A BREF SURVEY ON ARMENDARIZ AND CENTRAL ARMENDARIZ RINGS

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Summary. In this paper R is a ring with unity, and σ is endomorphism of the ring. We deal with central Armendariz rings as a natural generalization of Armendariz rings. We investigate a possibility of extending central Armendariz property from a ring to corresponding polynomial or matrix extension. At the end of this paper we consider an interesting note on reduced rings.

1 INTRODUCTION

Throughout this article R denotes a ring with unity, $R[x]$ is corresponding polynomial ring, σ denotes an endomorphism of R , $R[x; \sigma]$ denotes skew polynomial ring with the ordinary addition and the multiplication subject to the relation $xr = \sigma(r)x$, and $R[[x; \sigma]]$ denotes power series ring. The notion of Armendariz ring is introduced by Rege and Chhawchharia (see [2]). They defined a ring R to be Armendariz if $f(x)g(x) = 0$ implies $a_i b_j = 0$, for all polynomials $f(x) = \sum_{i=0}^n a_i x^i$ and $g(x) = \sum_{j=0}^m b_j x^j$ from $R[x]$. The motivation for those rings comes from the fact that Armendariz had shown that the above result can be extended to a class of reduced rings, i.e., rings without non-zero nilpotent elements. In [1] authors introduced a class of central Armendariz rings. A ring R is called central Armendariz ring if $f(x)g(x) = 0$ implies $a_i b_j \in C(R)$, for all polynomials $f(x) = \sum_{i=0}^n a_i x^i$ and $g(x) = \sum_{j=0}^m b_j x^j$ from $R[x]$, where $C(R)$ is center of a ring R . Clearly Armendariz rings are central Armendariz rings. It is known from [1] that a class of central Armendariz rings is closed for polynomial extensions and localizations, and that the central Armendariz rings are strictly between Armendariz rings and abelian rings. As a generalization of σ -skew Armendariz rings, Onyang (see [4]) introduced a notion of weak σ -skew Armendariz ring (see [3],[4],[5]). A weak σ -skew Armendariz ring R is a ring in which $f(x)g(x) = 0$ implies $a_i \sigma^i(b_j)$ is the nilpotent element of R for all $f(x) = \sum_{i=0}^n a_i x^i$ and $g(x) = \sum_{j=0}^m b_j x^j$ from $R[x; \sigma]$. Chain and Tong (see [5]) have proved that if R and S are rings and σ is an isomorphism of rings R and S and R is α -skew Armendariz ring, then S is $\sigma\alpha\sigma^{-1}$ -skew Armendariz ring. In this paper we give (see [3]) a variant of this theorem for weak skew-Armendariz rings. In our main result we give an example of central Armendariz matrix ring $T(n, R)$, for reduced ring R .

2 EXTENDING OF ARMENDARIZ PROPERTY

In this section we deal with possibility of extending the Armendariz property under ring isomorphism (see [3]). From universal algebra we know that every homomorphism σ of rings R and S can be extended to the homomorphism of the corresponding rings of polynomials $R[x]$ and $S[x]$ by $\sum_{i=0}^m a_i x^i \mapsto \sum_{i=0}^m \sigma(a_i) x^i$, which we also denote by σ . Chain and Tong in

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[5] prove that if σ is ring isomorphism of rings R and S and R is α -skew Armendariz, then S is $\sigma\alpha\sigma^{-1}$ skew Armendariz ring. We prove the weak skew Armendariz variant of this theorem.

Theorem 2.1 ([3]) *Let R and S be rings with a ring isomorphism $\sigma: R \rightarrow S$. If R is weak α -skew Armendariz then S is weak $\sigma\alpha\sigma^{-1}$ -skew Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^m a_i x^i$ and $g(x) = \sum_{j=0}^n b_j x^j$ be polynomials from the ring $S[x; \sigma\alpha\sigma^{-1}]$. We have to prove that $f(x)g(x) = 0$ implies $a_i(\sigma\alpha\sigma^{-1})^i b_j \in \text{nil}(S)$, for all i and j .

As we noted, σ extends to the isomorphism of corresponding polynomial rings, so that there exist polynomials $f_1(x) = \sum_{i=0}^m a'_i x^i$ and $g_1(x) = \sum_{j=0}^n b'_j x^j$ from $R[x]$ such that $f(x) = \sigma(f_1(x)) = \sum_{i=0}^m \sigma(a'_i) x^i$ and $g(x) = \sigma(g_1(x)) = \sum_{j=0}^n \sigma(b'_j) x^j$.

First, we shall show that $f(x)g(x) = 0$ implies $f_1(x)g_1(x) = 0$. If $f(x)g(x) = 0$, we have

$$a_0 b_k + a_1(\sigma\alpha\sigma^{-1})(b_{k-1}) + \dots + a_k(\sigma\alpha\sigma^{-1})^k(b_0) = 0,$$

for any $0 \leq k \leq m$. From the definition of $f_1(x)$ and $g_1(x)$, we have

$$\sigma(a'_0)\sigma(b'_k) + \sigma(a'_1)(\sigma\alpha\sigma^{-1})\sigma(b'_{k-1}) + \dots + \sigma(a'_k)(\sigma\alpha\sigma^{-1})^k\sigma(b'_0) = 0,$$

and using $(\sigma\alpha\sigma^{-1})^t = \sigma\alpha^t\sigma^{-1}$ we obtain

$$a'_0 b'_k + a'_1 \alpha(b'_{k-1}) + \dots + a'_k \alpha^k(b'_0) = 0,$$

which means that $f_1(x)g_1(x) = 0$ in the ring $R[x; \alpha]$.

It remains to prove that $f_1(x)g_1(x) = 0$ implies $a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in \text{nil}(S)$. From the fact that R is weak α -skew Armendariz we have $a'_i \alpha^i(b'_j) \in \text{nil}(R)$, and since

$$a'_i = \sigma^{-1}(a_i), b'_j = \sigma^{-1}(b_j), \text{ we have } \sigma^{-1}(a_i) \alpha^i \sigma^{-1}(b_j) \in \text{nil}(R).$$

This implies

$$\sigma^{-1}(a_i) \sigma^{-1} \sigma \alpha^i \sigma^{-1}(b_j) = \sigma^{-1}(a_i (\sigma\alpha\sigma^{-1})^i(b_j)) \in \text{nil}(R)$$

and finally we obtain

$$a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in \text{nil}(S), \quad 0 \leq i \leq m, 0 \leq j \leq n.$$

Hence S is weak $\sigma\alpha\sigma^{-1}$ -skew Armendariz.

3 MATRIX CENTRAL ARMENDARIZ RING $T(R, n)$

In this section we give an example of matrix central Armendariz ring. For a ring R consider a following set of triangular matrices

$$T_n(R) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \mid a_{ij} \in R \right\}.$$

We also consider the following set of triangular matrices over ring R

$$T(R, n) = \left\{ \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & a_0 & a_1 & \dots & a_{n-2} \\ 0 & 0 & a_0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_0 \end{bmatrix} \mid a_i \in R \right\},$$

which is subring of $T_n(R)$. It is well known that $T_n(R)$ and $T(R, n)$ are subrings of the triangular matrix rings with matrix addition and multiplication. Let α be endomorphism of ring R . It is well known that endomorphism α can be naturally extended to an endomorphism

$$\bar{\alpha}: T_n(R) \rightarrow T_n(R),$$

and

$$\bar{\alpha}: T(R, n) \rightarrow T(R, n),$$

with:

$$\bar{\alpha} \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \right) = \begin{bmatrix} \alpha(a_{11}) & \alpha(a_{12}) & \alpha(a_{13}) & \dots & \alpha(a_{1n}) \\ 0 & \alpha(a_{22}) & \alpha(a_{23}) & \dots & \alpha(a_{2n}) \\ 0 & 0 & \alpha(a_{33}) & \dots & \alpha(a_{3n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha(a_{nn}) \end{bmatrix},$$

and

$$\bar{\alpha} \left(\begin{bmatrix} a_0 & a_1 & a_{13} & \dots & a_{n-1} \\ 0 & a_0 & a_1 & \dots & a_{n-2} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_0 \end{bmatrix} \right) = \begin{bmatrix} \alpha(a_0) & \alpha(a_1) & \alpha(a_2) & \dots & \alpha(a_{n-1}) \\ 0 & \alpha(a_0) & \alpha(a_1) & \dots & \alpha(a_{n-2}) \\ 0 & 0 & \alpha(a_{33}) & \dots & \alpha(a_{3n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha(a_0) \end{bmatrix}.$$

Theorem 3.1 *If R is reduced ring then $T(R, n)$ is central Armendariz ring.*

Proof. From [1] we obtain that for reduced ring R , the factor ring $R[x]/(x^n)$ is central Armendariz, for all $n \geq 2$. We use the ring isomorphism $f: R[x]/(x^n) \rightarrow T(R, n)$ given by

$$f(a_0 + a_1x + \dots + a_{n-1}x^{n-1} + (x^n)) = (a_0, a_1, \dots, a_{n-1}),$$

where (x^n) is ideal in $R[x]$ generated with x^n , and $(a_0, a_1, \dots, a_{n-1})$ is a brief representation for a matrix from $T(R, n)$. Therefore $T(R, n)$ is central Armendariz ring.

We end this section with our result from [3], in which we give sufficient condition for the power series ring $R[[x; \sigma]]$ to be reduced.

Theorem 3.2 *If an endomorphism σ of a reduced ring R satisfies so called compatibility condition: $a\sigma(b) = 0 \Leftrightarrow ab = 0$, then the power series ring $R[[x; \sigma]]$ is reduced.*

Proof. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $(f(x))^2 = 0$. It is clear that $a_0^2 = 0$, so that $a_0 = 0$. Now from the compatibility condition $a_1 \sigma(a_1) = 0$ implies $a_1^2 = 0$, but since R is reduced we have $a_1 = 0$. By induction argument we have $a_i = 0$ for all i . This means that $f(x) = 0$ and so $R[[x; \sigma]]$ is reduced.

Without compatibility condition the previous theorem is not true. Since for the ring $R = Z_2 \oplus Z_2$ and σ defined by $\sigma(a, b) = (b, a)$, it is easy to check that $R[[x; \sigma]]$ is not reduced. Observe that $(1, 0)(0, 1) = (0, 0)$ but $(1, 0)\sigma(0, 1) \neq (0, 0)$.

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