Quality-Factor Enhancing Locations for Substrate Mounted Resonators

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An important but often overlooked factor that affects the performance of a meso/micro electro mechanical vibratory sensor is the structural interaction between the sensor's resonator and the substrate on which it is mounted. Situating resonators at node points eliminates this interaction and thereby helps to improve a resonator's quality-factor for a particular mode of vibration. This paper addresses the problem of locating a single degree of freedom spring-mass resonator on a generic cantilever substrate. The loci of natural frequencies obtained when the resonator's mounting location is varied are developed, and the nodal locations are identified. Thereafter a method to obtain these locations from the characteristic equation without solving the associated eigenvalue problem is described. Lookup tables detailing the nodal locations and the corresponding natural frequencies for various resonator parameters are presented. It is found that at these special nodal locations, the magnitude of the power transmitted through anchors is negligible, which ensures minimal structural interaction between the resonator and the substrate.

1. INTRODUCTION

Mechanical resonators are high performance devices designed to generate sinusoidal signals at precise frequencies. To excite these vibrations, energy is pumped into the device as sinusoidal forcing waves at the resonator's natural frequency. The resulting large amplitude resonant oscillations are subsequently utilized to generate the required signals.

The performance of micro/meso-scale resonators such as the ones used in Coriolis vibratory gyroscopes depends on the Q-factor of the resonators' operating mode of vibration.^{1,2} The accuracy of measurements made by such gyroscopes improves with an increase of its Q-factor and hence it is desirable to fabricate resonators with Q-factors as high as possible. To realize high Q-factors, it is a common engineering practice to choose quartz based meso-scale resonators designs.³ Although the Q-factor associated with quartz is extremely high (reported to be over one billion³), in reality, several factors reduce the effective Q-factor of a resonator mode. The various factors that affect the Q-factor of a resonator's vibratory mode are summarized in Table 1. The resultant Q-factor of the resonator's mode of operation can be expressed as a sum of the contributions from each of these factors in the following form:⁴

$$\frac{1}{Q} = \frac{1}{Q_{material}} + \frac{1}{Q_{pressure}} + \frac{1}{Q_{TED}} + \frac{1}{Q_{anchor}} + \frac{1}{Q_{other}}. \quad (1)$$

Therefore the resultant Q-factor of a resonator mode is dictated by the factor with the least contributory Q-factor. Among these, the reduction of Q-factor due to improper anchorage of the resonator to its substrate motivates this work.

The type, the number and the location of the anchors attaching the resonator to a substrate significantly affects the Q-factor of the resonator's mode of vibration.^{5,6} An improper anchor location leads to structural interaction between the resonator

and the substrate which degrades resonator performance. Several researchers have investigated anchor induced reduction of Q-factor and have developed methods to predict the energy transmitted away from the resonator through the anchors.⁷⁻⁹ The energy carried by structural waves through the anchor into the substrate on which the resonator is mounted per cycle, is expressed for a cycle of harmonic motion with a period T as:

$$\pi_L = \frac{1}{T} \int_0^T \underline{F} \cdot \underline{\dot{u}} dt; \quad (2)$$

where $\underline{F}$ is the vector of forces and moments acting at the anchor location and $\underline{\dot{u}}$ denotes the vector of velocities of the anchor point.¹⁰ The energy lost at an anchor depends on the velocity of the attachment point. Hence the obvious design strategy to be adopted to minimize anchor losses (and to maximize Q_{anchor}) is to locate the resonator such that the point of attachment is a node point (of the mode shape corresponding to the resonant frequency at which the resonator is designed to operate) of the entire assembly comprising the resonator as well as the substrate to which it is mounted.

Several researchers have investigated the direct problem of evaluating the vibratory characteristics of combined systems comprising masses elastically mounted to continuous media. Free vibrations of beams with elastically mounted masses has been explored in the past.¹¹⁻¹⁴ Multi-span beams carrying multiple spring mass systems have also been investigated.^{15,16} Non-uniform beams carrying an arbitrary number of spring mass systems have also investigated¹⁷ as well. However, the inverse problem of estimating a design parameter, such as a stiffness or a mass, so that the combined system meets certain requirements has received less attention. A few notable inverse problems investigated include the imposition of nodes at certain locations on a continuous structure by attaching resonators,¹⁸⁻²⁰ constructing a physically realizable system from a beam's eigen solutions²¹ and generating by design certain natural frequencies in a beam with an added mass.²²

Table 1. Summary of the various factors that contribute to the final Q-factor of a resonator.

Notation	Contributing factor	Description
$Q_{material}$	Material friction	The energy lost due to the friction inherent to the material constituting the resonator
$Q_{pressure}$	Viscosity of the gaseous medium	The energy utilized to overcome the resistance imparted by the gas surrounding the resonator
Q_{TED}	Thermoelastic effect	The loss which arises due to the localized vibratory heating of a region within the oscillating body
Q_{anchor}	Anchoring	The energy transmitted from the resonator to the substrate through the anchors
Q_{other}	Others	The dissipation due to other effects such as surface defects and Akhiezer effect

Table 2. Operating frequencies, Q-factors and damping levels for various high-Q resonators

Resonator geometry	Frequency (MHz)	Quality factor, Q	Damping factor, $\zeta = \frac{1}{2Q}$	Reference
Stemless disk	74.3	98000	5.1e-6	5
Disk	0.247	100000	5e-6	23
Hemispherical	0.0184	143000	3.45e-6	24
Dual ring	19.7	150000	3.33e-6	23
Cantilever	0.0083	403000	1.24e-6	24
Tuning fork	0.0022	800000	6.25e-7	25
Lame square beam	6.35	1700000	2.94e-7	26

Unlike previous investigations, this work addresses the inverse problem of finding the locations on a substrate where an attached resonator experiences no structural interaction with its substrate for an intended operating resonant mode. A designer who aims to locate a resonator on a substrate with established natural frequencies, modes and nodal locations may perhaps be tempted to locate the resonator at the node points of the (bare) substrate. However, with the addition of the resonator, the location of the node points (of the assembly) deviate from those of the bare substrate. The deviation of the nodal locations upon attachment of the resonator is less significant for resonators with low mass and becomes substantial as the resonator mass increases. The location of the (new) node points of the assembly for a particular mode of vibration depends on the location of the resonator itself. The work explores the possibility of finding locations on a substrate where an attached resonator of a predetermined mass and operating frequency makes the same location a node point of the assembly comprising the substrate and the resonator. The investigation considers the problem of locating a generic single degree of freedom (DoF) spring-mass resonator on the commonly encountered cantilever substrate. The equation of motion governing the dynamics of such a resonator-substrate assembly is presented in Section 2. Thereafter, in Section 3, the natural frequencies and mode shapes of the system are obtained by solving the eigenvalue problem, and the nodal locations are identified from their loci. In Section 4, a method is developed to calculate the nodal mounting locations from the characteristic equation without solving the eigenvalue problem. The nodal locations for various sets of resonator mass and stiffness are identified and tabulated.

Figure 1. Schematic representation of a resonator mounted on a continuous substrate.

2. THE MODEL

The Q-factors and the associated damping factors¹ for various high-Q resonators as reported in literature are presented in Table 2. It is evident that the damping factors associated with such high performance systems are extremely low. The time scale on which the resonator oscillations decay is much larger than the time scale associated with the resonator's oscillatory frequencies. The decay in the amplitude for each cycle can therefore be assumed to be negligible, and hence the vibratory characteristics of the high performance systems motivating this work is investigated in the following sections using an undamped structural model. The effect of damping is considered later in Section 5.

The system investigated here comprises a resonator mounted on a continuous elastic substrate. The resonator is modeled as a single-DoF spring-mass oscillator with a mass M_R and a spring stiffness K_R as represented in Fig. 1. The elastic substrate to which the resonator is mounted, is modeled as an Euler-Bernoulli beam²⁷ of length L with a flexural rigidity EI (where E is the modulus of elasticity of the homogeneous material constituting the beam and I the area moment of inertia with respect to the neutral axis of bending) and a mass per unit length ρA , where ρ is the uniform material density and A is the constant cross sectional area through out the length of the beam. The transverse deflection of the beam at a location X along the length of the beam at any time T is denoted by the variable $U(X, T)$. The resonator is assumed to be mounted at a location $X = X_0$. The displacement of mass M_R is denoted by the variable $Z(T)$. In terms of the dimensionless variables, $x = \frac{X}{L}$, $u = \frac{U}{L}$, $z = \frac{Z}{L}$, $x_0 = \frac{X_0}{L}$, $k = \frac{K_R L^3}{EI}$, $m = \frac{M_R}{\rho A L}$ and $t = T \sqrt{\frac{EI}{\rho A L^4}}$, the equations governing the motion of the system as presented in Reference¹² are the following:

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = k(z - y) \delta(x - x_0); \quad x \in [0, 1]; \quad (3)$$

$$m \frac{d^2 z}{dt^2} + k(z - y) = f \sin \Omega t. \quad (4)$$

¹For a single-DoF oscillator, the damping factor ζ and the Q are related as $Q = 1/2\zeta$. As ζ tends to zero, Q-factor tends to infinity.

Here the variable $y = u(x_0, t)$ denotes the instantaneous displacement of the point of attachment of the resonator to the continuous substrate. The following boundary conditions are employed to model the clamped-free (cantilever) elastic substrate.

$$\begin{aligned} u(0, t) &= 0; & \frac{\partial u}{\partial x}(0, t) &= 0; \\ \frac{\partial^2 u}{\partial x^2}(1, t) &= 0; & \frac{\partial^3 u}{\partial x^3}(1, t) &= 0. \end{aligned} \quad (5)$$

The equations presented above governing the dynamics of the system are solved and the solutions are examined in the following section.

3. EIGEN STRUCTURE

The Galerkin's method^{28,29} is employed to solve the above system of equations. Cantilever beam functions (denoted here by ϕ_i where $i = 1..n$), obtained from the solution of the equation governing the transverse vibrations of a classic cantilever Euler-Bernoulli beam,^{27,29} are used as comparison functions to expand the variable u . Note that these functions satisfy the boundary conditions presented in Eq. (5). The expansion:

$$u(x, t) = \sum_{i=1}^n \phi_i(x) \eta_i(t); \quad (6)$$

is substituted in Eqs. (3) and (4), and the resulting expressions are multiplied n times, each time by ϕ_i and integrated with respect to x from $x = 0$ to $x = 1$. The obtained system of coupled ordinary differential equations, for $f = 0$ can be expressed in the matrix form as:

$$\underline{\underline{M}} \ddot{\underline{\eta}} + \underline{\underline{K}} \underline{\eta} = \underline{0}; \quad (7)$$

where the vector $\underline{\eta}$ is the transpose of the vector $\{\eta_1 \ \eta_2 \ \dots \ \eta_n \ z\}$. A general solution of the form:

$$\underline{\eta} = \underline{u} e^{j\lambda t}; \quad (8)$$

is assumed and on substitution, Eq. (7) yields:

$$(-\lambda^2 \underline{\underline{M}} + \underline{\underline{K}}) \underline{u} = \underline{0}; \quad (9)$$

which is an algebraic eigenvalue problem, where λ^2 represents the eigenvalues and $\underline{u}$ the eigenvectors. The eigenvalue problem is solved; the results obtained are bench marked for certain parameter combinations in Appendix 1. The elements of the mass matrix $\underline{\underline{M}}$ and the stiffness matrix $\underline{\underline{K}}$ depend on the dimensionless system parameters m , k and x_0 . The values of the resonator mass m and stiffness k , depend on the designed resonator and is usually decided early in the product design cycle. For fixed values of m and k as the location x_0 changes, the natural frequencies, which are the square roots of the eigenvalues, and the mode shapes (obtained from the eigenvectors using expansion (6)) of the system vary. At $x_0 = 0$ the natural frequencies of a standard clamped-free beam are recovered. In addition, a frequency $\omega_r = \sqrt{\frac{k}{m}}$ corresponding to the uncoupled resonator is also obtained. The loci of the first

Figure 2. The loci of the lowest four natural frequencies as the attachment location x_0 is varied from 0 to 1 for mass $m = 1$ and stiffness $k = 150$. The (red) straight line parallel to the x axis marks the resonator natural frequency $\omega_r = \sqrt{\frac{k}{m}}$.

four natural frequencies as the resonator attachment location is changed from $x_0 = 0$ to $x_0 = 1$, while keeping parameters $m = 1$ and $k = 150$ are presented in Fig. 2. It is observed that the variation of x_0 causes the various modes of the system to interact with each other. The loci presented in Fig. 2 show significant interaction between the second and the third mode of the system. The natural frequencies approach each other and veer away at the location corresponding to the point marked 'V'. Note that at V, the mode shapes corresponding to these two natural frequencies are the same.

We search for the locations x_0 on the cantilever substrate where the attachment of a resonator of mass m and stiffness k makes x_0 a node point for a mode shape of the system. Such a nodal point would not oscillate if the system is forced at the frequency corresponding to that particular mode shape. For such a nodal location, $y(t) = 0$ and Eq. (4) becomes:

$$m \frac{d^2 z}{dt^2} + kz = f \sin \Omega t. \quad (10)$$

Equation (10) represents a single-DoF spring-mass oscillator with natural frequency $\sqrt{\frac{k}{m}}$, and which is forced at the frequency Ω . Therefore, if x_0 is a nodal location, one natural frequency of the system modeled by Eq. (7) has to equal $\sqrt{\frac{k}{m}}$. Moreover, to excite this particular mode, for which x_0 is a node point, the forcing speed Ω has to equal $\sqrt{\frac{k}{m}}$. The nodal location x_0^c is that value of x_0 at which any one of the natural frequencies equals $\sqrt{\frac{k}{m}}$. The frequency $\omega_r = \sqrt{\frac{k}{m}}$ is marked in Fig. 2 using the (red) solid line-type. The nodal location is obtained from the loci as the value of x_0 at which any locus intersects this horizontal line. The required nodal location for the pair of parameters $m = 1$ and $k = 150$ is found to be $x_0 = 0.5692$.

The loci of natural frequencies when location x_0 is varied for the case when $m = 1$ and $k = 300$ is presented in Fig. 3. The locus of the second mode intersects the frequency $\omega_r = \sqrt{\frac{k}{m}}$ at two distinct points, $x_0 = 0.6697$ and $x_0 = 0.9430$, and therefore there are two resonator mounting locations along the

Figure 3. The loci of the lowest four natural frequencies as the attachment location x_0 is varied from 0 to 1 for mass $m = 1$ and stiffness $k = 300$. The (red) straight line parallel to the x axis marks the resonator natural frequency $\omega_r = \sqrt{\frac{k}{m}}$.

Figure 4. The first four mode shapes corresponding to the nodal locations marked on Fig. 3; (a) when $x_0^c = 0.6697$ and (b) when $x_0^c = 0.9430$.

substrate where the attachment point of the resonator to the substrate is a node point. The mode shapes corresponding to the lowest four natural frequencies of the system when the resonator is attached to these locations are presented in Fig. 4(a) and Fig. 4(b) respectively. The (red) solid line-type corresponds to the mode shape for which the attachment point of the resonator is a node point. The resonator locations are marked on these mode shapes with the (red) circle marker.

For resonator designs with parameters m and k that correspond to multiple possible nodal locations x_0^c as presented in Fig. 3, the selection of one mounting location over the other is the designer's choice. However in most engineering cases, it is required that the oscillations of the resonator are of as high amplitude as possible when compared to the oscillations of the substrate. In other words, the total energy associated with the vibrating substrate (which is the sum of the kinetic and strain energies of the cantilever beam) should be as small as possible in comparison with the sum of the kinetic and potential energies of the resonator. A technique that can aid designers to choose one nodal location over the other is detailed in Ap-

Figure 5. The loci of the lowest four natural frequencies as the attachment location x_0 is varied from 0 to 1 for mass $m = 1$ and stiffness $k = 550$. The (red) straight line parallel to the x axis marks the resonator natural frequency $\omega_r = \sqrt{\frac{k}{m}}$.

pendix 2.

Figure 5 presents the loci of natural frequencies as location x_0 is varied for the values of $m = 1$ and $k = 550$. None of the loci crosses the line which marks the frequency $\omega_r = \sqrt{\frac{k}{m}}$. Therefore there are no locations x_0 along the beam where the attachment of a resonator with mass $m = 1$ and stiffness $k = 550$ makes the attachment point a node point.

4. NODAL ATTACHMENT LOCATIONS

The system under investigation has two points of fastening: the clamped boundary of the elastic substrate (at $x = 0$) and the location x_0 where the resonator is attached to the substrate. The displacement $u(0, t)$ is zero due to the imposed boundary conditions. Hence for minimum anchor loss, the condition that needs to be satisfied in locating an already designed resonator, of predefined values m and k , is to select x_0 such that it coincides with a node point of the combined assembly of the resonator and the substrate. A technique to obtain such nodal locations without tracing the loci of the natural frequencies is presented next.

The characteristic polynomial, associated with the eigenvalue problem presented in Eq. (9), is obtained by computing the determinant of the matrix $(-\lambda^2 \underline{M} + \underline{K})$ and is of degree $2(n + 1)$. The characteristic equation is of the form:

$$\lambda^{2(n+1)} + p_{(n+1)-1} \lambda^{2(n+1)-2} + p_{(n+1)-2} \lambda^{2(n+1)-4} + \dots + p_0 = 0. \quad (11)$$

For instance, if $n = 3$ we obtain $\lambda^8 + p_3 \lambda^6 + p_2 \lambda^4 + p_1 \lambda^2 + p_0 = 0$. The p_i of Eq. (11) are functions of the system parameters m , k and x_0 . For a particular resonator (with fixed values of m and k), the only system parameter that the coefficients of the characteristic equation depend on is the anchor location x_0 . At the nodal location, $x_0 = x_0^c$, as explained earlier, one of the natural frequencies of the system must equal the uncoupled resonator frequency $\omega_r = \sqrt{\frac{k}{m}}$. To find the value of x_0 at which the attachment location of the resonator to the substrate becomes

Figure 6. The number of possible nodal locations x_0^c for a resonator of mass m and stiffness k . The ‘ $\cdot$ ’ marker type represents parameter combinations with a single nodal location, the ‘ o ’ marker type with two, the ‘ $\bullet$ ’ marker type with three, the ‘ $\times$ ’ marker type with four and the ‘ $*$ ’ marker type with five. The unhatched regions correspond to parameter combinations without any possible nodal location x_0^c .

Figure 7. The variation of the nodal locations x_0^c as stiffness k changes for mass $m = 1$. These parameters correspond to the (red) dashed line of Fig. 6. Regimes P and S have one nodal location, Q has two and R has none.

a node point, the condition that one natural frequency should equal $\omega_r = \sqrt{\frac{k}{m}}$ is enforced. Therefore, $\lambda = \omega_r$ is substituted in the characteristic equation and the resulting nonlinear equation in x_0 is solved numerically in the domain $x_0 \in [0, 1]$ to obtain the required locations x_0^c . Here, the value of x_0 is computed using a combination of the bisection and the secant numerical methods.

As was seen in Section 3, it is possible that there exists multiple solutions for x_0^c (i.e. multiple nodal locations) for a resonator of mass m and stiffness k . The number of such nodal locations depends on the values of these resonator parameters. In general, as the value of the frequency ω_r increases, the number of possible solutions for x_0^c within the regime $0 < x < 1$ also increases. A chart depicting the number of possible values of x_0^c for various combinations of parameters m and k is presented in Fig. 6. The hatched regions represent parameter combinations with at least one possible value of x_0 . There is

Table 3. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass $m = 0.05$. The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

Stiffness	Nodal locations	Natural frequencies			
k	x_0^c	ω_1	ω_2	ω_3	ω_4
1	0.1460	3.5149	<u>4.4721</u>	22.0371	61.7024
10	0.6118	3.4341	<u>14.1421</u>	22.5181	61.7886
50	0.2024	3.5145	21.6437	<u>31.6228</u>	62.4997
150	0.4301	3.4921	20.8560	<u>54.7723</u>	64.9889
	0.5801	3.4506	21.1161	<u>54.7723</u>	65.0191
	0.8111	3.3370	22.0015	<u>54.7723</u>	64.2048
	0.9543	3.2411	20.8259	<u>54.7723</u>	70.2207
300	0.1241	3.5158	21.9894	60.0485	<u>77.4596</u>

an interesting unhatched region (marked R) banked by hatched regions on both sides. For combinations of values of m and k within this region, there are no locations x_0 along the substrate where the resonator can be mounted so that the attachment point is a node point for any mode shape of the system. The (red) dashed line corresponds to parameter $m = 1$. For this value of m , there are various regimes of stiffnesses within $0 < k < 1000$ for which the nodal location x_0 can have either no solution, one solution or two solutions. These nodal locations as a function of stiffness k are presented in Fig. 7. The stiffness values corresponding to regions marked P and S have one possible nodal location x_0^c , Q has two and R has none. The parameters chosen to develop Figs. 2, 3 and 5 fall in the regions marked P, Q and R of Fig. 7, respectively.

The nodal attachment locations x_0^c for various values of k when $m = 0.05$ is presented in Table 3. For $k = 1, 10, 50$ and 300 , the system has one nodal point each. However, for $k = 150$, there are four possible attachment locations at which the mounting point is a node point for the assembly comprising the resonator and the substrate. The table also presents the lowest four natural frequencies of the system for each of these combinations of mass m and stiffness k .

The nodal locations and natural frequencies for the cases when $m = 0.1, 0.25, 0.5, 1$ and 2 are presented in Tables 4, 5, 6, 7, and 8 respectively. As evident in Figs. 5, 6 and 7, a nodal location x_0^c does not exist for all combinations of resonator parameters m and k . For those values of m and k , which do not yield a nodal location x_0^c , the corresponding table entries for x_0^c are marked ‘ $\times$ ’ and the natural frequencies are left blank.

Throughout this investigation, the number of basis functions n was taken to be 6. The resulting system had 7 natural frequencies (6 of which originated from the previously mentioned basis functions ϕ_i and 1 from the resonator DoF) out of which the lowest four changed by less than 0.0005 percent when n was increased from 5 to 6. For instance, when $m = 1$ and $k = 150$, as n was incremented from 5 to 6, the value of x_0^c changed negligibly and the fourth natural frequency, ω_4 decreased by just 0.0002 percent from 62.3425 to 62.3423.

5. POWER TRANSMITTED THROUGH THE ANCHOR

The energy transmitted per second through the anchors is a measure of the structural interaction between a resonator and

Table 4. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass $m = 0.1$. The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

Stiffness	Nodal locations	Natural frequencies			
k	x_0^c	ω_1	ω_2	ω_3	ω_4
1	$\times$				
10	0.5042	3.4246	<u>10.0000</u>	22.6038	61.6970
50	$\times$				
150	0.2937	3.5038	20.5303	<u>38.7298</u>	65.9878
300	0.4301	3.4685	19.8956	<u>54.7723</u>	67.8976
	0.5800	3.3885	20.3772	<u>54.7723</u>	67.9836
	0.8111	3.1829	21.9757	<u>54.7723</u>	66.4487
	0.9543	3.0214	20.0900	<u>54.7723</u>	77.3137

Table 5. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass $m = 0.25$. The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

Stiffness	Nodal locations	Natural frequencies			
k	x_0^c	ω_1	ω_2	ω_3	ω_4
1	$\times$				
10	0.3242	3.4547	<u>6.3246</u>	22.3541	61.8747
50	0.6118	3.1569	<u>14.1421</u>	24.3135	62.1638
150	$\times$				
300	0.2445	3.5005	19.5896	<u>34.6410</u>	67.9733

Table 6. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass $m = 0.5$. The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

Stiffness	Nodal locations	Natural frequencies			
k	x_0^c	ω_1	ω_2	ω_3	ω_4
1	$\times$				
10	0.1460	3.5048	<u>4.4721</u>	22.0646	61.7514
50	0.5042	3.1208	<u>10.0000</u>	24.7245	61.6970
150	0.6697	2.7560	<u>17.3205</u>	26.9796	64.0120
	0.9430	2.1228	<u>17.3205</u>	35.2021	63.7346
300	$\times$				

Table 7. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass $m = 1$. The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

Stiffness	Nodal locations	Natural frequencies			
k	x_0^c	ω_1	ω_2	ω_3	ω_4
1	$\times$				
10	$\times$				
50	0.3730	3.1862	<u>7.0711</u>	24.0248	62.3493
150	0.5692	2.6426	<u>12.2474</u>	28.7684	62.3423
300	0.6697	2.3438	<u>17.3205</u>	30.4062	66.4971
	0.9430	1.6622	<u>17.3205</u>	42.7217	66.7933

Table 8. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass $m = 2$. The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

Stiffness	Nodal locations	Natural frequencies			
k	x_0^c	ω_1	ω_2	ω_3	ω_4
1	$\times$				
10	$\times$				
50	0.2084	3.3905	<u>5.0000</u>	22.5040	62.3368
150	0.4531	2.6377	<u>8.6603</u>	28.8844	62.1052
300	0.5692	2.2087	<u>12.2474</u>	33.7125	63.0963

Figure 8. (a) The frequency of forcing, and the magnitudes of (b) the force transmitted through the anchor, of (c) the velocity of the anchorage location and of (d) the power transmitted by the force, as the attachment location of the resonator on the substrate is varied from $x_0 = 0$ to $x_0 = 1$ for three different sets of damping coefficients $\alpha = 0.0122, \beta = 0$ (marked by the dashed line type and corresponds to a Q-factor of 1000 for the mode under forcing), $\alpha = 0.0122, \beta = 8.165e - 5$ (marked by the solid line type and corresponds to a Q-factor of 500), and $\alpha = 0.0122, \beta = 1.633e - 4$ (marked by the dot line type and corresponds to a Q-factor of 333).

the substrate on which it is mounted. It is important for the performance of vibratory sensors, such as the ones motivating this investigation, that this interaction is minimal. The ideal anchoring location is the value of x_0 for which the amplitude of the power transmitted through the anchor is minimum. The effect of the attachment location x_0 on the force and the power transmitted through the point of anchorage (of the resonator to the substrate) is investigated here.

Vibratory sensors are operated at the resonance corresponding a mode of interest. Here we select the forcing speed γ to be equal to the second natural frequency of the system. Note that γ changes with the attachment location x_0 as presented in Fig. 8(a). The parameters $m = 1$ and $k = 150$ are selected (the loci of natural frequencies for the same set parameters was presented in Fig. 2). Note that the undamped, unforced variant of the same discretized system was investigated previously in Sections 3 and 4. A lightly dissipative version of the system presented in Section 2 is explored here. Damping is introduced directly in the discretized governing equations (Eq. (7)) and is assumed to be of the proportional type.³⁰ The equations governing the forced system are obtained as:

$$\underline{M} \ddot{\eta} + \underline{C} \dot{\eta} + \underline{K} \eta = \underline{Q} \sin \gamma t; \quad (12)$$

where $\underline{Q}$ (which is of size $(n + 1, 1)$) is a matrix of zeros except the $n + 1^{th}$ element which is one, and $\underline{C} = \alpha \underline{M} + \beta \underline{K}$ where α and β are scalar constants. The viscous damping

Figure 9. (a) The instantaneous force transmitted through the anchor, (b) the velocity of the anchor, and (c) the rate of work done by the force when the resonator attachment locations are $x_0 = 0.5$ (dash line-type), $x_0 = 0.5692$ (solid line-type) and $x_0 = 0.75$ (dot line-type) as functions of time when $m = 1$, $k = 150$, $\alpha = 0.0122$, $\beta = 0$ and $f = 1$.

(c) experienced by mass m (in the dimensionless form) is the $(n + 1)^{th}$ diagonal element of the damping matrix, i.e. $c = \underline{\underline{C}}(n + 1, n + 1)$. The solution $\underline{\eta}(t)$ of Eq. (12) is obtained by numerical integration using the ode45 routine of MATLAB, and is mapped back to coordinates $u(x, t)$, $y(t)$ and $z(t)$ using Eq. (6). The instantaneous force transmitted through the anchor is:

$$F_t(t) = k(z - y) + c \left(\frac{dz}{dt} - \frac{dy}{dt} \right). \quad (13)$$

The forces transmitted through the anchor as functions of time for three different anchorage locations $x_0 = 0.5$, 0.5692 and 0.75 are presented in Fig. 9(a) when the constants of proportional damping take on values $\alpha = 0.0122$ and $\beta = 0$ (which correspond to a Q-factor of approximately 1000 for the mode of forcing). The velocity of the attachment points $\frac{dy}{dt}$ for these locations are presented in Fig. 9(b). The location $x_0 = 0.5692$ is a node point for the undamped version of the system and the introduced nominal damping has negligible effect on the nodal location as the amplitude of the velocity $\frac{dy}{dt}$ is nearly zero here. Note that the effect of structural damping on the location of the node points of an Euler-Bernoulli beam under resonant forcing for a similar physical system is investigated and found to be negligible in Reference.³¹ The power:

$$P_t(t) = F_t \frac{dy}{dt} = \left(k(z - y) + c \left(\frac{dz}{dt} - \frac{dy}{dt} \right) \right) \frac{dy}{dt}; \quad (14)$$

transmitted through the anchors as a function of time is presented in Fig. 9(c). The amplitude of the power transmitted is a function of the attachment location x_0 . Figures 8(b), (c) and (d) present the variation in the magnitudes of the force F_t , the velocity of the anchor $\frac{dy}{dt}$, and the power transmitted through the anchor P_t , as the anchoring location x_0 is varied from $x_0 = 0$ to $x_0 = 1$. There are two points along the length of the substrate where the amplitude of the power transmitted through the anchor approaches zero. The first point $x_0 = 0$ is a clamped boundary of the substrate and the second is the

location $x_0 = 0.5692$ which is a node point for the assembly of the resonator and the substrate. It is interesting to note the occurrence of a sharp peak (labeled 'M' on Fig. 8(d)). If the resonator is attached at the location corresponding to the peak, the frequency of forcing which corresponds to the second mode of vibration in this case is maximum and the power transmitted has greatest amplitude.

It is observed that attaching a resonator at a nodal location minimizes the power transferred and the structural interaction between the resonator and the substrate. Therefore, to achieve the expected performance metrics of high Q-factor vibratory sensors, it is recommended to locate resonators at the node points of the assembly.

6. SUMMARY

Nodal positioning helps to minimize anchor losses and therefore enhances the overall Q-factor of high performance resonators. This work discusses the problem of positioning resonators at the nodal locations of a substrate. The technique developed is demonstrated in the problem of positioning a single-DoF spring-mass resonator on a generic cantilever substrate. The equations governing the dynamics of the system are presented and the associated eigenvalue problem is solved. The loci of natural frequencies as a function of the resonator location are developed and the nodal locations are identified. The nodal locations are calculated numerically from the characteristic equation as well. A chart depicting the number of possible nodal locations for various values of resonator parameters is presented. For certain resonator parameters, multiple nodal locations exist along the substrate. It was found that there also are additional resonator parameter combinations without a possible nodal location anywhere along the span of the substrate. The nodal locations for various combinations of resonator mass and resonator stiffness are presented in the form of lookup tables. The ideas presented here can be extended to more complicated structures. The method employed is expected to assist designers who aim to position a high performance resonator on a substrate without compromising its Q-factor.

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Table 9. Comparison of frequencies obtained with existing literature for parameter $x_0 = 1$.

Parameters	Author(s)	Frequency			
		ω_1	ω_2	ω_3	ω_4
$m = 0.2, k = 0.1$	Rossit and Laura ¹³	0.6952	3.5747	22.044	61.701
	Banerjee ¹⁴	0.69517	3.5747	22.044	61.7
	Present	0.6952	3.5747	22.0431	61.7002
$m = 1, k = 1$	Rossit and Laura ¹³	0.8594	4.0711	22.1257	61.7293
	Banerjee ¹⁴	0.8592	4.0711	22.126	61.73
	Present	0.8594	4.0712	22.1254	61.7295
$m = 1, k = 10$	Rossit and Laura ¹³	1.4194	7.4469	22.9987	62.0267
	Banerjee ¹⁴	1.4194	7.447	22.999	62.027
	Present	1.4194	7.4488	22.9993	62.0268

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APPENDIX 1: BENCHMARKING

Several researchers have investigated the problem of evaluating the natural frequencies of a cantilever beam carrying a mass mounted elastically at its free end. To validate the solution procedure employed in this work, the natural frequencies of this system were obtained using the procedure presented in Section 3 and are compared with those of existing literature in Table 9. The results are in excellent agreement. Note that in Reference,¹³ the author presents the values of parameter β_i which need to be squared to obtain the natural frequencies being compared.

APPENDIX 2: CRITERION TO LOCATE RESONATORS WHEN MULTIPLE NODAL LOCATIONS EXISTS

The mode shapes and natural frequencies of the assembly are obtained from the eigenvalue problem presented in

Eq. (9). When multiple nodal locations exists (as for the case presented in Fig. 3), to pick one over the other, it is required to compare the eigenvectors (corresponding to the eigenvalues of magnitude $\sqrt{\frac{k}{m}}$) of these cases. Let the eigenvector corresponding to a nodal location x_{0q}^c be $\underline{u}_q = \{u_{q,1} \ u_{q,2} \ u_{q,3} \cdots u_{q,n} \ u_{q,n+1}\}^T$ where T denotes the transpose operation. The first n elements of this eigenvector correspond to the mode shape of the substrate and the $(n+1)$ th element to the displacement z of resonator mass. To compare distinct eigenvectors for the same resonator amplitude, these eigenvectors are normalized with respect to the $(n+1)$ th element. Therefore the z -normalized mode shapes are obtained as:

$$\psi_q(x) = \frac{1}{u_{q,n+1}} \sum_{i=1}^n u_{q,i} \cdot \phi_i(x). \quad (\text{A.1})$$

The z -normalized mode shapes are those configurations of the substrate which correspond to the same z displacement of magnitude unity. The most energy efficient mounting location out of all possible x_0^c is that for which the total energy associated with the oscillatory motion of the substrate at the corresponding z -normalized mode shape is minimum. The motion of the substrate at such a mode shape with the frequency $\Omega = \sqrt{\frac{k}{m}}$ can be expressed as:

$$u(x, t) = \psi(x) e^{j\Omega t}. \quad (\text{A.2})$$

The total energy associated with the cantilever beam is the sum of its kinetic (T) and potential (V) energies and can be expressed as:

$$E_{\text{substrate}} = T_{\text{substrate}} + V_{\text{substrate}} = \int_0^1 \frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 dx + \int_0^1 \frac{1}{2} \left(\frac{\partial^2 u}{\partial x^2} \right)^2 dx. \quad (\text{A.3})$$

Therefore the energy associated with a normalized mode shape (of Eq. (A.2)) of the substrate is:

$$E_{\text{substrate}} = \frac{1}{2} \int_0^1 \left(-\frac{k}{m} \psi^2 + \left(\frac{\partial^2 \psi}{\partial x^2} \right)^2 \right) dx. \quad (\text{A.4})$$

Out of all the possible nodal points x_0^c , it is recommended to select the location for which the associated z -normalized mode shape returns a minimum value for the integral presented in Eq. (A.4).