Response of Duffing's Oscillator to Harmonic Base Excitation and Significance of First Order Term

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Responses of systems with nonlinear stiffness subjected to base harmonic excitation are determined. An expression to estimate the amplitude in the fundamental frequency of oscillation is derived from first principles using Lindstedt's method. It is observed that the amplitude determined using the zero order approximation is in error at low frequencies. Therefore, an expression for the first order approximation of the amplitude of response at the fundamental frequency is derived. Zero order and first order approximation terms together form the response. Characteristics showing the variation of the amplitude with the excitation frequency for various nonlinear spring parameters are presented. The issue at low frequencies is resolved by the incorporation of the first order term. An expression for the phase difference and the expression of the asymptote where the responses converge are also derived.

NOMENCLATURE

k	stiffness
m	mass
c	coefficient of viscous damping
f_{spring}	spring restoring force
x	absolute displacement of the system
$\dot{x}$	velocity of the system
$\ddot{x}$	acceleration of the system
ϵ	small parameter, $\epsilon \ll 1$
α, β	parameters related to the nonlinear spring
$\epsilon\zeta$	damping factor
ω	parameter related to the natural frequency
y	base displacement
ϵY	amplitude of the base excitation
Ω	frequency of excitation
t	time
T	time period of oscillation
τ	modified time variable
φ	phase angle
ω_0	linearised natural frequency
x_i	i^{th} order term of the displacement, $i = 0, 1, 2, \dots$
φ_i	i^{th} order term of the phase angle, $i = 0, 1, 2, \dots$
A_i	i^{th} order term of the amplitude of the displacement, $i = 0, 1, 2, \dots$

1. INTRODUCTION

Many systems encountered in practice exhibit nonlinear behavior^{1,2} due to the change in their stiffness with deformation. This may be because of its geometrical construction or the characteristics of the material itself. Bi-linear and cubic are a couple of simple models that are used for representing such stiffness characteristics. A single degree-of-freedom (SDOF) system with a spring having cubic non-linearity is called a

Duffing's oscillator. Many equipment, machines, buildings etc. are provided with isolators to reduce their vibrations or the vibrations transmitted. Behavior of all such isolators can be represented by this model. Vaiana et. al.,³ Sudhir Kaul,⁴ Losanno et. al.⁵ and Madera Sierra et. al.⁶ used such models to represent the behavior of isolators, which show that such models are generally used for representing the nonlinear stiffness of isolators. These models find place even in the analysis of non-linear dampers⁷⁻⁹ and other devices.¹⁰

Determining the response of a Duffing's oscillator is a classic example of a nonlinear problem and a large amount of work is carried out on this subject. The text book by Meirovitch¹¹ gives a good insight into this problem. Considering the non-linearity to be weak, perturbation techniques are widely used for arriving at the solution. Lindstedt's method, method of averaging, harmonic balance method, method of multiple scales are the most commonly used perturbation techniques.^{12,13}

Using the above mentioned techniques, the free vibration characteristics and the response to harmonic force excitation are studied extensively. These studies have provided expressions for determining the frequency of the free vibration of the Duffing's oscillator. Expressions were also presented for the amplitude of the response when excited by a harmonic force.¹¹ The responses show 'jump phenomenon' when they are subjected to force excitation. There are several works that discuss on the jump phenomenon.¹⁴⁻¹⁸ Malatkar and Nayfeh¹⁶ obtained the minimum force that needs to be applied to make the system to have jumps in their responses. Brennan et.al.¹⁷ obtained expressions for determining the frequencies at which the jumps in the responses occur. Works are also reported in determining the parameters of the nonlinear system from its response characteristics using methods like nonlinear subspace identification¹⁸ etc. All these works are related to force excitation.

Many of such nonlinear systems exhibit hysteresis. A suitable model that represents this type of hysteresis is Bouc-Wen model. This model was proposed initially by Bouc¹⁹ and later generalized by Wen.^{20,21} In this model the restoring force is related to the displacement through a first order nonlinear differential equation. By assigning suitable values for the parameters, the required hysteresis loop is represented. Vaiana et. al.²²⁻²⁴ developed several models that provide significant computational advantages while representing the hysteresis behavior.

Though there are several works carried out on the response of nonlinear systems when subjected to force excitation as well as complex models that represent their hysteresis behavior, it is seen that the works reported on the response to base excitation are very few. Hundal²⁵ determined the response of systems to base excitation with nonlinear damping. Xiaojuan Sun and Jianrun Zhang²⁶ worked on the response of isolators with nonlinear damping. In these works, the stiffness was considered to be linear. Responses are also obtained for modulated sinusoidal excitation using method of multiple scales,²⁷ but it was for a force excitation and not for a base excitation. In another study²⁸ the response to base excitation of an oscillator is determined with a nonlinearity in the secondary spring. The main spring was still linear and the nonlinear spring connects only the damper.

Ravindra and Mallik^{29,30} obtained the response of a nonlinear system when subjected to base excitation. They used harmonic balance method for determining the solution. Zarko Milovanovic et.al.³¹ derived an expression for determining the amplitude of the response of a vibration isolator subjected to base excitation. Bahareh Zaghari et.al.³² determined the response to base excitation along with force excitation. In all the above works²⁹⁻³² the base excitation problem is written in the form of force excitation problems and the relative responses are determined using the expressions for force excitation. The absolute displacements are then determined from the relative displacements. While doing the above, the phase of the absolute displacement is assumed, thus it does not give an analytically correct solution. Other works reported on base excitation³³⁻³⁵ adopt numerical techniques to determine the solution and hence no expressions are derived. A few works reported^{36,37} on response to base excitation, deal with cantilevered beam with tip mass. In those works, numerical techniques are used to determine the solution and no expressions are derived. Also, the nonlinearities present in those investigations are not of the cubic type.

Though there are expressions reported for determining the amplitude of the response of Duffing's oscillator to base sine excitation,²⁹⁻³² they are derived with certain assumptions on the phase difference which is analytically not exact. In this work, the absolute response is derived from the first principles (not from the relative response) without any such assumption. Here, Lindstedt's perturbation technique which is mathematically rigorous is chosen to derive the expression for the amplitude. As in the previous studies the nonlinearity is assumed to be small. In previous works, only the zero order approximation was considered. It is shown that the first order approximation terms are also essential and expressions for these terms are derived. Using these expressions, characteristic curves for the

Figure 1. Duffing's oscillator with base excitation.

base sine excitation are obtained. These results are particularly useful for Isolators used in sensitive equipment / optical elements mounted in spacecraft / aircraft, isolators used in gyros, isolators in building etc.

2. ZERO ORDER APPROXIMATION OF RESPONSE

Lindstedt's method well established for determining the response to force excitation³ is used to obtain the response to base excitation.

Let us consider a quasi-harmonic system as shown in Fig. 1 subjected to a base harmonic excitation. It consists of a mass, a spring and a damper.

2.1. Equation of Motion

The spring force is given by¹¹:

$$f_{spring}(x) = kx + \epsilon k(\alpha x + \beta x^3); \quad (1)$$

where x is the absolute displacement. The first derivative of $f_{spring}(x)$ with respect to the displacement gives the stiffness of the spring and in general it is a function of the displacement. If β is +ve, stiffness increases with the displacement. Such springs are called hardening spring. In a softening spring the stiffness reduces with the displacement and β is -ve. If $\beta = 0$, the system will be linear and the corresponding stiffness is $k(1 + \epsilon\alpha)$.

The second-order ordinary differential equation of motion of a linear SDOF system having mass m , damping coefficient c and stiffness k , when subjected to a base displacement y , is:

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky. \quad (2)$$

When the spring force is as given in Eq. (1), the differential equation of motion becomes:

$$m\ddot{x} + c\dot{x} + k(1 + \epsilon\alpha)x + \epsilon k\beta x^3 = c\dot{y} + k(1 + \epsilon\alpha)y + \epsilon k\beta y^3. \quad (3)$$

The right hand side of the above equation has several terms with one nonlinear term whereas in the case of force excitation there is only one term and it is linear. Dividing Eq. (3) by the mass, we get:

$$\ddot{x} + \omega^2 x = \epsilon \left[-\omega^2 (\alpha x + \beta x^3) - 2\zeta \omega \dot{x} \right] + 2\epsilon \zeta \omega \dot{y} + \omega^2 y + \epsilon \omega^2 (\alpha y + \beta y^3); \quad (4)$$

where, $\omega^2 = k/m$ and $\epsilon\zeta = c/(2m\omega)$ is the damping factor.

For a harmonic base excitation of $y = \epsilon Y \cos \omega t$, the equation of motion in terms of absolute displacement can be shown to be:

$$\ddot{x} + \omega^2 x = \epsilon \left[-\omega^2 (\alpha x + \beta x^3) - 2\zeta\omega\dot{x} \right] - 2\epsilon\zeta\omega\Omega\epsilon Y \sin \Omega t + \omega^2 \epsilon Y \cos \Omega t + \epsilon\omega^2 (\epsilon\alpha Y \cos \Omega t + \beta\epsilon^3 Y^3 \cos^3 \Omega t). \quad (5)$$

Equation (5) represents the differential equation of motion of the Duffing's oscillator when subjected to base excitation.

2.2. Expression for Amplitude of the Absolute Displacement

Our aim is to obtain a periodic solution of Eq. (5). The period of excitation is $T = \frac{2\pi}{\Omega}$. Shifting the time scale t to τ , which is the new time variable so that the period of oscillation is 2π (and not $\frac{2\pi}{\Omega}$), the equation of motion changes to:

$$\Omega^2 x'' + \omega^2 x = \epsilon \left[-\omega^2 (\alpha x + \beta x^3) - 2\zeta\omega\Omega x' \right] - 2\epsilon\zeta\omega\Omega\epsilon Y \sin(\tau + \varphi) + \omega^2 \epsilon Y \cos(\tau + \varphi) + \epsilon\omega^2 [\epsilon\alpha Y \cos(\tau + \varphi) + \beta\epsilon^3 Y^3 \cos^3(\tau + \varphi)]; \quad (6)$$

where φ is the phase angle. While doing so we employed the transformations $\tau = \Omega t - \varphi$ and $\frac{d}{dt} = \Omega \frac{d}{d\tau}$.

For a solution as given below:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots; \quad (7)$$

$$\varphi = \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2 + \dots; \quad (8)$$

the equation of motion becomes (taking $x = x_0 + \epsilon x_1$):

$$\Omega^2 (x_0 + \epsilon x_1)'' + \omega^2 (x_0 + \epsilon x_1) = \epsilon \left[-\omega^2 (\alpha [x_0 + \epsilon x_1] + \beta [x_0 + \epsilon x_1]^3) - 2\zeta\omega\Omega (x_0 + \epsilon x_1)' \right] - 2\epsilon\zeta\omega\Omega\epsilon Y \sin(\tau + \varphi) + \omega^2 \epsilon Y \cos(\tau + \varphi) + \epsilon\omega^2 [\epsilon\alpha Y \cos(\tau + \varphi) + \beta\epsilon^3 Y^3 \cos^3(\tau + \varphi)]. \quad (9)$$

This can be further written as:

$$\begin{aligned} & \left(\Omega^2 x_0'' + \omega^2 x_0 \right) + \epsilon \left(\Omega^2 x_1'' + \omega^2 x_1 \right) = \\ & \epsilon \left[-\omega^2 (\alpha x_0 + \beta x_0^3) - 2\zeta\omega\Omega x_0' + \omega^2 Y \cos(\tau + \varphi_0) \right] \\ & + \epsilon^2 \left[-\omega^2 (\alpha x_1 + 3\beta x_0^2 x_1) - 2\zeta\omega\Omega x_1' + \right. \\ & \left. 2\zeta\omega\Omega Y \sin(\tau + \varphi) + \omega^2 \alpha Y \cos(\tau + \varphi) \right] \\ & + \epsilon^3 [3\beta x_0 x_1^2 + \beta Y^3 \cos^3(\tau + \varphi)] + \epsilon^4 \beta x_1^3. \end{aligned} \quad (10)$$

It can be shown that with $\cos(\epsilon\varphi_1) = 1$ and $\sin(\epsilon\varphi_1) = \epsilon\varphi_1$ for small values of ϵ (for $\epsilon\varphi_1 < 0.3$, the error in the trigonometric terms will be less than 5%):

$$\cos(\tau + \varphi) = \cos \tau \cos \varphi_0 - \sin \tau \sin \varphi_0 - \epsilon\varphi_1 \{\cos \tau \sin \varphi_0 + \sin \tau \cos \varphi_0\}; \quad (11)$$

$$\sin(\tau + \varphi) = \sin \tau \cos \varphi_0 + \cos \tau \sin \varphi_0 - \epsilon\varphi_1 \{\sin \tau \sin \varphi_0 - \cos \tau \cos \varphi_0\}. \quad (12)$$

Using the above relations in Eq. (10) and equating the coefficients of terms having the same powers of ϵ , we get:

$$\Omega^2 x_0'' + \omega^2 x_0 = 0; \quad (13)$$

$$\Omega^2 x_1'' + \omega^2 x_1 = -2\zeta\omega\Omega x_0' - \omega^2 (\alpha x_0 + \beta x_0^3) + \omega^2 Y \{\cos \tau \cos \varphi_0 - \sin \tau \sin \varphi_0\}. \quad (14)$$

The $x_i(\tau)$, $i = 0, 1, 2, \dots$ are solved sequentially, subject to the periodicity condition $x_i(\tau + 2\pi) = x_i(\tau)$ and the initial conditions $x_i'(0) = 0$.

The solution of the first equation is:

$$x_0 = A_0 \cos\left(\frac{\omega}{\Omega}\right) \tau. \quad (15)$$

To have a periodic solution, $\omega = \Omega$ and hence the zero order solution becomes:

$$x_0 = A_0 \cos \tau. \quad (16)$$

A_0 is determined from the second differential equation, i.e., Eq. (14).

$$\begin{aligned} x_1'' + x_1 &= 2\zeta A_0 \sin \tau \\ &- \left(\alpha A_0 \cos \tau + \beta (A_0 \cos \tau)^3 \right) + Y \cos(\tau + \varphi_0). \end{aligned} \quad (17)$$

Recalling that $\cos^3 \tau = \frac{1}{4} (3 \cos \tau + \cos 3\tau)$ we get:

$$\begin{aligned} x_1'' + x_1 &= (2\zeta A_0 - Y \sin \varphi_0) \sin \tau \\ &+ \left(-\alpha A_0 - \frac{3}{4} \beta A_0^3 + Y \cos \varphi_0 \right) \cos \tau - \frac{1}{4} \beta A_0^3 \cos 3\tau. \end{aligned} \quad (18)$$

The solution must satisfy the periodicity condition to prevent the occurrence of secular term.¹¹ Therefore, coefficients of $\sin \tau$ and $\cos \tau$ should vanish. This leads to:

$$2\zeta A_0 = Y \sin \varphi_0; \quad (19)$$

$$\alpha A_0 + \frac{3}{4} \beta A_0^3 = Y \cos \varphi_0. \quad (20)$$

To determine A_0 , multiply Eqs. (19) and (20) by ϵ . Squaring both sides of Eqs. (19) and (20) and then adding them we get:

$$(2\epsilon\zeta A_0)^2 + \left(\epsilon\alpha A_0 + \frac{3}{4} \epsilon\beta A_0^3 \right)^2 = (\epsilon Y)^2. \quad (21)$$

By solving Eq. (21) the amplitude of the response can be obtained. The parameter $\epsilon\alpha$ is not an independent parameter and hence needs to be eliminated. This can be done by defining $\omega_0^2 = \Omega^2 (1 + \epsilon\alpha)$ where ω_0 is the natural frequency of the associated linear system, that is with $\beta = 0$. Using the above

definition and re-arranging the terms, the zero order frequency-response function becomes:

$$(2\epsilon\zeta A_0 \Omega^2)^2 + \left(\omega_0^2 \left[A_0 + \frac{3}{4}\epsilon\beta A_0^3 \right] - \Omega^2 A_0 \right)^2 = (\epsilon Y)^2 \Omega^4. \quad (22)$$

The above equation can be expanded to form a polynomial equation as given below, and then can be solved for A_0 .

$$\frac{9}{16} (\epsilon\beta)^2 \omega_0^4 A_0^6 + \frac{3}{2} (\epsilon\beta) \omega_0^2 (\omega_0^2 - \Omega^2) A_0^4 + \left((\omega_0^2 - \Omega^2)^2 + (2\epsilon\zeta \Omega^2)^2 \right) A_0^2 - (\epsilon Y)^2 \Omega^4 = 0. \quad (23)$$

Taking $S = A_0^2$, it becomes a cubic equation as:

$$\frac{9}{16} (\epsilon\beta)^2 \omega_0^4 S^3 + \frac{3}{2} (\epsilon\beta) \omega_0^2 (\omega_0^2 - \Omega^2) S^2 + \left((\omega_0^2 - \Omega^2)^2 + (2\epsilon\zeta \Omega^2)^2 \right) S - (\epsilon Y)^2 \Omega^4 = 0. \quad (24)$$

Equation (24) gives the zero order frequency-response function for the absolute displacement of a damped nonlinear system for a harmonic base excitation. The zero order solution is given by Eq. (16) and A_0 can be determined by solving Eq. (24).

The phase difference, which is the difference with respect to the phase of the excitation, can be shown to be:

$$\tan \varphi_0 = \frac{2\epsilon\zeta}{\epsilon\alpha + \frac{3}{4}\epsilon\beta A_0^2}. \quad (25)$$

3. RESPONSE FOR VARIOUS PARAMETERS OF THE SYSTEM

The responses of the system for its various parameters are determined through solving Eq. (24) and the results are given in the following figures. The responses shown are the amplitudes of the zero order approximation (A_0).

The response characteristics are shown in Figs. 2–6. The damping factor considered is 0.05. In all these figures the amplitudes of the absolute displacement are plotted against the frequency of excitation for a system having a particular value of linearised natural frequency. The characteristics are given for five different linearised natural frequencies (ω_0) 200, 400, 600, 800, 1000 rad/s. For a particular value of linearised natural frequency of the oscillator, the responses are shown for 6 values of amplitudes of base excitation. The amplitudes of excitation are 0.1 mm, 0.5 mm, 1 mm, 1.5 mm, 2 mm and 2.5 mm. All the above curves are presented in one figure. All the curves in the same figure are drawn for a particular value of nonlinear spring parameter $\epsilon\beta$ in N/mm² and it is indicated in the title of the figure. The amplitudes of responses are given in millimeters.

Figures 2–7 provide the above characteristics for different values of nonlinear spring parameter $\epsilon\beta$. Figures 2–4 give the responses respectively for $\epsilon\beta = 0.001$, $\epsilon\beta = 0.005$ and

$\epsilon\beta = 0.01$, when the spring is a hardening type. The results of systems with a softening type are shown in Figs. 5 and 6, Fig. 5 is for $\epsilon\beta = 0.005$ and Fig. 6 is for $\epsilon\beta = 0.01$.

The jump phenomenon in the response is seen as in forced excitation. The curves showing the variation of amplitude with the excitation frequency converge to the curve corresponding to $\epsilon Y = 0$. In a linear system, they asymptotically approach $\Omega = \omega_0$, when the damping is absent. In a nonlinear system, that is $\epsilon\beta \neq 0$, they asymptotically approach a parabola, the equation of which can be derived as:

$$A_0^2 = \frac{4}{3\epsilon\beta} \left\{ \left(\frac{\Omega}{\omega_0} \right)^2 - 1 \right\}. \quad (26)$$

The asymptote intersects the Ω -axis at $\Omega = \omega_0$.

It is important to note that the responses estimated using the expression derived in this work show very low values of responses when the frequency of excitation is low. The amplitude of response should have been approximately equal to the amplitude of excitation. The amplitude of response estimated is the zero order approximation. This issue is resolved by incorporating the first order approximation terms. This is presented subsequently.

4. FIRST ORDER APPROXIMATION OF RESPONSE

The issue seen in the results at lower frequencies is investigated here.

As discussed previously the displacement response can be expanded as per Eq. (7) and the phase of the response can be expanded as per Eq. (8). The zero order approximation to the solution is $x_0 = A_0 \cos \tau$. Solving the differential equation

$$x_1'' + x_1 = -\frac{1}{4}\beta A_0^3 \cos 3\tau; \quad (27)$$

the first order approximation to the solution can be obtained as:

$$x_1 = A_1 \cos \tau + \frac{1}{32}\beta A_0^3 \cos 3\tau. \quad (28)$$

Combining both, the solution becomes:

$$x(\tau) = (A_0 + \epsilon A_1) \cos \tau + \frac{1}{32}\epsilon\beta A_0^3 \cos 3\tau + \dots \quad (29)$$

The value of A_0 can be determined using Eq. (24). The response estimated using A_0 alone, that is without A_1 , is found to be not correct in the low frequencies. Therefore, A_1 is determined and investigated whether incorporation of it improves the results.

4.1. Expression for A_1

The differential equation of motion is:

$$\begin{aligned} \Omega^2 x'' + \omega^2 x = & \epsilon \left[-\omega^2 (\alpha x + \beta x^3) - 2\zeta \omega \Omega x' \right] \\ & - 2\epsilon\zeta \omega \Omega \epsilon Y \sin(\tau + \varphi) + \omega^2 \epsilon Y \cos(\tau + \varphi) \\ & + \epsilon \omega^2 [\epsilon \alpha Y \cos(\tau + \varphi) + \beta \epsilon^3 Y^3 \cos^3(\tau + \varphi)]. \end{aligned} \quad (30)$$

Figure 2. Frequency response of hard spring ($\epsilon\beta = 0.001$) for various values of excitations.

Figure 3. Frequency response of hard spring ($\epsilon\beta = 0.005$) for various values of excitations.

Figure 4. Frequency response of hard spring ($\epsilon\beta = 0.01$) for various values of excitations.

Figure 5. Frequency response of soft spring ($\epsilon\beta = 0.005$) for various values of excitations.

Figure 6. Frequency response of soft spring ($\epsilon\beta = 0.01$) for various values of excitations.

Figure 7. Frequency response (ϵA_1) of hard spring ($\epsilon\beta = 0.001$).

Using the solution for x in terms of x_1 , x_2 etc., the differential equation becomes:

$$\begin{aligned} \Omega^2 (x_0 + \epsilon x_1 + \epsilon^2 x_2)'' + \omega^2 (x_0 + \epsilon x_1 + \epsilon^2 x_2) = \\ \epsilon \left[-\omega^2 (\alpha [x_0 + \epsilon x_1 + \epsilon^2 x_2] + \beta [x_0 + \epsilon x_1 + \epsilon^2 x_2]^3) \right. \\ \left. - 2\zeta\omega\Omega (x_0 + \epsilon x_1 + \epsilon^2 x_2)' + \omega^2 Y \cos(\tau + \varphi) \right] \\ + \epsilon^2 (-2\zeta\omega\Omega Y \sin(\tau + \varphi) + \omega^2 \alpha Y \cos(\tau + \varphi)) \\ + \epsilon^4 \omega^2 \beta Y^3 \cos^3(\tau + \varphi). \quad (31) \end{aligned}$$

For small values of ϵ , the values of higher powers of ϵ are negligible and hence:

$$[x_0 + \epsilon x_1 + \epsilon^2 x_2]^3 = (x_0^3 + \epsilon 3x_0^2 x_1). \quad (32)$$

Using Eq. (32) the differential equation becomes:

$$\begin{aligned} \Omega^2 (x_0 + \epsilon x_1 + \epsilon^2 x_2)'' + \omega^2 (x_0 + \epsilon x_1 + \epsilon^2 x_2) = \\ \epsilon \left[-\omega^2 (\alpha [x_0 + \epsilon x_1 + \epsilon^2 x_2] + \beta (x_0^3 + \epsilon 3x_0^2 x_1)) \right. \\ \left. - 2\zeta\omega\Omega (x_0 + \epsilon x_1 + \epsilon^2 x_2)' + \omega^2 Y \cos(\tau + \varphi) \right] \\ + \epsilon^2 (-2\zeta\omega\Omega Y \sin(\tau + \varphi) + \omega^2 \alpha Y \cos(\tau + \varphi)) \\ + \epsilon^4 \omega^2 \beta Y^3 \cos^3(\tau + \varphi). \quad (33) \end{aligned}$$

Expanding the differentials, we get:

$$\begin{aligned} (\Omega^2 x_0'' + \omega^2 x_0) + \epsilon (\Omega^2 x_1'' + \omega^2 x_1) \\ + \epsilon^2 (\Omega^2 x_2'' + \omega^2 x_2) = \\ \epsilon [-\omega^2 (\alpha x_0 + \beta x_0^3) - 2\zeta\omega\Omega x_0' + \omega^2 Y \cos(\tau + \varphi)] \\ + \epsilon^2 [-\omega^2 (\alpha x_1 + 3\beta x_0^2 x_1) - 2\zeta\omega\Omega x_1' \\ - 2\zeta\omega\Omega Y \sin(\tau + \varphi) + \omega^2 \alpha Y \cos(\tau + \varphi)]. \quad (34) \end{aligned}$$

Expanding φ as given by Eq. (8) and using the trigonometric expansions we get:

$$\begin{aligned} \cos(\tau + \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2) = \\ \cos(\tau + \varphi_0) - \epsilon\varphi_1 \sin(\tau + \varphi_0) - \epsilon^2\varphi_2 \sin(\tau + \varphi_0) \\ - \epsilon\varphi_1 * \epsilon^2\varphi_2 \cos(\tau + \varphi_0); \quad (35) \end{aligned}$$

$$\begin{aligned} \sin(\tau + \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2) = \\ \sin(\tau + \varphi_0) + \epsilon\varphi_1 \cos(\tau + \varphi_0) + \epsilon^2\varphi_2 \cos(\tau + \varphi_0) \\ - \epsilon\varphi_1 * \epsilon^2\varphi_2 \sin(\tau + \varphi_0). \quad (36) \end{aligned}$$

by approximating for the sin and cos of small angles. As value of ϵ is very small, the values of ϵ^2 and ϵ^3 can be neglected. Therefore, one can approximate the cosine and sine expansions in Eq. (35) and (36) as:

$$\cos(\tau + \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2) = \cos(\tau + \varphi_0) - \epsilon\varphi_1 \sin(\tau + \varphi_0); \quad (37)$$

$$\sin(\tau + \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2) = \sin(\tau + \varphi_0) + \epsilon\varphi_1 \cos(\tau + \varphi_0). \quad (38)$$

Applying Eq. (37) and (38) the differential equation becomes:

$$\begin{aligned} (\Omega^2 x_0'' + \omega^2 x_0) + \epsilon (\Omega^2 x_1'' + \omega^2 x_1) \\ + \epsilon^2 (\Omega^2 x_2'' + \omega^2 x_2) = \\ \epsilon [-\omega^2 (\alpha x_0 + \beta x_0^3) - 2\zeta\omega\Omega x_0' \\ + \omega^2 Y \{\cos(\tau + \varphi_0) - \epsilon\varphi_1 \sin(\tau + \varphi_0) - \epsilon^2\varphi_2 \sin(\tau + \varphi_0)\}] \\ + \epsilon^2 [-\omega^2 (\alpha x_1 + 3\beta x_0^2 x_1) - 2\zeta\omega\Omega x_1' \\ - 2\zeta\omega\Omega Y \{\sin(\tau + \varphi_0) + \epsilon\varphi_1 \cos(\tau + \varphi_0) \\ + \epsilon^2\varphi_2 \cos(\tau + \varphi_0)\} + \omega^2 \alpha Y \{\cos(\tau + \varphi_0) \\ - \epsilon\varphi_1 \sin(\tau + \varphi_0) - \epsilon^2\varphi_2 \sin(\tau + \varphi_0)\}]. \quad (39) \end{aligned}$$

On neglecting higher powers of ϵ :

$$\begin{aligned} (\Omega^2 x_0'' + \omega^2 x_0) + \epsilon (\Omega^2 x_1'' + \omega^2 x_1) \\ + \epsilon^2 (\Omega^2 x_2'' + \omega^2 x_2) = \\ \epsilon [-\omega^2 (\alpha x_0 + \beta x_0^3) - 2\zeta\omega\Omega x_0' + \omega^2 Y \cos(\tau + \varphi_0)] \\ + \epsilon^2 [-\omega^2 (\alpha x_1 + 3\beta x_0^2 x_1) - 2\zeta\omega\Omega x_1' \\ - \omega^2 Y \varphi_1 \sin(\tau + \varphi_0) - 2\zeta\omega\Omega Y \sin(\tau + \varphi_0) \\ + \omega^2 \alpha Y \cos(\tau + \varphi_0)]. \quad (40) \end{aligned}$$

Equating the coefficients of ϵ^0 , the zero order approximation of the differential equation becomes:

$$\Omega^2 x_0'' + \omega^2 x_0 = 0. \quad (41)$$

Equating the coefficients of ϵ^1 we get:

$$\begin{aligned} \Omega^2 x_1'' + \omega^2 x_1 = \\ -\omega^2 (\alpha x_0 + \beta x_0^3) - 2\zeta\omega\Omega x_0' + \omega^2 Y \cos(\tau + \varphi_0). \quad (42) \end{aligned}$$

Equating the coefficients of ϵ^2 , the differential equation becomes:

$$\begin{aligned} \Omega^2 x_2'' + \omega^2 x_2 = -\omega^2 (\alpha x_1 + 3\beta x_0^2 x_1) \\ - 2\zeta\omega\Omega x_1' - \omega^2 Y \varphi_1 \sin(\tau + \varphi_0) - 2\zeta\omega\Omega Y \\ \sin(\tau + \varphi_0) + \omega^2 \alpha Y \cos(\tau + \varphi_0). \quad (43) \end{aligned}$$

The zero order approximation to the solution is as given by Eq. (16) and the first order approximation is as given by Eq. (28). Second order approximation can be obtained from Eq. (43), which can be written as:

$$\begin{aligned} \Omega^2 x_2'' + \omega^2 x_2 = -\omega^2 (\alpha x_1 + 3\beta x_0^2 x_1) - 2\zeta\omega\Omega x_1' \\ - \{\omega^2 Y \varphi_1 + 2\zeta\omega\Omega Y\} [\cos(\tau) * \sin(\varphi_0) + \cos(\varphi_0) * \sin(\tau)] \\ + \omega^2 \alpha Y [\cos(\tau) * \cos(\varphi_0) - \sin(\tau) * \sin(\varphi_0)]. \quad (44) \end{aligned}$$

Using the solutions for x_0 and x_1 and dividing Eq. (44) by

ω^2 we get:

$$\begin{aligned} x_2'' + x_2 = & - \left(\alpha \{ A_1 \cos \tau + \frac{1}{32} \beta A_0^3 \cos 3\tau \} \right. \\ & + 3\beta \{ A_0 \cos \tau \}^2 \{ A_1 \cos \tau + \frac{1}{32} \beta A_0^3 \cos 3\tau \} \\ & - 2\zeta \left\{ -A_1 \sin \tau - \frac{3}{32} \beta A_0^3 \sin 3\tau \right\} \\ & - \{ Y \varphi_1 + 2\zeta Y \} (\cos \tau \sin \varphi_0 + \cos \varphi_0 \sin \tau) \\ & \left. + \alpha Y (\cos \tau \cos \varphi_0 - \sin \tau \sin \varphi_0) \right). \quad (45) \end{aligned}$$

Re-arranging the terms we get:

$$\begin{aligned} x_2'' + x_2 = & - \left(\alpha A_1 \cos \tau + \frac{1}{32} \alpha \beta A_0^3 \cos 3\tau + \right. \\ & + 3\beta A_0^2 A_1 \cos^3 \tau \\ & + \frac{3}{32} \beta^2 A_0^5 \cos 3\tau \cos^2 \tau \left. \right) - \left(-2\zeta A_1 \sin \tau - \frac{3}{16} \zeta \beta A_0^3 \sin 3\tau \right) \\ & - \{ Y \varphi_1 + 2\zeta Y \} (\cos \tau \sin \varphi_0 + \cos \varphi_0 \sin \tau) \\ & + \alpha Y (\cos \tau \cos \varphi_0 - \sin \tau \sin \varphi_0). \quad (46) \end{aligned}$$

Using the trigonometric relations $\cos 3\tau \cos^2 \tau = \frac{1}{4}(\cos \tau + 2 \cos 3\tau + \cos 5\tau)$ and $\cos^3 \tau = \frac{1}{4}(3 \cos \tau + \cos 3\tau)$ the differential equation changes to:

$$\begin{aligned} x_2'' + x_2 = & - \left(\alpha A_1 \cos \tau + \frac{1}{32} \alpha \beta A_0^3 \cos 3\tau + \right. \\ & \beta A_0^2 A_1 \left(\frac{1}{4} (3 \cos \tau + \cos 3\tau) \right) + \\ & \frac{3}{32} \beta^2 A_0^5 \left(\frac{1}{4} (\cos \tau + 2 \cos 3\tau + \cos 5\tau) \right) \\ & - \left(-2\zeta A_1 \sin \tau - \frac{3}{16} \zeta \beta A_0^3 \sin 3\tau \right) \\ & - \{ Y \varphi_1 + 2\zeta Y \} (\cos \tau \sin \varphi_0 + \cos \varphi_0 \sin \tau) \\ & \left. + \alpha Y (\cos \tau \cos \varphi_0 - \sin \tau \sin \varphi_0) \right). \quad (47) \end{aligned}$$

Collecting all $\cos \tau$, $\cos 3\tau$ and neglecting $\cos 5\tau$ terms, we get:

$$\begin{aligned} x_2'' + x_2 = & \cos \tau \left\{ -\alpha A_1 - \frac{9}{4} \beta A_0^2 A_1 - \frac{3}{128} \beta^2 A_0^5 \right. \\ & - (Y \varphi_1 + 2\zeta Y) \sin \varphi_0 + \alpha Y \cos \varphi_0 \left. \right\} \\ & + \sin \tau \left\{ 2\zeta A_1 - (Y \varphi_1 + 2\zeta Y) \cos \varphi_0 - \alpha Y \sin \varphi_0 \right\} \\ & + \cos 3\tau \left\{ -\frac{1}{32} \alpha \beta A_0^3 - \frac{3}{4} \beta A_0^2 A_1 - \frac{6}{128} \beta^2 A_0^5 \right\} \\ & + \frac{3}{16} \zeta \beta A_0^3 \sin 3\tau. \quad (48) \end{aligned}$$

To avoid the secular terms, equating the coefficients of $\cos \tau$ and $\sin \tau$ to zero:

$$\begin{aligned} -\alpha A_1 - \frac{9}{4} \beta A_0^2 A_1 - \frac{3}{128} \beta^2 A_0^5 \\ - (Y \varphi_1 + 2\zeta Y) \sin \varphi_0 + \alpha Y \cos \varphi_0 = 0; \quad (49) \end{aligned}$$

$$2\zeta A_1 - (Y \varphi_1 + 2\zeta Y) \cos \varphi_0 - \alpha Y \sin \varphi_0 = 0. \quad (50)$$

The aim now is to solve for A_1 . Re-arranging the terms and removing the dependence on the phase angle:

$$\begin{aligned} (Y \varphi_1 + 2\zeta Y)^2 + (\alpha Y)^2 = \\ (\alpha A_1 + \frac{9}{4} \beta A_0^2 A_1 + \frac{3}{128} \beta^2 A_0^5)^2 + (2\zeta A_1)^2. \quad (51) \end{aligned}$$

Introducing ϵ and considering $\epsilon \varphi_1 \ll 2\epsilon \zeta$, we get:

$$\begin{aligned} (2\epsilon \zeta Y)^2 + (\epsilon \alpha Y)^2 = \\ (\epsilon \alpha A_1 + \frac{9}{4} \epsilon \beta A_0^2 A_1 + \frac{3}{128} (\epsilon \beta)^2 A_0^5)^2 + (2\epsilon \zeta A_1)^2. \quad (52) \end{aligned}$$

As $\frac{\omega_0^2 - \Omega^2}{\Omega^2} = \epsilon \alpha$, Eq. (52) gets modified as:

$$\begin{aligned} (2\epsilon \zeta Y)^2 + \left(\frac{\omega_0^2 - \Omega^2}{\Omega^2} \epsilon Y \right)^2 = \\ \left(\frac{\omega_0^2 - \Omega^2}{\Omega^2} \epsilon A_1 + \frac{9}{4} \epsilon \beta A_0^2 A_1 + \frac{3}{128} (\epsilon \beta)^2 A_0^5 \right)^2 \\ + (2\epsilon \zeta A_1)^2. \quad (53) \end{aligned}$$

Expanding the square of the expression and collecting the terms involving ϵA_1 and $(\epsilon A_1)^2$ we get:

$$\begin{aligned} (2\epsilon \zeta Y)^2 + \left(\frac{\omega_0^2 - \Omega^2}{\Omega^2} \epsilon Y \right)^2 = \\ \left(\left[\frac{\omega_0^2 - \Omega^2}{\Omega^2} \right]^2 + \left[\frac{9}{4} \epsilon \beta A_0^2 \right]^2 \right. \\ \left. + 2 \left[\frac{\omega_0^2 - \Omega^2}{\Omega^2} \right] \left[\frac{9}{4} \epsilon \beta A_0^2 \right] + [2\epsilon \zeta]^2 \right) (\epsilon A_1)^2 \\ + \left(\frac{27}{256} (\epsilon \beta)^3 A_0^7 + \frac{6}{128} \left[\frac{\omega_0^2 - \Omega^2}{\Omega^2} \right] (\epsilon \beta)^2 A_0^5 \right) (\epsilon A_1) \\ + \left(\frac{3}{128} (\epsilon \beta)^2 A_0^5 \right)^2. \quad (54) \end{aligned}$$

Simplifying the coefficient of $(\epsilon A_1)^2$, and re-arranging the terms:

$$\begin{aligned} \left(\left\{ \left[\frac{\omega_0^2 - \Omega^2}{\Omega^2} \right] + \left[\frac{9}{4} \epsilon \beta A_0^2 \right] \right\}^2 + [2\epsilon \zeta]^2 \right) (\epsilon A_1)^2 \\ + \left(\frac{27}{256} (\epsilon \beta)^3 A_0^7 + \frac{6}{128} \left[\frac{\omega_0^2 - \Omega^2}{\Omega^2} \right] (\epsilon \beta)^2 A_0^5 \right) (\epsilon A_1) \\ + \left(\frac{3}{128} (\epsilon \beta)^2 A_0^5 \right)^2 - (2\epsilon \zeta Y)^2 - \left(\frac{\omega_0^2 - \Omega^2}{\Omega^2} \epsilon Y \right)^2 = 0. \quad (55) \end{aligned}$$

The above relation is a quadratic equation in ϵA_1 as given be-

low:

$$\begin{aligned} & \left(\left\{ \left[\frac{\omega_0^2 - \Omega^2}{\Omega^2} \right] + \left[\frac{9}{4} \epsilon \beta A_0^2 \right] \right\}^2 + [2\epsilon \zeta]^2 \right) (\epsilon A_1)^2 \\ & + \frac{3}{128} (\epsilon \beta)^2 \left(\frac{9}{2} \epsilon \beta A_0^2 + 2 \left[\frac{\omega_0^2 - \Omega^2}{\Omega^2} \right] \right) A_0^5 (\epsilon A_1) \\ & + \left(\frac{3}{128} (\epsilon \beta)^2 A_0^5 \right)^2 - (2\epsilon \zeta \epsilon Y)^2 - \left(\frac{\omega_0^2 - \Omega^2}{\Omega^2} \epsilon Y \right)^2 = 0. \end{aligned} \quad (56)$$

Solving the above quadratic equation, ϵA_1 can be determined. The amplitude of the response in the fundamental frequency is $A_0 + \epsilon A_1$.

4.2. Results of ϵA_1

The values of ϵA_1 for various values of nonlinear spring parameter, damping and excitation etc., are shown in Figs. 7–10 for hard and soft springs with linearised natural frequency of 500 rad/s. The results show that the amplitude of response at low frequency is approximately the same as the amplitude of excitation. The problem in the low frequency can be resolved by incorporating the first order term.

5. RESPONSE WITH ZERO AND FIRST ORDER APPROXIMATIONS

It was discussed previously that the responses estimated using the expression derived in this work show very low value of response when the frequency of excitation is low. The amplitude of response should have been approximately equal to the amplitude of excitation. The amplitude of response estimated was with the zero order approximation and the first order approximation term was not included.

Here the amplitude of the response in the fundamental frequency of oscillation considering both zero order as well as the first order approximations ($A_0 + \epsilon A_1$), is plotted at various excitation frequencies and various parameters of the system. They are shown in Figs. 11–18 and are presented in the same way as before.

Figure 11 gives the responses for the nonlinear spring parameter $\epsilon \beta = 0.001$, Fig. 12 gives the results for $\epsilon \beta = 0.005$, Fig. 13 gives the results for $\epsilon \beta = 0.01$ and Fig. 14 shows the results for $\epsilon \beta = 0.05$ when the spring is hardening type. Figures 15–18 show similar result for a system with softening spring.

It can be seen that the problems encountered at low frequencies are resolved by incorporating the first order approximation term and the responses are on expected lines. The response near resonance is contributed by zero order term and the response at low frequencies are from the first order term. The values of A_0 can be determined by solving Eq. (24) and ϵA_1 can be determined by solving Eq. (56).

Following points can be noted from all the above response curves

- As the frequency of excitation is increased or decreased, a jump is seen in the magnitude of the response. As the

excitation frequency is being increased the jump occurs at a higher frequency and when the excitation frequency is being decreased the jump occurs at a lower frequency.

- In the case of a hard spring with the increase in the magnitude of excitation (ϵY) the frequency at which the response is the maximum increases (for constant $\epsilon \beta$ and $\epsilon \zeta$). For the soft spring, the characteristics are reversed.
- In the case of a hard spring, the frequency at which the response is the maximum increases considerably with the increase in the non-linearity ($\epsilon \beta$).
- For small values of nonlinearities, the amplification remains approximately the same for different values of excitations. Consequently, the maximum response will be proportional to the excitation and will be independent of nonlinear spring parameters. The behavior of a soft spring is identical to that of a hard spring with reverse characteristics.
- In the absence of damping, the response asymptotically converges to a parabola. This parabola intersects the Ω -axis at $\Omega = \omega_0$.
- The response at the fundamental frequency leads the excitation by 90 degrees. This is considering the zero order approximation. The phase difference is zero if damping is absent, as in linear systems.

6. CONCLUSIONS

Responses of a system with nonlinear stiffness subjected to base harmonic excitation are determined using Lindstedt's method. An expression to estimate the amplitude of the fundamental frequency of oscillation is derived. It exhibits the jump phenomena as shown for forced excitations. The amplitudes of responses at various values of excitation frequencies are obtained for various nonlinear parameters of the oscillator. The existing expression for the amplitude of response for base excitation is derived from relative displacement making certain assumptions on phase. But the present expression is derived from first principles without these assumptions. Additionally, the expression for the phase difference and the expression for the asymptote where the responses converge are also derived.

The responses estimated using the expression derived in this work show very low value of response when the frequency of excitation is low. At low frequencies, amplitude of response should have been approximately equal to the amplitude of excitation. The amplitude of response estimated is the zero order approximation and the first order approximation was not included. Therefore, an expression for the first order approximation is derived. Its characteristics for various parameters of the system are obtained. By incorporating the first order term, the problem in the low frequency is resolved. Variation of the amplitude, incorporating both zero order term as well as the first order term, with the excitation frequency are presented for various nonlinear spring parameters of the oscillator.

Figure 8. Frequency response (ϵA_1) of hard spring ($\epsilon\beta = 0.01$).

Figure 9. Frequency response (ϵA_1) of soft spring ($\epsilon\beta = 0.001$).

Figure 10. Frequency response (ϵA_1) of soft spring ($\epsilon\beta = 0.01$).

Figure 11. Frequency response ($A_0 + \epsilon A_1$) of hard spring ($\epsilon\beta = 0.001$).

Figure 12. Frequency response ($A_0 + \epsilon A_1$) of hard spring ($\epsilon\beta = 0.005$).

Figure 13. Frequency response ($A_0 + \epsilon A_1$) of hard spring ($\epsilon\beta = 0.01$).

Figure 14. Frequency response ($A_0 + \epsilon A_1$) of hard spring ($\epsilon\beta = 0.05$).

Figure 15. Frequency response ($A_0 + \epsilon A_1$) of soft spring ($\epsilon\beta = 0.001$).

Figure 16. Frequency response ($A_0 + \epsilon A_1$) of soft spring ($\epsilon\beta = 0.005$).

Figure 17. Frequency response ($A_0 + \epsilon A_1$) of soft spring ($\epsilon\beta = 0.01$).

Figure 18. Frequency response ($A_0 + \epsilon A_1$) of soft spring ($\epsilon\beta = 0.05$).

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