ON THE COMPOSITION OF MAPPINGS POSSESSING THE INTERSECTION PROPERTY

Sevryuk M.B. Email: Sevryuk1145@scientifictext.ru

Sevryuk Mikhail Borisovich — Doctor of Physical and Mathematical Sciences, Principal Researcher,
LABORATORY FOR DYNAMICS OF ELEMENTARY PROCESSES,
V.L. TALROZE INSTITUTE OF ENERGY PROBLEMS OF CHEMICAL PHYSICS
RUSSIAN ACADEMY OF SCIENCES, MOSCOW

Abstract: it is well known that in the theory of dynamical systems, many theorems concerning area-preserving mappings can be carried over to mappings possessing the so-called intersection property: any simple closed curve that is not nullhomotopic intersects its image. However, the mappings with the intersection property do not constitute a group. In this note, we construct an explicit example of two analytic mappings of a two-dimensional annulus such that they possess the intersection property but their composition does not. One of these mappings is the rotation through an angle of 90° .

Keywords: intersection property, annulus, analytic mapping, composition.

О КОМПОЗИЦИИ ОТОБРАЖЕНИЙ СО СВОЙСТВОМ ПЕРЕСЕЧЕНИЯ Севрюк М.Б.

Севрюк Михаил Борисович — доктор физико-математических наук, главный научный сотрудник,
лаборатория динамики элементарных процессов,
Институт энергетических проблем химической физики им. В.Л. Тальрозе
Российская академия наук, г. Москва

Аннотация: хорошо известно, что многие теоремы теории динамических систем, касающиеся отображений, сохраняющих площади, можно перенести на отображения, обладающие т.н. свойством пересечения: любая простая замкнутая кривая, не гомотопная нулю, пересекается со своим образом. Однако отображения со свойством пересечения не образуют группу. В настоящей заметке строится явный пример двух аналитических отображений двумерного кольца, обладающих свойством пересечения, композиция которых свойством пересечения не обладает. Одно из этих отображений — поворот на угол 90° .

Ключевые слова: свойство пересечения, кольцо, аналитическое отображение, композиция.

UDC: 517.938.5

It is well known that in the theory of planar dynamical systems, many theorems concerning globally area-preserving mappings can be carried over to mappings possessing the so-called intersection property: any simple closed curve that is not nullhomotopic intersects its image (see e.g. [1, 2]). However, the mappings possessing the intersection property do not constitute a group: the composition of two mappings with this property does not necessarily possess it. Of course, all the experts realize this fact but I have failed to find any explicit counterexample in the literature. In this note, we construct two explicit analytic diffeomorphisms of the annulus

$A = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 25\}$ such that each of them possesses the intersection property but their composition does not. Moreover, one of them is the (counterclockwise, for definiteness) rigid rotation $R: (x, y) \mapsto (-y, x)$ over an angle of $\pi/2$.

Let Γ be the ellipse $\{x^2/4 + y^2/16 = 1\} \subset A$. The key ingredient of our example is an explicit analytic diffeomorphism $G: A \rightarrow A$ that leaves the ellipse $R(\Gamma) = \{x^2/16 + y^2/4 = 1\}$ and the boundary circles $\{x^2 + y^2 = 1\}$ and $\{x^2 + y^2 = 25\}$ invariant and takes the ellipse Γ to a curve Δ lying strictly inside Γ (see fig. 1).

Fig. 1. The key curves in our counterexample

In the annulus A , introduce the new coordinate frame $(\lambda, \varphi \bmod 2\pi)$ with $1 \leq \lambda \leq 5$ according to the formulas

$$x = f(\lambda) \cos \varphi, \quad y = g(\lambda) \sin \varphi, \quad f(\lambda) = \lambda + \xi(\lambda), \quad g(\lambda) = \lambda - \xi(\lambda), \quad (1)$$

$$\xi(\lambda) = \sin \frac{\pi}{4} (\lambda - 1). \quad (2)$$

Since the derivative $\xi'(\lambda) = \frac{\pi}{4} \cos \frac{\pi}{4} (\lambda - 1)$ never exceeds $\pi/4 < 1$ in absolute value, $\xi(1) = \xi(5) = 0$, and $\xi(3) = 1$, one concludes that $f'(\lambda) > 0$ and $g'(\lambda) > 0$ for all λ and $f(1) = g(1) = 1$, $f(5) = g(5) = 5$, $f(3) = 4$, and $g(3) = 2$. Consequently, the coordinate frame (λ, φ) is well defined and $\{x^2 + y^2 = 1\} = \{\lambda = 1\}$, $\{x^2 + y^2 = 25\} = \{\lambda = 5\}$, $R(\Gamma) = \{\lambda = 3\}$.

Now set

$$u(\lambda) = \frac{\cos \pi(\lambda - 1) - 1}{4}. \quad (3)$$

The derivative $u'(\lambda) = \frac{\pi}{4} \sin \pi(1 - \lambda)$ never exceeds $\pi/4 < 1$ in absolute value. The function u itself is negative everywhere on the interval $[1, 5]$ with the exception of the points $\lambda = 1, 3, 5$ where it vanishes. Thus, the mapping $\lambda \mapsto \lambda + u(\lambda)$ is a diffeomorphism of the interval $[1, 5]$ that leaves the points $\lambda = 1, 3, 5$ fixed and moves all the other points towards the left.

Finally, set

$$v(\varphi) = \frac{\sin 2\varphi}{3}. \quad (4)$$

Then the derivative $v'(\varphi) = \frac{2}{3} \cos 2\varphi$ never exceeds $2/3 < 1$ in absolute value. Moreover, the function v on the interval $[0, 2\pi]$ vanishes at the points $0, \pi/2, 3\pi/2, 2\pi$, is positive for $0 < \varphi < \pi/2$ and $\pi < \varphi < 3\pi/2$, and is negative for $\pi/2 < \varphi < \pi$ and $3\pi/2 < \varphi < 2\pi$. Therefore, the mapping $\varphi \mapsto \varphi + v(\varphi)$ is a diffeomorphism of the circle $\mathbf{R}/2\pi\mathbf{Z}$ with fixed points $0, \pi$ (repellers) and $\pi/2, 3\pi/2$ (attractors).

The diffeomorphism

$$G : (\lambda, \varphi) \mapsto (\lambda + u(\lambda), \varphi + v(\varphi)) \quad (5)$$

of the annulus A is the desired one.

The diffeomorphism $S = GRG^{-1} : A \rightarrow A$ possesses the intersection property (because R does) and takes $R(\Gamma)$ to Δ and vice versa:

$$SR(\Gamma) = GRG^{-1}R(\Gamma) = GRR(\Gamma) = G(\Gamma) = \Delta, \quad (6)$$

$$S(\Delta) = GRG^{-1}(\Delta) = GR(\Gamma) = R(\Gamma). \quad (7)$$

The composition RS takes the curve $R(\Gamma)$ to the curve $R(\Delta)$ which lies strictly inside $R(\Gamma)$ and takes the curve Δ to the curve Γ which lies strictly outside Δ (see fig. 1). Thus, this composition does not possess the intersection property.

In conclusion, we present a list [3–13] of several well known collections of counterexamples in various branches of mathematics.

References / Список литературы

1. Moser J. On invariant curves of area-preserving mappings of an annulus // Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, 1962. № 1. P. 1–20.
2. Le Calvez P. Dynamical properties of diffeomorphisms of the annulus and of the torus. Providence, RI: American Mathematical Society; Paris: Société Mathématique de France, 2000. x+105 p.
3. Boss V. Lectures on mathematics. V. 12. Counterexamples and paradoxes. 3rd ed. Moscow: Lenand, 2016. 224 p. (in Russian).
4. Bourchtein A., Bourchtein L. Counterexamples. From elementary calculus to the beginnings of analysis. Boca Raton, FL: CRC Press, 2015. xxix+331 p.
5. Bourchtein A., Bourchtein L. Counterexamples on uniform convergence. Sequences, series, functions, and integrals. Hoboken, NJ: John Wiley and Sons, 2017. xlix+220 p.
6. Gelbaum B.R., Olmsted J.M.H. Counterexamples in analysis. 3rd ed. Mineola, NY: Dover Publications, 2003. xxiv+195 p.
7. Meyer K.R. Counterexamples in dynamical systems via normal form theory // SIAM Rev. 1986. V. 28. № 1. P. 41–51.
8. Rassias J.M. Counterexamples in differential equations and related topics. Teaneck, NJ: World Scientific Publ., 1991. viii+184 p.
9. Shibinskii V.M. Examples and counterexamples in a course of calculus. Moscow: Vysshaya Shkola, 2007. 543 p. (in Russian).
10. Steen L.A., Seebach J.A. Counterexamples in topology. 3rd ed. Mineola, NY: Dover Publications, 1995. xii+244 p.
11. Stoyanov J.M. Counterexamples in probability. 2nd ed. Chichester: John Wiley and Sons, 1997. xxviii+342 p.
12. Székely G.J. Paradoxes in probability theory and mathematical statistics. Dordrecht: D. Reidel Publishing Co., 1986. xii+250 p.
13. Wise G.L., Hall E.B. Counterexamples in probability and real analysis. New York: The Clarendon Press, Oxford University Press, 1993. xii+211 p.