A NEW FORM OF THE ANTISYMMETRIC TENSOR AND THE COMPLETE SYSTEM OF EQUATIONS OF THE ELECTROMAGNETIC FIELD

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Abstract: a new form of the equivalent representation of the canonical antisymmetric tensor of the electromagnetic field is obtained. This form of representation is based on the decomposition of the asymmetric tensor of a general form into a symmetric and antisymmetric part. It follows from this expansion that the canonical antisymmetric tensor of the electromagnetic field can be equivalently represented as the difference between an asymmetric tensor of a general form and a symmetric tensor. Then Maxwell's equations can be written in the form of four-dimensional divergences of these tensors. From this representation, in addition to the Maxwell equations, new equations of the electromagnetic field also follow, expanding knowledge of it. One of these equations is the equation of motion of the electromagnetic field in the form of the dynamic Navier-Stokes equation.

Keywords: asymmetric tensor of the electromagnetic field, a symmetric tensor of the electromagnetic field, Maxwell equations.

НОВАЯ ФОРМА АНТИСИММЕТРИЧНОГО ТЕНЗОРА И ПОЛНАЯ СИСТЕМА УРАВНЕНИЙ ЭЛЕКТРОМАГНИТНОГО ПОЛЯ

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Аннотация: получена новая форма эквивалентного представления канонического антисимметричного тензора электромагнитного поля. Эта форма представления основана на разложении несимметричного тензора общего вида на симметричную и антисимметричную части. Из этого разложения следует, что канонический антисимметричный тензор электромагнитного поля можно эквивалентно представить в виде разности несимметричного тензора общего вида и симметричного тензора. Тогда уравнения Максвелла можно записать в виде четырехмерных дивергенций этих тензоров. Из этого представления кроме уравнений Максвелла следуют и новые уравнения электромагнитного поля, расширяющие знания о нем. Одним из таких уравнений является уравнение движения электромагнитного поля в форме динамического уравнения Навье – Стокса.

Ключевые слова: несимметричный тензор электромагнитного поля, симметричный тензор электромагнитного поля, уравнения Максвелла.

1. Introduction

The canonical representation of the antisymmetric electromagnetic field tensor is $F_{[\nu\mu]} = \partial_\nu A_\mu - \partial_\mu A_\nu$, where A_μ - is the four-dimensional electromagnetic potential, and ∂_ν - is the four-dimensional differentiation operator. However, the canonical antisymmetric tensor of the electromagnetic field $F_{[\nu\mu]}$ may have another form of equivalent representation. In accordance with the rules of tensor algebra, using the operations of alternation and symmetrization, an asymmetric tensor of the second rank $F_{\nu\mu}$ can be uniquely decomposed into antisymmetric and symmetric tensors $F_{\nu\mu} = F_{[\nu\mu]} + F_{(\nu\mu)}$. Then the antisymmetric tensor $F_{[\nu\mu]}$ of the electromagnetic field can be uniquely and equivalently written in the form of a difference of tensors $F_{[\nu\mu]} = 2F_{\nu\mu} - F_{(\nu\mu)}$, where $F_{(\nu\mu)}$ - the symmetric tensor. The Maxwell's equations follow from the antisymmetric tensor $F_{[\nu\mu]}$ of the electromagnetic field in the form of its four-dimensional divergence. It follows from the new form of the tensor $F_{[\nu\mu]}$ representation that Maxwell's equations can be equivalently written in the form of a difference of the

equations that are four-dimensional divergences of the tensors $F_{\nu\mu}$ and $F_{(\nu\mu)}$. New equations that expand knowledge of the electromagnetic field follow from the new form of the antisymmetric tensor of electromagnetic field.

The purpose of this article is to obtain of the electromagnetic field equations from the new form the antisymmetric tensor $F_{[\nu\mu]} = 2F_{\nu\mu} - F_{(\nu\mu)}$.

The electromagnetic field and electric charges are considered in a vacuum. The geometry of space-time is taken in the form of pseudo-Euclidean Minkowski space (ct, ix, iy, iz) [1]. The electromagnetic potential will be written in the form $\mathbf{A}_\mu$ ($\varphi / c, i\mathbf{A}$), where φ and $\mathbf{A}$ are the scalar and vector potentials of the electromagnetic field.

2. A new form of the antisymmetric electromagnetic field tensor.

The canonical antisymmetric tensor $F_{[\nu\mu]}$ for the electromagnetic field will be written in the matrix representation:

$$F_{[\nu\mu]} = \begin{pmatrix} 0 & \frac{1}{c}i \cdot (\partial_t A_x + \partial_x \varphi) & \frac{1}{c}i \cdot (\partial_t A_y + \partial_y \varphi) & \frac{1}{c}i \cdot (\partial_t A_z + \partial_z \varphi) \\ -\frac{1}{c}i \cdot (\partial_t A_x + \partial_x \varphi) & 0 & (\partial_x A_y - \partial_y A_x) & (\partial_x A_z - \partial_z A_x) \\ -\frac{1}{c}i \cdot (\partial_t A_y + \partial_y \varphi) & -(\partial_x A_y - \partial_y A_x) & 0 & (\partial_y A_z - \partial_z A_y) \\ -\frac{1}{c}i \cdot (\partial_t A_z + \partial_z \varphi) & -(\partial_x A_z - \partial_z A_x) & -(\partial_y A_z - \partial_z A_y) & 0 \end{pmatrix} \quad (1)$$

An asymmetric tensor of the second rank can be decomposed into symmetric and antisymmetric parts $F_{\nu\mu} = (\partial_\nu \mathbf{A}_\mu + \partial_\mu \mathbf{A}_\nu) / 2 + (\partial_\nu \mathbf{A}_\mu - \partial_\mu \mathbf{A}_\nu) / 2$. Using this decomposition and the tensor (1), the symmetric tensor $F_{(\nu\mu)}$ will be written as:

$$F_{(\nu\mu)} = \begin{pmatrix} 2\frac{1}{c^2}\partial_t \varphi & \frac{1}{c}i \cdot (\partial_t A_x - \partial_x \varphi) & \frac{1}{c}i \cdot (\partial_t A_y - \partial_y \varphi) & \frac{1}{c}i \cdot (\partial_t A_z - \partial_z \varphi) \\ \frac{1}{c}i \cdot (\partial_t A_x - \partial_x \varphi) & 2\partial_x A_x & (\partial_x A_y + \partial_y A_x) & (\partial_x A_z + \partial_z A_x) \\ \frac{1}{c}i \cdot (\partial_t A_y - \partial_y \varphi) & (\partial_x A_y + \partial_y A_x) & 2\partial_y A_y & (\partial_y A_z + \partial_z A_y) \\ \frac{1}{c}i \cdot (\partial_t A_z - \partial_z \varphi) & (\partial_x A_z + \partial_z A_x) & (\partial_y A_z + \partial_z A_y) & 2\partial_z A_z \end{pmatrix} \quad (2)$$

Then the asymmetric tensor $F_{\nu\mu}$ is obtained by adding tensors (1) and (2):

$$F_{\nu\mu} = \partial_\nu A_\mu = \begin{pmatrix} \frac{1}{c^2}\partial_t \varphi & \frac{1}{c}i \cdot \partial_t A_x & \frac{1}{c}i \cdot \partial_t A_y & \frac{1}{c}i \cdot \partial_t A_z \\ -\frac{1}{c}i \cdot \partial_x \varphi & \partial_x A_x & \partial_x A_y & \partial_x A_z \\ -\frac{1}{c}i \cdot \partial_y \varphi & \partial_y A_x & \partial_y A_y & \partial_y A_z \\ -\frac{1}{c}i \cdot \partial_z \varphi & \partial_z A_x & \partial_z A_y & \partial_z A_z \end{pmatrix} \quad (3)$$

Now, using the form $F_{[\nu\mu]} = 2F_{\nu\mu} - F_{(\nu\mu)}$ of the antisymmetric tensor of the electromagnetic field, it will be written in the form:

$$F_{[\nu\mu]} = 2 \cdot \begin{pmatrix} \frac{1}{c^2}\partial_t \varphi & \frac{1}{c}i \cdot \partial_t A_x & \frac{1}{c}i \cdot \partial_t A_y & \frac{1}{c}i \cdot \partial_t A_z \\ -\frac{1}{c}i \cdot \partial_x \varphi & \partial_x A_x & \partial_x A_y & \partial_x A_z \\ -\frac{1}{c}i \cdot \partial_y \varphi & \partial_y A_x & \partial_y A_y & \partial_y A_z \\ -\frac{1}{c}i \cdot \partial_z \varphi & \partial_z A_x & \partial_z A_y & \partial_z A_z \end{pmatrix} - \begin{pmatrix} 2\frac{1}{c^2}\partial_t \varphi & \frac{1}{c}i \cdot (\partial_t A_x - \partial_x \varphi) & \frac{1}{c}i \cdot (\partial_t A_y - \partial_y \varphi) & \frac{1}{c}i \cdot (\partial_t A_z - \partial_z \varphi) \\ \frac{1}{c}i \cdot (\partial_t A_x - \partial_x \varphi) & 2\partial_x A_x & (\partial_x A_y + \partial_y A_x) & (\partial_x A_z + \partial_z A_x) \\ \frac{1}{c}i \cdot (\partial_t A_y - \partial_y \varphi) & (\partial_x A_y + \partial_y A_x) & 2\partial_y A_y & (\partial_y A_z + \partial_z A_y) \\ \frac{1}{c}i \cdot (\partial_t A_z - \partial_z \varphi) & (\partial_x A_z + \partial_z A_x) & (\partial_y A_z + \partial_z A_y) & 2\partial_z A_z \end{pmatrix}$$

3. New equations of the electromagnetic field

The Maxwell's equations follow from the antisymmetric tensor in the form of its four-dimensional divergence. In the absence of sources $\partial^\nu F_{[\nu\mu]} = 0$ (here covariant and contravariant indices may not be distinguished). We will write a new representation of the Maxwell's equations in the form of a four-dimensional divergence:

$$\partial^\nu F_{[\nu\mu]} = 2\partial^\nu F_{\nu\mu} - \partial^\nu F_{(\nu\mu)} = 0 \quad (4)$$

This equation can be written in an expanded form:

$$2\left(\frac{1}{c^2}\partial_{tt}\varphi - \Delta\varphi\right) - \left(2\frac{1}{c^2}\partial_{tt}\varphi + \partial_t\nabla\cdot\mathbf{A} - \Delta\varphi\right) = \nabla\cdot(-\nabla\varphi - \partial_t\mathbf{A}) = \nabla\cdot\mathbf{E} = 0 \quad (5)$$

$$\begin{aligned} & 2\left(\frac{1}{c^2}\partial_{tt}\mathbf{A} - \Delta\mathbf{A}\right) - \left(\frac{1}{c^2}\partial_{tt}\mathbf{A} - \frac{1}{c^2}\partial_t\nabla\varphi - \nabla(\nabla\cdot\mathbf{A}) - \Delta\mathbf{A}\right) = \\ & = \frac{1}{c^2}\partial_t(\partial_t\mathbf{A} + \nabla\varphi) + \nabla\times\nabla\times\mathbf{A} = -\frac{1}{c^2}\partial_t\mathbf{E} + \nabla\times\mathbf{B} = 0 \end{aligned} \quad (6)$$

where, $\mathbf{E}$ and $\mathbf{B}$ is the strength of the electric field and the induction of the magnetic field respectively.

For a second-rank tensor, there are divergences for each of the indices. Equation (4) is a divergence with respect to the index ν . Let us find the divergence of the antisymmetric tensor $F_{[\nu\mu]}$ with respect to the index μ :

$$\partial^\mu F_{[\nu\mu]} = 2\partial^\mu F_{\nu\mu} - \partial^\mu F_{(\nu\mu)} = 0 \quad (7)$$

Obviously, because of the antisymmetric of the tensor $F_{[\nu\mu]}$, we must obtain the same result as in the previous case, but with the opposite sign. Equation (7) can be written in expanded form:

$$2\partial_t\left(\frac{1}{c^2}\partial_t\varphi + \nabla\cdot\mathbf{A}\right) - \left(2\frac{1}{c^2}\partial_{tt}\varphi + \partial_t\nabla\cdot\mathbf{A} - \Delta\varphi\right) = -\nabla\cdot(-\nabla\varphi - \partial_t\mathbf{A}) = -\nabla\cdot\mathbf{E} = 0 \quad (8)$$

$$\begin{aligned} & 2(-\nabla\left(\frac{1}{c^2}\partial_t\varphi + \nabla\cdot\mathbf{A}\right) - \left(\frac{1}{c^2}\partial_{tt}\mathbf{A} - \frac{1}{c^2}\partial_t\nabla\varphi - \nabla(\nabla\cdot\mathbf{A}) - \Delta\mathbf{A}\right) = \\ & = \frac{1}{c^2}\partial_t(-\partial_t\mathbf{A} - \nabla\varphi) - \nabla\times\nabla\times\partial_t\mathbf{A} = \frac{1}{c^2}\partial_t\mathbf{E} - \nabla\times\mathbf{B} = 0 \end{aligned} \quad (9)$$

Thus, we also obtained the Maxwell's equations. Equations (5) and (6) differ from equations (8) and (9). This is due to the fact that the divergences of the asymmetric tensor (3) $F_{\nu\mu}$ with respect to different indices are different. Let us find the divergence of this tensor from equations (4) and (7):

$$\partial^\nu F_{\nu\mu} = \partial^\nu F_{[\nu\mu]} / 2 + \partial^\nu F_{(\nu\mu)} / 2 = 0 \quad \text{and} \quad \partial^\mu F_{\nu\mu} = \partial^\mu F_{[\nu\mu]} / 2 + \partial^\mu F_{(\nu\mu)} / 2 = 0$$

We write these equations in expanded form [2]:

$$\frac{1}{c^2}\partial_{tt}\varphi - \Delta\varphi = 0 \quad (10)$$

$$\frac{1}{c^2}\partial_{tt}\mathbf{A} - \Delta\mathbf{A} = 0 \quad (11)$$

$$\partial_t\left(\frac{1}{c^2}\partial_t\varphi + \nabla\cdot\mathbf{A}\right) = 0 \quad (12)$$

$$-\nabla\left(\frac{1}{c^2}\partial_t\varphi + \nabla\cdot\mathbf{A}\right) = 0 \quad (13)$$

Equations (10) and (11) are the Maxwell's equations in the Lorentz gauge [2], and equations (12) and (13) are the four-dimensional derivative of the Lorentz gauge condition $\partial_t\varphi/c^2 + \nabla\cdot\mathbf{A} = 0$. In electrodynamics equations (10) and (11) are obtained by applying this condition. From the asymmetric tensor (3) Maxwell's equations and this gauge condition follow as the equations of the electromagnetic field, without additional assumptions and justifications. This suggests that the Lorentz gauge condition is not just a mathematical device, but is a rightful equation of the electromagnetic field and has a deep physical meaning, which will be discussed below.

From the equation (4) we find the divergence of the symmetric tensor:

$$\partial^\nu F_{(\nu\mu)} = 2\partial^\nu F_{\nu\mu} - \partial^\nu F_{[\nu\mu]} = 0$$

Let's write it in expanded form:

$$2\frac{1}{c^2}\partial_{tt}\varphi + \partial_t\nabla\cdot\mathbf{A} - \Delta\varphi = 0 \quad (14)$$

$$\frac{1}{c^2}\partial_{tt}\mathbf{A} - \frac{1}{c^2}\partial_t\nabla\varphi - \nabla(\nabla\cdot\mathbf{A}) - \Delta\mathbf{A} = 0 \quad (15)$$

Equation (15) is the electromagnetic analog of the equation of motion of an isotropic elastic medium [3] or the dynamic equation of Navier-Stokes motion. Here, this equation shows the generality of the laws of motion of different types of matter.

The canonical antisymmetric tensor $F_{[\nu\mu]}$ of the electromagnetic field is a four-dimensional rotor that is it describes the four-dimensional rotation of the electromagnetic field. In the physics of continuous media, a symmetric tensor describes the deformation of a medium. Then, for an electromagnetic field, the symmetric tensor $F_{(\nu\mu)}$ (2) and the equations (14) and (15) following from it describe the four-dimensional deformation of the field. Thus, these tensors complement one another in describing the four-dimensional motion of the electromagnetic field. The diagonal terms of the symmetric tensor (2) are the terms of the Lorentz gauge condition. It follows that this condition describes a four-dimensional volume deformation of the expansion-contraction of the electromagnetic field.

The Maxwell's equations with field sources in the form of the electric charge density ρ and the current density $\mathbf{J}$ are written in the form [2]:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho / \varepsilon_0 & \text{or} & & -\partial_t \nabla \cdot \mathbf{A} - \Delta \varphi &= \rho / \varepsilon_0 \\ \mu_0 \cdot \mathbf{J} + \frac{1}{c^2} \partial_t \mathbf{E} &= \nabla \times \mathbf{B} & \text{or} & & \frac{1}{c^2} \partial_{tt} \mathbf{A} + \frac{1}{c^2} \partial_t \nabla \varphi + \nabla \times \nabla \times \mathbf{A} &= \mu_0 \cdot \mathbf{J} \\ \frac{1}{c^2} \partial_{tt} \varphi - \Delta \varphi &= \rho / \varepsilon_0 & & & \frac{1}{c^2} \partial_{tt} \mathbf{A} - \Delta \mathbf{A} &= \mu_0 \cdot \mathbf{J} \end{aligned}$$

Then equations (14) and (15) with the sources of the field will have the form:

$$2 \frac{1}{c^2} \partial_{tt} \varphi + \partial_t \nabla \cdot \mathbf{A} - \Delta \varphi = \rho / \varepsilon_0 \quad \frac{1}{c^2} \partial_{tt} \mathbf{A} - \frac{1}{c^2} \partial_t \nabla \varphi - \nabla (\nabla \cdot \mathbf{A}) - \Delta \mathbf{A} = \mu_0 \cdot \mathbf{J} \quad \text{Thus,}$$

Maxwell's equations (5), (6), (10), (11) form a connected system with equations (12), (13), (14) and (15) and these equations have the same right to exist in Electrodynamics, as well as Maxwell's equations. All equations of this system can be written in two compressed equivalent versions:

$$1) \quad \partial^\nu F_{\nu\mu} = \mathbf{J}_\nu \quad \partial^\mu F_{\nu\mu} = 0 \quad \text{and} \quad 2) \quad \partial^\nu F_{[\nu\mu]} = \mathbf{J}_\nu \quad \partial^\nu F_{(\nu\mu)} = \mathbf{J}_\nu$$

To transfer from one variant to another we should add and deduct in pairs equations of corresponding variant. The first version of these equations in expanded form is:

$$\begin{aligned} \frac{1}{c^2} \partial_{tt} \varphi - \Delta \varphi &= \rho / \varepsilon_0 & \frac{1}{c^2} \partial_{tt} \mathbf{A} - \Delta \mathbf{A} &= \mu_0 \cdot \mathbf{J} \\ \partial_t \left(\frac{1}{c^2} \partial_t \varphi + \nabla \cdot \mathbf{A} \right) &= 0 & -\nabla \left(\frac{1}{c^2} \partial_t \varphi + \nabla \cdot \mathbf{A} \right) &= 0 \end{aligned}$$

The second version of these equations in expanded form is:

$$\begin{aligned} -\partial_t \nabla \cdot \mathbf{A} - \Delta \varphi &= \rho / \varepsilon_0 & \frac{1}{c^2} \partial_{tt} \mathbf{A} + \frac{1}{c^2} \partial_t \nabla \varphi + \nabla \times \nabla \times \mathbf{A} &= \mu_0 \cdot \mathbf{J} \\ 2 \frac{1}{c^2} \partial_{tt} \varphi + \partial_t \nabla \cdot \mathbf{A} - \Delta \varphi &= \rho / \varepsilon_0 & \frac{1}{c^2} \partial_{tt} \mathbf{A} - \frac{1}{c^2} \partial_t \nabla \varphi - \nabla (\nabla \cdot \mathbf{A}) - \Delta \mathbf{A} &= \mu_0 \cdot \mathbf{J} \end{aligned}$$

Each of these versions includes Maxwell's canonical equations. Which of these equivalent options to choose is determined by the specific task of the study. The second variant is interesting in that, it describes not only the electromagnetic field rotation, but also its deformation, which is not considered in the courses of electrodynamics. The dynamic Navier-Stokes equation also describes the wave motions of a continuous medium, hence, this also applies to its electromagnetic analog (15). This is obvious, since the application of the Lorentz gauge or equation (13) to it leads to Maxwell's wave equation (11).

4. Conclusion

A new form of representation of the canonical antisymmetric tensor of the electromagnetic field in the form $F_{[\nu\mu]} = 2F_{\nu\mu} - F_{(\nu\mu)}$ has new consequences. It follows that Maxwell's equations can be equivalently written in the form of a difference of equations that are four-dimensional divergences of tensors $F_{\nu\mu}$ and $F_{(\nu\mu)}$. The new form of the tensor $F_{[\nu\mu]}$ is followed by the equation of motion of the electromagnetic field in the form of the dynamic Navier-Stokes equation, which shows the unity of the laws of nature. This form of the antisymmetric electromagnetic field tensor makes it possible to give a physical interpretation of the Lorentz gauge condition as a

description of the four-dimensional volumetric deformation of the electromagnetic field, and equitably include this condition in the system of equations of electrodynamics.

Thus, the new form of the canonical antisymmetric tensor and the equations that follow from it expand the possibilities of describing the electromagnetic field and its understanding.

References / Список литературы

1. *Tonella M.A.*, Fundamentals of Electromagnetism and Relativity Theory. «Foreign Literature», 1962.
2. *Landau L.D., Lifshits E.M.* The Classical Theory of Fields. Oxford: Pergamon Press, 1983.
3. *Landau L.D., Lifshits E.M.* The Theory of elasticity. Oxford: Pergamon Press, 1983.