mu_A(ab)$ and $\gamma_A(a - b) = 0 = \gamma_A(ab)$, i.e., $a - b$ and $ab \in A$. Hence A is an LA-subring of R .

(2) Let A be a left ideal of R and $a, b \in R$. If $a, b \in A$, then by definition $\mu_A(a) = 1 = \mu_A(b)$ and $\gamma_A(a) = 0 = \gamma_A(b)$. Since $a - b$ and $ab \in A$, A being a left ideal of R , this implies that $\mu_A(a - b) = 1 = \mu_A(ab)$ and $\gamma_A(a - b) = 0 = \gamma_A(ab)$. Thus

$$\begin{aligned} \mu_A(a - b) &\geq \mu_A(a) \wedge \mu_A(b), \quad \mu_A(ab) \geq \mu_A(b) \\ \text{and } \gamma_A(a - b) &\leq \gamma_A(a) \vee \gamma_A(b), \quad \gamma_A(ab) \leq \gamma_A(b). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mu_A(a - b) &\geq \mu_A(a) \wedge \mu_A(b), \quad \mu_A(ab) \geq \mu_A(b) \\ \text{and } \gamma_A(a - b) &\leq \gamma_A(a) \vee \gamma_A(b), \quad \gamma_A(ab) \leq \gamma_A(b). \end{aligned}$$

when $a, b \notin A$. Therefore χ_A is an intuitionistic fuzzy left ideal of R .

Conversely, assume that χ_A is an intuitionistic fuzzy left ideal of R and let $a, b \in A$ and $z \in R$. This means that $\mu_A(a) = 1 = \mu_A(b)$ and $\gamma_A(a) = 0 = \gamma_A(b)$. Since

$$\begin{aligned} \mu_A(a - b) &\geq \mu_A(a) \wedge \mu_A(b) = 1 \wedge 1 = 1, \\ \mu_A(zb) &\geq \mu_A(b) = 1, \\ \gamma_A(a - b) &\leq \gamma_A(a) \vee \gamma_A(b) = 0 \vee 0 = 0, \\ \gamma_A(zb) &\leq \gamma_A(b) = 0, \end{aligned}$$

χ_A being an intuitionistic fuzzy left ideal of R . Thus $\mu_A(a - b) = 1 = \mu_A(zb)$ and $\gamma_A(a - b) = 0 = \gamma_A(zb)$, i.e., $a - b$ and $zb \in A$. Therefore A is a left ideal of R .

Remark 2.4: (i) A is an additive LA-subgroup of R if and only if χ_A is an intuitionistic fuzzy additive LA-subgroup of R .

(ii) A is an LA-subsemigroup of R if and only if χ_A is an intuitionistic fuzzy LA-subsemigroup of R .

Lemma 2.5: If A and B are two intuitionistic fuzzy LA-subrings (resp. (left, right, two-sided) ideals) of an LA-ring R , then $A \cap B$ is also an intuitionistic fuzzy LA-subring (resp. (left, right, two-sided) ideal) of R .

Proof: Obvious.

Lemma 2.6: If A and B are two intuitionistic fuzzy LA-subrings of an LA-ring R , then $A \circ B$ is also an intuitionistic fuzzy LA-subring of R .

Proof: Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy LA-subrings of R . We have to show that $A \circ B$ is also an intuitionistic fuzzy LA-subring of R . Now

$$\begin{aligned} (\mu_A \circ \mu_B)^2 &= (\mu_A \circ \mu_B) \circ (\mu_A \circ \mu_B) = (\mu_A \circ \mu_A) \circ (\mu_B \circ \mu_B) \subseteq \mu_A \circ \mu_B \\ \text{and } (\gamma_A \circ \gamma_B)^2 &= (\gamma_A \circ \gamma_B) \circ (\gamma_A \circ \gamma_B) = (\gamma_A \circ \gamma_A) \circ (\gamma_B \circ \gamma_B) \supseteq \gamma_A \circ \gamma_B. \end{aligned}$$

Since $\mu_B - \mu_B \subseteq \mu_B$ and $\gamma_B - \gamma_B \supseteq \gamma_B$, $B = (\mu_B, \gamma_B)$ being an intuitionistic fuzzy LA-subring of R . This implies that $\mu_A \circ (\mu_B - \mu_B) \subseteq \mu_A \circ \mu_B$ and

$\gamma_A \circ (\gamma_B - \gamma_B) \supseteq \gamma_A \circ \gamma_B$, i.e., $\mu_A \circ \mu_B - \mu_A \circ \mu_B \subseteq \mu_A \circ \mu_B$ and $\gamma_A \circ \gamma_B - \gamma_A \circ \gamma_B \supseteq \gamma_A \circ \gamma_B$. Therefore $A \circ B$ is an intuitionistic fuzzy LA-subring of R .

Remark 2.7: If A is an intuitionistic fuzzy LA-subring of an LA-ring R , then $A \circ A$ is also an intuitionistic fuzzy LA-subring of R .

Lemma 2.8: Let R be an LA-ring with left identity e . Then $RR = R$ and $eR = R = Re$.

Proof: Since $RR \subseteq R$ and $x = ex \in RR$, where $x \in R$, i.e., $RR = R$. Since e is the left identity of R , i.e., $eR = R$. Now $Re = (RR)e = (eR)R = RR = R$.

Lemma 2.9: Let R be an LA-ring with left identity e . Then every intuitionistic fuzzy right ideal of R is an intuitionistic fuzzy ideal of R .

Proof: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right ideal of R and $x, y \in R$. Now

$$\begin{aligned} \mu_A(xy) &= \mu_A((ex)y) = \mu_A((yx)e) \geq \mu_A(yx) \geq \mu_A(y) \\ \text{and } \gamma_A(xy) &= \gamma_A((ex)y) = \gamma_A((yx)e) \leq \gamma_A(yx) \leq \gamma_A(y). \end{aligned}$$

Thus A is an intuitionistic fuzzy ideal of R .

Lemma 2.10: If A and B are two intuitionistic fuzzy left (resp. right) ideals of an LA-ring R with left identity e , then $A \circ B$ is also an intuitionistic fuzzy left (resp. right) ideal of R .

Proof: Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy left ideals of R . We have to show that $A \circ B$ is also an intuitionistic fuzzy left ideal of R . Since $\mu_A \circ \mu_B - \mu_A \circ \mu_B \subseteq \mu_A \circ \mu_B$ and $\gamma_A \circ \gamma_B - \gamma_A \circ \gamma_B \supseteq \gamma_A \circ \gamma_B$. Now

$$\begin{aligned} R \circ (\mu_A \circ \mu_B) &= (R \circ R) \circ (\mu_A \circ \mu_B) = (R \circ \mu_A) \circ (R \circ \mu_B) \subseteq (\mu_A \circ \mu_B) \\ \text{and } R \circ (\gamma_A \circ \gamma_B) &= (R \circ R) \circ (\gamma_A \circ \gamma_B) = (R \circ \gamma_A) \circ (R \circ \gamma_B) \supseteq (\gamma_A \circ \gamma_B). \end{aligned}$$

Hence $A \circ B$ is an intuitionistic fuzzy left ideal of R .

Remark 2.11: If A is an intuitionistic fuzzy left (resp. right) ideal of an LA-ring R with left identity e , then $A \circ A$ is an intuitionistic fuzzy ideal of R .

Lemma 2.12: If A and B are two intuitionistic fuzzy ideals of an LA-ring R , then $A \circ B \subseteq A \cap B$.

Proof: Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy ideals of R and $x \in R$. If x cannot expressible as $x = \sum_{i=1}^n a_i b_i$, where $a_i, b_i \in R$ and n is any positive integer, then obviously $A \circ B \subseteq A \cap B$, otherwise we have

$$\begin{aligned} (\mu_A \circ \mu_B)(x) &= \bigvee_{x = \sum_{i=1}^n a_i b_i} \{ \bigwedge_{i=1}^n \{ \mu_A(a_i) \wedge \mu_B(b_i) \} \} \\ &\leq \bigvee_{x = \sum_{i=1}^n a_i b_i} \{ \bigwedge_{i=1}^n \{ \mu_A(a_i b_i) \wedge \mu_B(a_i b_i) \} \} \\ &= \bigvee_{x = \sum_{i=1}^n a_i b_i} \{ \bigwedge_{i=1}^n (\mu_A \cap \mu_B)(a_i b_i) \} = (\mu_A \cap \mu_B)(x). \\ &\Rightarrow \mu_A \circ \mu_B \subseteq \mu_A \cap \mu_B. \end{aligned}$$

Similarly, we can prove $\gamma_A \circ \gamma_B \supseteq \gamma_A \cup \gamma_B$.

Therefore $A \circ B \subseteq A \cap B$ for all intuitionistic fuzzy ideals A and B of R .

Remark 2.13: If A is an intuitionistic fuzzy ideal of an LA-ring R , then $A \circ A \subseteq A$.

Lemma 2.14: Let R be an LA-ring. Then $A \circ B \subseteq A \cap B$ for every intuitionistic fuzzy right ideal A and every intuitionistic fuzzy left ideal B of R .

Proof: Same as Lemma 2.12

Theorem 2.15: Let A be a non-empty subset of an LA-ring R . Then A is an interior (resp. quasi-, bi-, generalized bi-) ideal of R if and only if χ_A is an intuitionistic fuzzy interior (resp. quasi-, bi-, generalized bi-) ideal of R .

Proof: Let A be an interior ideal of R , this implies that A is an additive LA-subgroup of R . Then χ_A is an intuitionistic fuzzy additive LA-subgroup of R by the Remark 2.4. Let $x, y, a \in R$. If $a \in A$, then by definition $\mu_{\chi_A}(a) = 1$ and $\gamma_{\chi_A}(a) = 0$. Since $(xa)y \in A$, A being an interior ideal of R , this means that $\mu_{\chi_A}((xa)y) = 1$ and $\gamma_{\chi_A}((xa)y) = 0$. Thus $\mu_{\chi_A}((xa)y) \geq \mu_{\chi_A}(a)$ and $\gamma_{\chi_A}((xa)y) \leq \gamma_{\chi_A}(a)$. Similarly, we have $\mu_{\chi_A}((xa)y) \geq \mu_{\chi_A}(a)$ and $\gamma_{\chi_A}((xa)y) \leq \gamma_{\chi_A}(a)$, when $a \notin A$. Hence χ_A is an intuitionistic fuzzy interior ideal of R .

Conversely, suppose that χ_A is an intuitionistic fuzzy interior ideal of R , this means that χ_A is an intuitionistic fuzzy additive LA-subgroup of R . Then A is an additive LA-subgroup of R by the Remark 2.4. Let $x, y \in R$ and $a \in A$, so $\mu_{\chi_A}(a) = 1$ and $\gamma_{\chi_A}(a) = 0$. Since $\mu_{\chi_A}((xa)y) \geq \mu_{\chi_A}(a) = 1$ and $\gamma_{\chi_A}((xa)y) \leq \gamma_{\chi_A}(a) = 0$, χ_A being an intuitionistic fuzzy interior ideal of R . Thus $\mu_{\chi_A}((xa)y) = 1$ and $\gamma_{\chi_A}((xa)y) = 0$, i.e., $(xa)y \in A$. Hence A is an interior ideal of R . Similarly, we can prove for (quasi-, bi-, generalized bi-) ideal.

Lemma 2.16: If A and B are two intuitionistic fuzzy bi- (resp. generalized bi-, quasi-, interior) ideals of an LA-ring R , then $A \cap B$ is also an intuitionistic fuzzy bi- (resp. generalized bi-, quasi-, interior) ideal of R .

Proof: Obvious.

Lemma 2.17: If A and B are two intuitionistic fuzzy bi- (resp. generalized bi-, interior) ideals of an LA-ring R with left identity e , then $A \circ B$ is also an intuitionistic fuzzy bi- (resp. generalized bi-, interior) ideal of R .

Proof: Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy bi-ideals of R . We have to show that $A \circ B$ is also an intuitionistic fuzzy bi-ideal of R . Since A and B are two intuitionistic fuzzy LA-subrings of R , then $A \circ B$ is also an intuitionistic fuzzy LA-subring of R by the Lemma 2.6. Now

$$\begin{aligned} ((\mu_A \circ \mu_B) \circ R) \circ (\mu_A \circ \mu_B) &= ((\mu_A \circ \mu_B) \circ (R \circ R)) \circ (\mu_A \circ \mu_B) \\ &= ((\mu_A \circ R) \circ (\mu_B \circ R)) \circ (\mu_A \circ \mu_B) \\ &= ((\mu_A \circ R) \circ \mu_A) \circ ((\mu_B \circ R) \circ \mu_B) \\ &\subseteq \mu_A \circ \mu_B. \end{aligned}$$

Similarly, we have $((\gamma_A \circ \gamma_B) \circ R) \circ (\gamma_A \circ \gamma_B) \supseteq \gamma_A \circ \gamma_B$. Therefore $A \circ B$ is an intuitionistic fuzzy bi-ideal of R .

Lemma 2.18: Every intuitionistic fuzzy ideal of an LA-ring R is an intuitionistic fuzzy interior ideal of R . The converse is not true in general.

Proof: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of R and $x, y, z \in R$. Thus $\mu_A((xy)z) = \mu_A(xy) \geq \mu_A(y)$ and $\gamma_A((xy)z) = \gamma_A(xy) \leq \gamma_A(y)$. Hence A is an intuitionistic fuzzy interior ideal of R .

The converse is not true in general, giving an example:

Example 2.19: Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is an LA-ring.

+	0	1	2	3	4	5	6	7		·	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7		0	0	0	0	0	0	0	0	0
1	2	0	3	1	6	4	7	5		1	0	4	4	0	0	4	4	0
2	1	3	0	2	5	7	4	6		2	0	4	4	0	0	4	4	0
3	3	2	1	0	7	6	5	4	and	3	0	0	0	0	0	0	0	0
4	4	5	6	7	0	1	2	3		4	0	3	3	0	0	3	3	0
5	6	4	7	5	2	0	3	1		5	0	7	7	0	0	7	7	0
6	5	7	4	6	1	3	0	2		6	0	7	7	0	0	7	7	0
7	7	6	5	4	3	2	1	0		7	0	3	3	0	0	3	3	0

Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R . We define

$$\mu_A(0) = \mu_A(4) = 0.7, \quad \mu_A(1) = \mu_A(2) = \mu_A(3) = \mu_A(5) = \mu_A(6) = \mu_A(7) = 0$$

$$\text{and } \gamma_A(0) = \gamma_A(4) = 0, \quad \gamma_A(1) = \gamma_A(2) = \gamma_A(3) = \gamma_A(5) = \gamma_A(6) = \gamma_A(7) = 0.7.$$

$A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal of R , but not an intuitionistic fuzzy ideal of R , because A is not an intuitionistic fuzzy right ideal of R , as

$$\mu_A(41) = \mu_A(3) = 0.$$

$$\mu_A(4) = 0.7.$$

$$\Rightarrow \mu_A(41) \not\geq \mu_A(4).$$

$$\text{and } \gamma_A(41) = \gamma_A(3) = 0.7.$$

$$\gamma_A(4) = 0.$$

$$\Rightarrow \gamma_A(41) \not\leq \gamma_A(4).$$

Proposition 2.20: Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R with left identity e . Then A is an intuitionistic fuzzy ideal of R if and only if A is an intuitionistic fuzzy interior ideal of R .

Proof: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy interior ideal of R and $x, y \in R$. Thus $\mu_A(xy) = \mu_A((ex)y) \geq \mu_A(x)$ and $\gamma_A(xy) = \gamma_A((ex)y) \leq \gamma_A(x)$, i.e., A is an intuitionistic fuzzy right ideal of R . Hence A is an intuitionistic fuzzy ideal of R by the Lemma 2.9. Converse is true by the Lemma 2.18.

Lemma 2.21: Every intuitionistic fuzzy left (resp. right, two-sided) ideal of an LA-ring R is an intuitionistic fuzzy bi-ideal of R .

Proof: Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of R and $x, y, z \in R$. Thus

$$\mu_A((xy)z) = \mu_A(xy) \geq \mu_A(x) \text{ and } \mu_A((xy)z) = \mu_A((zy)x) \geq \mu_A(zy) \geq \mu_A(z),$$

this implies that $\mu_A((xy)z) \geq \mu_A(x) \wedge \mu_A(z)$. Similarly, we have $\gamma_A((xy)z) \leq \gamma_A(x) \vee \gamma_A(z)$. Therefore A is an intuitionistic fuzzy bi-ideal of R .

The converse is not true in general, giving an example:

Using Example 2.19, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of R , but not an intuitionistic fuzzy right ideal of R , as

$$\begin{aligned} \mu_A(41) &= \mu_A(3) = 0. \\ \mu_A(4) &= 0.7. \\ &\Rightarrow \mu_A(41) \not\geq \mu_A(4). \\ \text{and } \gamma_A(41) &= \gamma_A(3) = 0.7. \\ \gamma_A(4) &= 0. \\ &\Rightarrow \gamma_A(41) \not\leq \gamma_A(4). \end{aligned}$$

Lemma 2.22: Every intuitionistic fuzzy bi-ideal of an LA-ring R is an intuitionistic fuzzy generalized bi-ideal of R .

Proof: Obvious.

Lemma 2.23: Every intuitionistic fuzzy left (resp. right, two-sided) ideal of an LA-ring R is an intuitionistic fuzzy quasi-ideal of R .

Proof: Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of R . Now $\mu_A \circ R \cap R \circ \mu_A \subseteq R \circ \mu_A \subseteq \mu_A$ and $\gamma_A \circ R \cup R \circ \gamma_A \supseteq R \circ \gamma_A \supseteq \gamma_A$. So A is an intuitionistic fuzzy quasi-ideal of R .

Lemma 2.24: Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then every intuitionistic fuzzy quasi-ideal of R is an intuitionistic fuzzy bi-ideal of R .

Proof: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy quasi-ideal of R and $A \circ A \subseteq A$ by the Proposition 2.20. Now

$$\begin{aligned} (\mu_A \circ R) \circ \mu_A &\subseteq (R \circ R) \circ \mu_A \subseteq R \circ \mu_A \\ \text{and } (\mu_A \circ R) \circ \mu_A &\subseteq (\mu_A \circ R) \circ R = (\mu_A \circ R) \circ (e \circ R) \\ &= (\mu_A \circ e) \circ (R \circ R) \subseteq (\mu_A \circ e) \circ R = \mu_A \circ R. \\ &\Rightarrow (\mu_A \circ R) \circ \mu_A \subseteq \mu_A \circ R \cap R \circ \mu_A \subseteq \mu_A. \end{aligned}$$

Similarly, $(\gamma_A \circ R) \circ \gamma_A \supseteq \gamma_A \circ R \cup R \circ \gamma_A \supseteq \gamma_A$. Hence A is an intuitionistic fuzzy bi-ideal of R .

3 REGULAR LA-RINGS

An LA-ring R is called a regular if for every $x \in R$, there exists an element $a \in R$ such that $x = (xa)x$. In this section, we characterize regular LA-rings by the properties of intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals.

Lemma 3.1: Every intuitionistic fuzzy right ideal of a regular LA-ring R is an intuitionistic fuzzy ideal of R .

Proof: Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of R . Let $x, y \in R$, this implies that there exists an element $a \in R$, such that $x = (xa)x$. Thus

$$\mu_A(xy) = \mu_A(((xa)x)y) = \mu_A((yx)(xa)) \geq \mu_A(yx) \geq \mu_A(y)$$

and

$$\gamma_A(xy) = \gamma_A(((xa)x)y) = \gamma_A((yx)(xa)) \leq \gamma_A(yx) \leq \gamma_A(y).$$

Hence A is an intuitionistic fuzzy ideal of R .

Lemma 3.2: Let $A = (\mu_A, \gamma_A)$ be an IFS of a regular LA-ring R . Then A is an intuitionistic fuzzy ideal of R if and only if A is an intuitionistic fuzzy interior ideal of R .

Proof: Consider that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal of R . Let $x, y \in R$, then there exists an element $a \in R$, such that $x = (xa)x$. Thus

$$\mu_A(xy) = \mu_A(((xa)x)y) = \mu_A((yx)(xa)) \geq \mu_A(x)$$

and

$$\gamma_A(xy) = \gamma_A(((xa)x)y) = \gamma_A((yx)(xa)) \leq \gamma_A(x),$$

i.e., A is an intuitionistic fuzzy right ideal of R . So A is an intuitionistic fuzzy ideal of R by the Lemma 3.1. Converse is true by the Lemma 2.18.

Remark 3.3: The concept of intuitionistic fuzzy (interior, two-sided) ideals coincides with the same concept in regular LA-rings.

Proposition 3.4: Let R be a regular LA-ring. Then $(A \circ R) \cap (R \circ A) = A$ for every intuitionistic fuzzy right ideal A of R .

Proof: Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of R . This implies that $(A \circ R) \cap (R \circ A) \subseteq A$, because every intuitionistic fuzzy right ideal of R is an intuitionistic fuzzy quasi-ideal of R by the Lemma 2.23. Let $x \in R$, this implies that there exists an element $a \in R$, such that $x = (xa)x$. Thus

$$\begin{aligned} (\mu_A \circ R)(x) &= \bigvee_{x = \sum_{i=1}^n a_i b_i} \{ \bigwedge_{i=1}^n \{ \mu_A(a_i) \wedge R(b_i) \} \} \\ &\geq \mu_A(xa) \wedge R(x) \geq \mu_A(x) \wedge 1 = \mu_A(x) \\ \text{and } (\gamma_A \circ R)(x) &= \bigwedge_{x = \sum_{i=1}^n a_i b_i} \{ \bigvee_{i=1}^n \{ \gamma_A(a_i) \vee R(b_i) \} \} \\ &\leq \gamma_A(xa) \vee R(x) \leq \gamma_A(x) \vee 0 = \gamma_A(x). \\ &\Rightarrow A \subseteq A \circ R. \end{aligned}$$

Similarly, we have $A \subseteq R \circ A$, i.e., $A \subseteq (A \circ R) \cap (R \circ A)$. Hence $(A \circ R) \cap (R \circ A) = A$.

Lemma 3.5: Let R be a regular LA-ring. Then $D \circ L = D \cap L$ for every intuitionistic fuzzy right ideal D and every intuitionistic fuzzy left ideal L of R .

Proof: Since $D \circ L \subseteq D \cap L$, for every intuitionistic fuzzy right ideal $D = (\mu_D, \gamma_D)$ and every intuitionistic fuzzy left ideal $L = (\mu_L, \gamma_L)$ of R by the Lemma 2.14. Let $x \in R$, this means that there exists an element $a \in R$ such that $x = (xa)x$. Thus

$$\begin{aligned} (\mu_D \circ \mu_L)(x) &= \bigvee_{x = \sum_{i=1}^n a_i b_i} \{ \bigwedge_{i=1}^n \{ \mu_D(a_i) \wedge \mu_L(b_i) \} \} \\ &\geq \mu_D(xa) \wedge \mu_L(x) \geq \mu_D(x) \wedge \mu_L(x) = (\mu_D \cap \mu_L)(x) \\ \text{and } (\gamma_D \circ \gamma_L)(x) &= \bigwedge_{x = \sum_{i=1}^n a_i b_i} \{ \bigvee_{i=1}^n \{ \gamma_D(a_i) \vee \gamma_L(b_i) \} \} \\ &\leq \gamma_D(xa) \vee \gamma_L(x) \leq \gamma_D(x) \vee \gamma_L(x) = (\gamma_D \cup \gamma_L)(x). \end{aligned}$$

Therefore $D \circ L = D \cap L$.

Lemma 3.6: Let R be an LA-ring with left identity e . Then Ra is the smallest left ideal of R containing a .

Proof: Let $x, y \in Ra$ and $r \in R$. This implies that $x = r_1 a$ and $y = r_2 a$, where $r_1, r_2 \in R$. Now

$$\begin{aligned} x - y &= r_1 a - r_2 a = (r_1 - r_2) a \in Ra \\ \text{and } rx &= r(r_1 a) = (er)(r_1 a) = ((r_1 a)r)e = ((r_1 a)(er))e \\ &= ((r_1 e)(ar))e = (e(ar))(r_1 e) = (ar)(r_1 e) \\ &= ((r_1 e)r)a \in Ra. \end{aligned}$$

Since $a = ea \in Ra$. Thus Ra is a left ideal of R containing a . Let I be another left ideal of R containing a . Since $ra \in I$, where $ra \in Ra$, i.e., $Ra \subseteq I$. Hence Ra is the smallest left ideal of R containing a .

Lemma 3.7: Let R be an LA-ring with left identity e . Then aR is a left ideal of R .

Proof: Straight forward.

Proposition 3.8: Let R be an LA-ring with left identity e . Then $aR \cup Ra$ is the smallest right ideal of R containing a .

Proof: Let $x, y \in aR \cup Ra$, this means that $x, y \in aR$ or Ra . Since aR and Ra both are left ideals of R , so $x - y \in aR$ and Ra , i.e., $x - y \in aR \cup Ra$. We have to show that $(aR \cup Ra)R \subseteq (aR \cup Ra)$. Now

$$\begin{aligned} (aR \cup Ra)R &= (aR)R \cup (Ra)R = (RR)a \cup (Ra)(eR) \\ &\subseteq Ra \cup (Re)(aR) = Ra \cup R(aR) \\ &= Ra \cup a(RR) \subseteq Ra \cup aR = aR \cup Ra. \\ &\Rightarrow (aR \cup Ra)R \subseteq aR \cup Ra. \end{aligned}$$

Since $a \in Ra$, i.e., $a \in aR \cup Ra$. Let I be another right ideal of R containing a . Since $aR \in IR \subseteq I$ and $Ra = (RR)a = (aR)R \in (IR)R \subseteq IR \subseteq I$, i.e., $aR \cup Ra \subseteq I$. Therefore $aR \cup Ra$ is the smallest right ideal of R containing a .

Theorem 3.9: Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

- (1) R is a regular.
- (2) $D \cap L = D \circ L$ for every intuitionistic fuzzy right ideal D and every intuitionistic fuzzy left ideal L of R .
- (3) $C = (C \circ R) \circ C$ for every intuitionistic fuzzy quasi-ideal C of R .

Proof: Suppose that (1) holds and $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy quasi-ideal of R . Then $(C \circ R) \circ C \subseteq C$, because every intuitionistic fuzzy quasi-ideal of R is an intuitionistic fuzzy bi-ideal of R by the Lemma 2.24. Let $x \in R$, this implies that there exists an element a of R such that $x = (xa)x$. Thus

$$\begin{aligned} ((\mu_C \circ R) \circ \mu_C)(x) &= \bigvee_{a = \sum_{i=1}^n a_i b_i} \{ \bigwedge_{i=1}^n \{ (\mu_C \circ R)(a_i) \wedge \mu_C(b_i) \} \} \\ &\geq (\mu_C \circ R)(xa) \wedge \mu_C(x) \\ &= \bigvee_{xa = \sum_{i=1}^n p_i q_i} \{ \bigwedge_{i=1}^n \{ \mu_C(p_i) \wedge R(q_i) \} \} \wedge \mu_C(x) \\ &\geq \mu_C(x) \wedge R(a) \wedge \mu_C(x) = \mu_C(x). \\ &\Rightarrow \mu_C \subseteq (\mu_C \circ R) \circ \mu_C. \end{aligned}$$

Similarly, we have $\gamma_C \supseteq (\gamma_C \circ R) \circ \gamma_C$, i.e., $C = (C \circ R) \circ C$. Hence (1) implies (3). Assume that (3) holds. Let D be an intuitionistic fuzzy right ideal and L be an intuitionistic fuzzy left ideal of R . This means that D and L be intuitionistic fuzzy quasi-ideals of R by the Lemma 2.23, so $D \cap L$ be also an intuitionistic fuzzy quasi-ideal of R . Then by our assumption, $D \cap L = ((D \cap L) \circ R) \circ (D \cap L) \subseteq (D \circ R) \circ L \subseteq D \circ L$, i.e., $D \cap L \subseteq D \circ L$. Since $D \circ L \subseteq D \cap L$. Therefore $D \circ L = D \cap L$, i.e., (3) $\Rightarrow$ (2). Suppose that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 3.7 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.8. This implies that χ_{Ra} is an intuitionistic fuzzy left ideal and $\chi_{aR \cup Ra}$ is an intuitionistic fuzzy right ideal of R , by the Theorem 2.3. Then by our supposition

$$\chi_{aR \cup Ra} \cap \chi_{Ra} = \chi_{aR \cup Ra} \circ \chi_{Ra}, \text{ i.e., } \chi_{(aR \cup Ra) \cap Ra} = \chi_{(aR \cup Ra) Ra}$$

by the Theorem 2.2. Thus $(aR \cup Ra) \cap Ra = (aR \cup Ra)Ra$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in (aR \cup Ra)Ra$, so $a \in (aR)(Ra) \cup (Ra)(Ra)$. This implies that

$$a \in (aR)(Ra) \text{ or } a \in (Ra)(Ra).$$

If $a \in (aR)(Ra)$, then

$a = (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a = ((xe)(ay))a = (a((xe)y))a$ for any $x, y \in R$.

If $a \in (Ra)(Ra)$, then $(Ra)(Ra) = ((Re)a)(Ra) = ((ae)R)(Ra) = (aR)(Ra)$, i.e., $a \in (aR)(Ra)$. So a is a regular, i.e., R is a regular. Hence (2) $\Rightarrow$ (1).

Theorem 3.10: Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

- (1) R is a regular.
- (2) $A = (A \circ R) \circ A$ for every intuitionistic fuzzy quasi-ideal A of R .
- (3) $B = (B \circ R) \circ B$ for every intuitionistic fuzzy bi-ideal B of R .
- (4) $C = (C \circ R) \circ C$ for every intuitionistic fuzzy generalized bi-ideal C of R .

Proof: (1) $\Rightarrow$ (4), is obvious. Since (4) $\Rightarrow$ (3), every intuitionistic fuzzy bi-ideal of R is an intuitionistic fuzzy generalized bi-ideal of R by the Lemma 2.22. Since (3) $\Rightarrow$ (2), every intuitionistic fuzzy quasi-ideal of R is an intuitionistic fuzzy bi-ideal of R by the Lemma 15. (2) $\Rightarrow$ (1), by the Theorem 3.9.

Theorem 3.11: Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

- (1) R is a regular.
- (2) $A \cap I = (A \circ I) \circ A$ for every intuitionistic fuzzy quasi-ideal A and every intuitionistic fuzzy ideal I of R .
- (3) $B \cap I = (B \circ I) \circ B$ for every intuitionistic fuzzy bi-ideal B and every intuitionistic fuzzy ideal I of R .
- (4) $C \cap I = (C \circ I) \circ C$ for every intuitionistic fuzzy generalized bi-ideal C and every intuitionistic fuzzy ideal I of R .

Proof: Assume that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal and $I = (\mu_I, \gamma_I)$ be an intuitionistic fuzzy ideal of R . Now $(C \circ I) \circ C \subseteq (R \circ I) \circ R \subseteq I \circ R \subseteq I$ and $(C \circ I) \circ C \subseteq (C \circ R) \circ C \subseteq C$, i.e., $(C \circ I) \circ C \subseteq C \cap I$. Let $x \in R$, this means that there exists an element $a \in R$ such that $x = (xa)x$. Now $xa = ((xa)x)a = (ax)(xa) = x((ax)a)$. Thus

$$\begin{aligned}
 ((\mu_C \circ \mu_I) \circ \mu_C)(x) &= \bigvee_{x = \sum_{i=1}^n a_i b_i} \{ \bigwedge_{i=1}^n \{ (\mu_C \circ \mu_I)(a_i) \wedge \mu_C(b_i) \} \} \\
 &\geq (\mu_C \circ \mu_I)(xa) \wedge \mu_C(x) \\
 &= \bigvee_{xa = \sum_{i=1}^n p_i q_i} \{ \bigwedge_{i=1}^n \{ \mu_C(p_i) \wedge \mu_I(q_i) \} \} \wedge \mu_C(x) \\
 &\geq \mu_C(x) \wedge \mu_I((ax)a) \wedge \mu_C(x) \\
 &\geq \mu_C(x) \wedge \mu_I(x) = (\mu_C \cap \mu_I)(x). \\
 &\Rightarrow \mu_C \cap \mu_I \subseteq (\mu_C \circ \mu_I) \circ \mu_C.
 \end{aligned}$$

Similarly, we have $\gamma_C \cup \gamma_I \supseteq (\gamma_C \circ \gamma_I) \circ \gamma_C$. Hence $C \cap I = (C \circ I) \circ C$, i.e., (1) $\Rightarrow$ (4). It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Suppose that (2) is true. Then

$A \cap R = (A \circ R) \circ A$, where R itself is an intuitionistic fuzzy two-sided ideal of R . So $A = (A \circ R) \circ A$, because every intuitionistic fuzzy two-sided ideal of R is an intuitionistic fuzzy quasi-ideal of R . Hence R is a regular by the Theorem 3.9, i.e., (2) $\Rightarrow$ (1).

Theorem 3.12: Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

- (1) R is a regular.
- (2) $A \cap D \subseteq D \circ A$ for every intuitionistic fuzzy quasi-ideal A and every intuitionistic fuzzy right ideal D of R .
- (3) $B \cap D \subseteq D \circ B$ for every intuitionistic fuzzy bi-ideal B and every intuitionistic fuzzy right ideal D of R .
- (4) $C \cap D \subseteq D \circ C$ for every intuitionistic fuzzy generalized bi-ideal C and every intuitionistic fuzzy right ideal D of R .

Proof: (1) $\Rightarrow$ (4), is obvious. It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Suppose that (2) holds, this implies that $D \cap A = A \cap D \subseteq D \circ A$, where A is an intuitionistic fuzzy left ideal of R . Since $D \circ A \subseteq D \cap A$, i.e., $D \cap A = D \circ A$. Hence R is a regular by the Theorem 3.9, i.e., (2) $\Rightarrow$ (1).

4 INTRA-REGULAR LA-RINGS

An LA-ring R is called an intra-regular if for every $x \in R$, there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^n (a_i x^2) b_i$. In this section, we characterize intra-regular LA-rings by the properties of intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals.

Lemma 4.1: Every intuitionistic fuzzy left (right) ideal of an intra-regular LA-ring R is an intuitionistic fuzzy ideal of R .

Proof: Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of R . Let $x, y \in R$, this implies that there exist elements $a_i, b_i \in R$, such that $x = \sum_{i=1}^n (a_i x^2) b_i$. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A(((a_i x^2) b_i) y) = \mu_A((y b_i)(a_i x^2)) \\ &\geq \mu_A(y b_i) \geq \mu_A(y) \\ \text{and } \gamma_A(xy) &= \gamma_A(((a_i x^2) b_i) y) = \gamma_A((y b_i)(a_i x^2)) \\ &\leq \gamma_A(y b_i) \leq \gamma_A(y). \end{aligned}$$

Hence A is an intuitionistic fuzzy ideal of R .

Proposition 4.2: Let A be an IFS of an intra-regular LA-ring R with left identity e . Then A is an intuitionistic fuzzy ideal of R if and only if A is an intuitionistic fuzzy interior ideal of R .

Proof: Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal of R . Let $x, y \in R$, this implies that there exist elements $a_i, b_i \in R$, such that $x = \sum_{i=1}^n (a_i x^2) b_i$. Thus

$$\begin{aligned}
 \mu_A(xy) &= \mu_A(((a_i x^2)b_i)y) = \mu_A((yb_i)(a_i x^2)) \\
 &= \mu_A((yb_i)(a_i(xx))) = \mu_A((yb_i)(x(a_i x))) \\
 &= \mu_A((yx)(b_i(a_i x))) \geq \mu_A(x).
 \end{aligned}$$

Similarly, we have $\gamma_A(xy) \leq \gamma_A(x)$, i.e., A is an intuitionistic fuzzy right ideal of R . Hence A is an intuitionistic fuzzy ideal of R by the Lemma 4.1. Converse is true by the Lemma 2.18.

Remark 4.3: The concept of intuitionistic fuzzy (interior, two-sided) ideals coincides in intra-regular LA-rings with left identity.

Lemma 4.4: Let R be an intra-regular LA-ring with left identity e . Then $D \cap L \subseteq L \circ D$ for every intuitionistic fuzzy left ideal L and every intuitionistic fuzzy right ideal D of R .

Proof: Let $L = (\mu_L, \gamma_L)$ be an intuitionistic fuzzy left ideal and $D = (\mu_D, \gamma_D)$ be an intuitionistic fuzzy right ideal of R . Let $x \in R$, this means that there exist elements such that $a_i, b_i \in R$ such that $x = \sum_{i=1}^n (a_i x^2)b_i$. Now

$$\begin{aligned}
 x &= (a_i x^2)b_i = (a_i(xx))b_i = (x(a_i x))b_i \\
 &= (x(a_i x))(eb_i) = (xe)((a_i x)b_i) = (a_i x)((xe)b_i).
 \end{aligned}$$

Thus

$$\begin{aligned}
 (\mu_L \circ \mu_D)(x) &= \bigvee_{x = \sum_{i=1}^n p_i q_i} \{ \bigwedge_{i=1}^n \{ \mu_L(p_i) \wedge \mu_D(q_i) \} \} \\
 &\geq \mu_L(a_i x) \wedge \mu_D((xe)b_i) \geq \mu_L(x) \wedge \mu_D(x) \\
 &= \mu_D(x) \wedge \mu_L(x) = (\mu_D \cap \mu_L)(x) \\
 \text{and } (\gamma_L \circ \gamma_D)(x) &= \bigwedge_{x = \sum_{i=1}^n p_i q_i} \{ \bigvee_{i=1}^n \{ \gamma_L(p_i) \vee \gamma_D(q_i) \} \} \\
 &\leq \gamma_L(a_i x) \vee \gamma_D((xe)b_i) \leq \gamma_L(x) \vee \gamma_D(x) \\
 &= \gamma_D(x) \vee \gamma_L(x) = (\gamma_D \cup \gamma_L)(x). \\
 &\Rightarrow D \cap L \subseteq L \circ D.
 \end{aligned}$$

Theorem 4.5: Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

- (1) R is an intra-regular.
- (2) $D \cap L \subseteq L \circ D$ for every intuitionistic fuzzy left ideal L and every intuitionistic fuzzy right ideal D of R .

Proof: (1) $\Rightarrow$ (2) is true by the Lemma 4.4. Suppose that (2) holds. Let $a \in R$, then Ra is a left ideal of R containing a by the Lemma 3.6 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.8. So χ_{Ra} is an intuitionistic fuzzy left ideal and $\chi_{aR \cup Ra}$ is an intuitionistic fuzzy right ideal of R , by the Theorem 1.3. By our supposition

$$\chi_{aR \cup Ra} \cap \chi_{Ra} \subseteq \chi_{Ra} \circ \chi_{aR \cup Ra}, \text{ i.e., } \chi_{(aR \cup Ra) \cap Ra} \subseteq \chi_{(Ra)(aR \cup Ra)}$$

by the Theorem 1.2. Thus $(aR \cup Ra) \cap Ra \subseteq Ra(aR \cup Ra)$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in Ra(aR \cup Ra) = (Ra)(aR) \cup (Ra)(Ra)$. This implies that $a \in (Ra)(aR)$ or $a \in (Ra)(Ra)$. If $a \in (Ra)(aR)$, then

$$\begin{aligned} (Ra)(aR) &= (Ra)((ea)(RR)) = (Ra)((RR)(ae)) \\ &= (Ra)((ae)R) = (Ra)((aR)R) \\ &= (Ra)((RR)a) = (Ra)(Ra) = ((Ra)a)R \\ &= ((Ra)(ea))R = ((Re)(aa))R = (Ra^2)R. \end{aligned}$$

So $a \in (Ra^2)R$. If $a \in (Ra)(Ra)$, then obvious $a \in (Ra^2)R$. This implies that a is an intra-regular. Hence R is an intra-regular, i.e., (2) $\Rightarrow$ (1).

Theorem 4.6: Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

- (1) R is an intra-regular.
- (2) $A \cap I = (A \circ I) \circ A$ for every intuitionistic fuzzy quasi-ideal A and every intuitionistic fuzzy ideal I of R .
- (3) $B \cap I = (B \circ I) \circ B$ for every intuitionistic fuzzy bi-ideal B and every intuitionistic fuzzy ideal I of R .
- (4) $C \cap I = (C \circ I) \circ C$ for every intuitionistic fuzzy generalized bi-ideal C and every intuitionistic fuzzy ideal I of R .

Proof: Suppose that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal and $I = (\mu_I, \gamma_I)$ be an intuitionistic fuzzy ideal of R . Now $(C \circ I) \circ C \subseteq (R \circ I) \circ R \subseteq I \circ R \subseteq I$ and $(C \circ I) \circ C \subseteq (C \circ R) \circ C \subseteq C$, thus $(C \circ I) \circ C \subseteq C \cap I$. Let $x \in R$, this implies that there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^n (a_i x^2) b_i$. Now

$$\begin{aligned} x &= (a_i x^2) b_i = (a_i (xx)) b_i = (x(a_i x)) b_i = (b_i (a_i x)) x. \\ b_i (a_i x) &= b_i (a_i ((a_i x^2) b_i)) = b_i ((a_i x^2)(a_i b_i)) = b_i ((a_i x^2) c_i) \\ &= (a_i x^2)(b_i c_i) = (a_i x^2) d_i = (a_i x^2)(e d_i) = (d_i e)(x^2 a_i) \\ &= m_i (x^2 a_i) = x^2 (m_i a_i) = (xx) l_i = (l_i x) x = (l_i x)(ex) \\ &= (xe)(x l_i) = x((xe) l_i). \end{aligned}$$

Thus

$$\begin{aligned} ((\mu_C \circ \mu_I) \circ \mu_C)(x) &= \vee_{x = \sum_{i=1}^n p_i q_i} \{ \wedge_{i=1}^n \{ (\mu_C \circ \mu_I)(p_i) \wedge \mu_C(q_i) \} \} \\ &\geq (\mu_C \circ \mu_I)(b_i (a_i x)) \wedge \mu_C(x) \\ &= \vee_{b_i (a_i x) = \sum_{i=1}^n m_i n_i} \{ \wedge_{i=1}^n \{ \mu_C(m_i) \wedge \mu_I(n_i) \} \} \wedge \mu_C(x) \\ &\geq \mu_C(x) \wedge \mu_I((xe) l_i) \wedge \mu_C(x) \\ &\geq \mu_C(x) \wedge \mu_I(x) = (\mu_C \cap \mu_I)(x). \\ &\Rightarrow \mu_C \cap \mu_I \subseteq (\mu_C \circ \mu_I) \circ \mu_C. \end{aligned}$$

Similarly, we have $\gamma_C \cup \gamma_I \supseteq (\gamma_C \circ \gamma_I) \circ \gamma_C$. Hence $C \cap I = (C \circ I) \circ C$, i.e. (1) $\Rightarrow$ (4). It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Assume that (2) is true. Let A be an intuitionistic fuzzy right ideal and I be an intuitionistic fuzzy two-sided ideal of R . Since every intuitionistic fuzzy right ideal of R is an intuitionistic fuzzy quasi-ideal of R by the Lemma 2.23, so A is an intuitionistic fuzzy quasi-ideal of R . By our assumption $A \cap I = (A \circ I) \circ A \subseteq (R \circ I) \circ A \subseteq I \circ A$, i.e., $A \cap I \subseteq I \circ A$. Hence R is an intra-regular by the Theorem 4.5, i.e., (2) $\Rightarrow$ (1).

Theorem 4.7: Let R be an LA-ring with left identity e , such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

- (1) R is an intra-regular.
- (2) $A \cap L \subseteq L \circ A$ for every intuitionistic fuzzy quasi-ideal A and every intuitionistic fuzzy left ideal L of R .
- (3) $B \cap L \subseteq L \circ B$ for every intuitionistic fuzzy bi-ideal B and every intuitionistic fuzzy left ideal L of R .
- (4) $C \cap L \subseteq L \circ C$ for every intuitionistic fuzzy generalized bi-ideal C and every intuitionistic fuzzy left ideal L of R .

Proof: Assume that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal and $L = (\mu_L, \gamma_L)$ be an intuitionistic fuzzy left ideal of R . Let $x \in R$, this means that there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^n (a_i x^2) b_i$. Now $x = (a_i (xx)) b_i = (x(a_i x)) b_i = (b_i (a_i x)) x$. Thus

$$\begin{aligned} (\mu_L \circ \mu_C)(x) &= \bigvee_{x = \sum_{i=1}^n p_i q_i} \{ \bigwedge_{i=1}^n \{ \mu_L(p_i) \wedge \mu_C(q_i) \} \} \\ &\geq \mu_L(b_i(a_i x)) \wedge \mu_C(x) \geq \mu_L(x) \wedge \mu_C(x) \\ &= \mu_C(x) \wedge \mu_L(x) = (\mu_C \cap \mu_L)(x). \\ &\Rightarrow \mu_C \cap \mu_L \subseteq \mu_L \circ \mu_C. \end{aligned}$$

Similarly, we have $\gamma_C \cup \gamma_L \supseteq \gamma_L \circ \gamma_C$. Hence $C \cap L \subseteq L \circ C$, i.e., (1) $\Rightarrow$ (4). It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Suppose that (2) holds. Let A be an intuitionistic fuzzy right ideal and L be an intuitionistic fuzzy left ideal of R . Since every intuitionistic fuzzy right ideal of R is an intuitionistic fuzzy quasi-ideal of R , this implies that A is an intuitionistic fuzzy quasi-ideal of R . By our supposition, $A \cap L \subseteq L \circ A$. Thus R is an intra-regular by the Theorem 4.5, i.e., (2) $\Rightarrow$ (1).

5 CONCLUSION

Our ambition is to inspire the study and maturity of non associative algebraic structure (LA-ring). The objective is to explain original methodological developments on ordered LA-rings, which will be very helpful for upcoming theory of algebraic structure. The ideal of fuzzy set to the characterizations of LA-rings are captivating a great attention of algebraist.

The aim of this paper is to investigate, the study of (regular, intra-regular) LA-rings by using of fuzzy left (right, interior, quasi-, bi-, generalized bi-) ideals.

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