The Exact Frequency Equations for the Euler-Bernoulli Beam Subject to Boundary Damping

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The Euler-Bernoulli (E-B) beam is the most commonly utilized model in the study of vibrating beams. The exact frequency equations for this problem, subject to energy-conserving boundary conditions, are well-known; however, the corresponding dissipative problem has been solved only approximately, via asymptotic methods. These methods, of course, are not accurate when looking at the low end of the spectrum. Here, we solve for the exact frequency equations for the E-B beam subject to boundary damping. Numerous numerical examples are provided, showing plots of both the complex wave numbers and the exponential damping rates for the first five frequencies in each case. Some of these results are surprising.

1. INTRODUCTION

The Euler-Bernoulli beam is, by far, still the most commonly implemented beam element in engineering applications, because most beams in these applications possess aspect ratios that allow one to neglect the effects of rotary inertia and shear deformation.¹ It follows that the homogeneous E-B equation subject to energy-conserving boundary conditions has been well-studied. The exact frequency equations for the problem, subject to various combinations of the four standard energy-conserving boundary conditions — clamped, simply supported, roller-supported and free — are presented in Pilkey² and Han et al.³ (the latter which seems to be the first reference to include the roller-supported condition). The E-B beam subject to boundary damping also has been well-studied; however, to date, the problem has only been solved approximately, using asymptotic methods.^{4–7} (The same, of course, can be said for the Timoshenko beam, for which numerous references can be found using an asymptotic approach.) By their nature, asymptotic methods are not accurate for the lowest modes, the modes possessing the greatest energies of vibration — thus, the most important modes, both mathematically and physically. For example, for the cantilever E-B beam, the first wave number is 1.875,³ while an asymptotic estimate for their number is 1.571 (which we have computed using the methodology from Chen et al.⁴), the error being about 16%.

Here, then, we derive the exact frequency equations for the complex wave numbers of the E-B beam subject to standard energy-dissipative boundary conditions, then solve several special cases and present the results graphically, using Mathematica. Along the way, we find that the dissipative results do match the conservative results, when the system is subjected to the appropriate zero or infinite damping at each end.

This paper, then, is organized as follows. In Section 2, the partial differential equation and energy-dissipative boundary conditions are derived using a standard energy method, the problem is recast in dimensionless form, and diagrams for the physical implementation of the boundary conditions are presented. In Section 3, the system's exact frequency equation

is derived and compared with the exact equations for the ten cases involving energy-conserving boundary conditions. In Section 4 we present numerical results, providing, for a somewhat exhaustive set of special cases, plots of the complex wave numbers as functions of the damping parameters, which include some interesting results, for which we provide a physical explanation. In addition, we plot the (real) exponential damping rates for these same examples; some of these results are surprising, as we shall see.

2. THE PROBLEM

We consider a uniform E-B beam of length L , with the following constant physical parameters: σ , the mass-per-unit-volume; A , the cross-sectional area; E , Young's modulus; and I , the area moment of inertia. Letting $W = W(x, t)$, $0 \leq x \leq L$, $t \geq 0$, be the lateral deflection, the energy of the beam at time t is:

$$E(t) = \frac{1}{2} \int_0^L [\sigma A W_t^2(x, t) + E I W_{xx}^2(x, t)] dx; \quad t \geq 0; \quad (1)$$

(see Han et al.,³ for example). Here and below we use the notation $W_t = \frac{\partial W}{\partial t}$, etc. We require boundary dissipation of energy, so that the boundary conditions must be such that $E'(t) < 0$. But:

$$E'(t) = \int_0^L [\sigma A W_t(x, t) W_{tt}(x, t) + E I W_{xx}(x, t) W_{xxt}(x, t)] dx; \quad (2)$$

and performing integration by parts twice on the second term results in:

$$E'(t) = E I [W_{xx}(x, t) W_{xt}(x, t) - W_{xxx}(x, t) W_t(x, t)] \Big|_{x=0}^{x=L} + \int_0^L W_t(x, t) [\sigma A W_{tt}(x, t) + E I W_{xxxx}(x, t)] dx. \quad (3)$$

Requiring the integral to be zero yields the well known E-B beam equation:

$$\sigma AW_{tt}(x, t) + EIW_{xxxx}(x, t) = 0 \quad 0 < x < L; \quad t > 0. \quad (4)$$

Then, we can ensure that $E'(t) < 0$ by choosing the following standard boundary conditions:

$$-EIW_{xxx}(0, t) = \alpha_1 W(0, t); \quad (5)$$

$$EIW_{xx}(0, t) = \beta_1 W_{xt}(0, t); \quad (6)$$

$$-EIW_{xxx}(L, t) = -\alpha_2 W_t(L, t); \quad (7)$$

$$EIW_{xx}(L, t) = -\beta_2 W_{xt}(L, t); \quad t > 0; \quad (8)$$

where each *damping constant* $\alpha_1, \beta_1, \alpha_2, \beta_2$ is nonnegative and $\alpha_1 + \beta_1 + \alpha_2 + \beta_2 > 0$. We have chosen to write (5)–(8) in the given form in order to reflect the fact that the *shear force* is $Q(x, t) = -EIW_{xxx}(x, t)$ and the *bending moment* is $M(x, t) = EIW_{xx}(x, t)$ (depending on the sign convention chosen). It is easy to implement these boundary conditions physically—we illustrate at the end of this Section.

We next perform the standard separation of variables:

$$W(x, t) = e^{-i\xi^2 t} w(x); \quad (9)$$

where ξ will represent the dimensional wave numbers. Applying Eqs. (9) to Eqs. (4)–(8), the resulting system is:

$$EIw^{(4)}(x) - \sigma A\xi^4 w(x) = 0; \quad 0 < x < L; \quad (10)$$

$$EIw'''(0) = i\xi^2 \alpha_1 w(0); \quad (11)$$

$$EIw''(0) = -i\xi^2 \beta_1 w'(0); \quad (12)$$

$$EIw'''(L) = -i\xi^2 \alpha_2 w(L); \quad (13)$$

$$EIw''(L) = i\xi^2 \beta_2 w'(L). \quad (14)$$

Lastly, we transform the system to dimensionless form by introducing the dimensionless quantities:

$$y = \frac{x}{L}; \quad v = \frac{w}{L}; \quad k^2 = \sqrt{\frac{\sigma A}{EI}} L^2 \xi^2; \quad (15)$$

$$a_j = \frac{L}{\sqrt{\sigma A EI}} \alpha_j; \quad b_j = \frac{1}{L \sqrt{\sigma A EI}} \beta_j; \quad j = 1, 2;$$

and, letting $y \rightarrow x$ and $v \rightarrow w$, we arrive at the dimensionless system:

$$w^{(4)}(x) - k^4 w(x) = 0; \quad 0 < x < 1; \quad (16)$$

$$w'''(0) - ia_1 k^2 w(0) = 0; \quad (17)$$

$$w''(0) + ib_1 k^2 w'(0) = 0; \quad (18)$$

$$w'''(1) + ia_2 k^2 w(1) = 0; \quad (19)$$

$$w''(1) - ib_2 k^2 w'(1) = 0. \quad (20)$$

Before solving the problem, we present here, in Figures 1 and 2, diagrams of passive damping mechanisms for realizing the dissipative boundary conditions given by Eqs. (7) and (8), respectively.

In Figure 1, we have a vertical linear damper, with damping coefficient c_V , at $x = L$. As the velocity is $W_t(L, t)$, the resulting shear force is:

$$Q(L, t) = -c_V W_t(L, t). \quad (21)$$

Figure 1. Vertical damper at right end $x = L$, for realizing the boundary condition given by Eq. (7).

Figure 2. Horizontal damper at right end $x = L$, for realizing the boundary condition given by Eq. (8).

In Figure 2, we have a horizontal linear damper, with damping coefficient c_H , at $x = L$, at a distance $\frac{h}{2}$ from the beam's neutral axis, where h is the beam's vertical thickness. As the horizontal velocity at the damper is $W_{xt}(L, t) \cdot \frac{h}{2}$, the resulting bending moment is:

$$M(L, t) = c_H \frac{h}{2} \text{velocity} = c_H W_{xt}(L, t) \frac{h^2}{4}. \quad (22)$$

(See Chen et al.⁴) (Please note that these are standard linear dampers. Versions of these same dampers but as connectors between two Euler-Bernoulli beams can be found in Chen et al.,⁸ along with asymptotic and experimental results.)

3. EXACT EQUATION FOR THE WAVE NUMBERS

We wish to derive the exact equation satisfied by the complex wave numbers k . To this end, we apply the boundary conditions given by Eqs. (17)–(20) to:

$$w(x) = A \cos kx + B \sin kx + C \cosh kx + D \sinh kx; \quad (23)$$

the general solution of Eq. (16). The resulting equation is:

$$0 = \begin{vmatrix} -1 & ib_1 k & 1 & ib_1 k \\ ia_1 & k & ia_1 & -k \\ -\cos k & -\sin k & \cosh k & \sinh k \\ +ib_2 k \sin k & -ib_2 k \cos k & -ib_2 k \sinh k & -ib_2 k \cosh k \\ k \sin k & -k \cos k & k \sinh k & k \cosh k \\ +ia_2 \cos k & +ia_2 \sin k & +ia_2 \cosh k & +ia_2 \sinh k \end{vmatrix}; \quad (24)$$

Table 1. The first five positive wave numbers for Eqs. (30)–(35).

Boundary Conditions	k_1	k_2	k_3	k_4	k_5
$C - C, F - F$	4.730	7.853	10.996	14.137	17.279
$C - S, S - F$	3.927	7.069	10.210	13.352	16.493
$C - R, R - F$	2.365	5.498	8.639	11.781	14.923
$C - F$	1.875	4.694	7.855	10.996	14.137
$S - S, R - R$	π	2π	3π	4π	5π
$S - R$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$

or, upon simplification:

$$\begin{aligned}
 0 = & k^2(\cos k \cosh k - 1) \\
 & + (a_1 b_1 + a_2 b_2) k^2 (\cos k \cosh k + 1) \\
 & + 2(a_1 b_2 + a_2 b_1) k^2 \cos k \cosh k \\
 & + 2(a_1 a_2 - b_1 b_2 k^4) \sin k \sinh k \\
 & + a_1 b_1 a_2 b_2 k^2 (\cos k \cosh k - 1) \\
 & + ik[(a_1 + a_2)(\cos k \sinh k - \sin k \cosh k) \\
 & - (b_1 + b_2)k^2(\cos k \sinh k + \sin k \cosh k) \\
 & + a_1 a_2 (b_1 + b_2)(\cos k \sinh k - \sin k \cosh k) \\
 & - b_1 b_2 (a_1 + a_2)k^2(\cos k \sinh k + \sin k \cosh k)].
 \end{aligned} \quad (25)$$

Now, we should be able to recover from Eq. (25) the frequency equations for the various conservative E–B beam problems using the following facts, which follow from the boundary conditions given by Eqs. (17)–(20):

$$a_j = b_j = 0 \Rightarrow \text{that end is free (F);} \quad (26)$$

$$a_j = b_j = \infty \Rightarrow \text{that end is clamped (C);} \quad (27)$$

$$a_j = \infty, b_j = 0 \Rightarrow \text{that end is simply supported (S);} \quad (28)$$

$$a_j = 0, b_j = \infty \Rightarrow \text{that end is roller-supported (R).} \quad (29)$$

There are ten such conservative E–B beam problems and, for the sake of comparison, we have derived or re-derived (as many appear in Pilkey² or Han et al.³) their frequency equations:

$$C - C, F - F: \cos k \cosh k - 1 = 0; \quad (30)$$

$$C - S, S - F: \cos k \sinh k - \sin k \cosh k = 0; \quad (31)$$

$$C - R, R - F: \cos k \sinh k + \sin k \cosh k = 0; \quad (32)$$

$$C - F: \cos k \cosh k + 1 = 0; \quad (33)$$

$$S - S, R - R: \sin k \sinh k = 0; \quad (34)$$

$$S - R: \cos k \cosh k = 0; \quad (35)$$

where, e.g., $C - F$ means that one end is clamped and the other is free.

It is easy to see that each of the Eqs. (30)–(35) can be recovered from Eq. (25) using Eqs. (26)–(29). Here, we also provide the first five positive roots of Eqs. (30)–(35) in Table 1, as we shall need them below. (These are provided in Han et al.,³ and we have re-derived them, using Mathematica.)

4. FREQUENCY CURVES AND DAMPING RATES FOR ISOLATED DAMPING CONSTANTS

In order to present the results in a reasonable manner, and given the expected symmetry that interchanging a_1 with a_2 and b_1 and b_2 leaves Eq. (25) unchanged, we consider the four cases where the left end boundary condition is energy-conserving, i.e., where the pair a_1, b_1 satisfies one of the Eqs. (26)–(29). In addition, we isolate the damping constants by considering separately the cases $b_2 = 0$, $b_2 = \infty$, $a_2 = 0$, and $a_2 = \infty$. Thus, we are led to the following 16 frequency equations:

Case 1: $a_1 = b_1 = 0$ (left end free)

$$\begin{aligned}
 1A: (b_2 = 0) \\
 k(\cos k \cosh k - 1) + ia_2(\cos k \sinh k - \sin k \cosh k) = 0;
 \end{aligned} \quad (36)$$

$$\begin{aligned}
 1B: (b_2 = \infty) \\
 a_2(1 + \cos k \cosh k) - ik(\cos k \sinh k + \sin k \cosh k) = 0;
 \end{aligned} \quad (37)$$

$$\begin{aligned}
 1C: (a_2 = 0) \\
 \cos k \cosh k - 1 - ib_2 k(\cos k \sinh k + \sin k \cosh k) = 0;
 \end{aligned} \quad (38)$$

$$\begin{aligned}
 1D: (a_2 = \infty) \\
 b_2 k(1 + \cos k \cosh k) + i(\cos k \sinh k - \sin k \cosh k) = 0;
 \end{aligned} \quad (39)$$

Case 2: $a_1 = b_1 = \infty$ (left end clamped)

$$\begin{aligned}
 2A: (b_2 = 0) \\
 k(1 + \cos k \cosh k) + ia_2(\cos k \sinh k - \sin k \cosh k) = 0;
 \end{aligned} \quad (40)$$

$$\begin{aligned}
 2B: (b_2 = \infty) \\
 a_2(\cos k \cosh k - 1) - ik \cos k \sinh k + \sin k \cosh k = 0;
 \end{aligned} \quad (41)$$

$$\begin{aligned}
 2C: (a_2 = 0) \\
 1 + \cos k \cosh k - ib_2 k(\cos k \sinh k + \sin k \cosh k) = 0;
 \end{aligned} \quad (42)$$

$$\begin{aligned}
 2D: (a_2 = \infty) \\
 b_2 k(\cos k \cosh k - 1) + i(\cos k \sinh k - \sin k \cosh k) = 0;
 \end{aligned} \quad (43)$$

Case 3: $a_1 = 0, b_1 = \infty$ (left end roller-supported)

$$\begin{aligned}
 3A: (b_2 = 0) \\
 2a_2 \cos k \cosh k - ik(\cos k \sinh k + \sin k \cosh k) = 0;
 \end{aligned} \quad (44)$$

$$\begin{aligned}
 3B: (b_2 = \infty) \\
 2k \sin k \sinh k + ia_2(\cos k \sinh k + \sin k \cosh k) = 0;
 \end{aligned} \quad (45)$$

$$\begin{aligned}
 3C: (a_2 = 0) \\
 2b_2 k \sin k \sinh k + i(\cos k \sinh k + \sin k \cosh k) = 0;
 \end{aligned} \quad (46)$$

$$\begin{aligned}
 3D: (a_2 = \infty) \\
 2 \cos k \cosh k - ib_2 k(\cos k \sinh k + \sin k \cosh k) = 0;
 \end{aligned} \quad (47)$$

Figure 3. Curves for the first five wave numbers, k , for $0 \leq \alpha \leq \infty$, for Case 1A, see Eq. (36).

Figure 4. Curves for the first five wave numbers, k , for $0 \leq \alpha \leq \infty$, for Case 1B, see Eq. (37).

Case 4: $a_1 = \infty, b_1 = 0$ (left end simply supported)

$$4A: (b_2 = 0)$$

$$2a_2 \sin k \sinh k + ik(\cos k \sinh k - \sin k \cosh k) = 0; \quad (48)$$

$$4B: (b_2 = \infty)$$

$$2k \cos k \cosh k + ia_2(\cos k \sinh k - \sin k \cosh k) = 0; \quad (49)$$

$$4C: (a_2 = 0)$$

$$2b_2 k \cos k \cosh k + i(\cos k \sinh k - \sin k \cosh k) = 0; \quad (50)$$

$$4D: (a_2 = \infty)$$

$$2 \sin k \sinh k + ib_2 k(\cos k \sinh k - \sin k \cosh k) = 0. \quad (51)$$

We would like to plot the complex wave numbers k , in each of the Eqs. (36)–(51), as a function of the damping parameter α (representing a_2 or b_2). First, let us note that, if we identify a_2 with $\frac{1}{b_2}$, the following cases will be identical: 1A and 2D, 1B and 2C, 1C and 2B, 1D and 2A, 3A and 3D, 3B and 3C, 4A and 4D, and 4B and 4C.

So in Figures 3–10, we plot the curves for the first five wave numbers for cases 1A–1D, 3A, 3B, 4A, and 4B, respectively. In each, we plot $\text{Im}(k)$ versus $\text{Re}(k)$, and we label some points with the values of α corresponding to those points. (These would, of course, correspond to values of $1/\alpha$ in the paired cases not appearing.)

Looking at Figure 3, we have a beam that is free at the left end, while the right end “goes from free ($\alpha = 0$) to simply supported ($\alpha = \infty$).” Thus, e.g., the leftmost curve “starts at” the first wave number for the $F - F$ beam ($k_1 = 4.730$ in Table 1) and “ends at” the first wave number for the $F - S$ beam ($k_1 = 3.927$ in Table 1). The remaining four curves behave similarly, as do all five curves in each of Figures 4, 7, 8, and 9.

In contrast, we see that Figures 5, 6, and 10 are qualitatively different from the above five figures. Indeed, here we see that

Figure 5. Curves for the first five wave numbers, k , for $0 \leq \alpha \leq \infty$, for Case 1C, see Eq. (38).

Figure 6. Curves for the first five wave numbers, k , for $0 \leq \alpha \leq \infty$, for Case 1D, see Eq. (39).

Figure 7. Curves for the first five wave numbers, k , for $0 \leq \alpha \leq \infty$, for Case 3A, see Eq. (44).

Figure 8. Curves for the first five wave numbers, k , for $0 \leq \alpha \leq \infty$, for Case 3B, see Eq. (45).

the first curve of each is much larger than the remaining curves, and is oddly shaped. In addition, each “starts” or “ends at” $k = 0$; similarly, the other four curves are shifted in the same way, as we describe presently.

Let us look at Figure 5, where we have a beam that is free at the left end, while the right end goes from free ($\alpha = 0$) to roller-supported ($\alpha = \infty$). The first curve starts at the wave number $k = 0$ for the $F - F$ beam and ends at $k_1 = 2.365$ for the $F - R$ beam. Similarly, the second curve starts at $k_1 =$

Figure 9. Curves for the first five wave numbers, k , for $0 \leq \alpha \leq \infty$, for Case 4A, see Eq. (48).

Figure 10. Curves for the first five wave numbers, k , for $0 \leq \alpha \leq \infty$, for Case 4B, see Eq. (49).

4.730 for the $F - F$ beam and ends at $k_2 = 5.498$ for the $F - R$ beam. (Again, see Table 1 for the given values of k .) The remaining curves behave similarly, as do those in Figures 6 and 10.

We believe that one can explain this shifting by looking at the vibration modes. Let us compare modes for Cases 1A (Figure 3) and 1C (Figure 5). For Case 1A, we plot the modes corresponding to $k_0 = 0$, k_1 , k_2 , and k_3 for the $F - S$ beam and, below each, the corresponding mode for the $F - F$ beam; see Figure 11. Let us begin by considering a vibration at the k_1 -mode for the $F - S$ beam, corresponding to $\alpha = \infty$, and slowly “relax” the right end boundary condition by letting $\alpha \rightarrow 0$. The resulting $F - F$ beam will then be vibrating at its k_1 -mode. Similarly, as $\alpha \rightarrow 0$, each $F - S$ mode “goes to” the $F - F$ mode directly below it. This behavior is consistent with the curves in Figure 3.

For Case 1C, we plot the same modes as above for the $F - R$ beam and, below them, the $F - F$ beam, in Figure 12. Now, consider the k_2 -mode for the $F - R$ beam, corresponding to $\alpha = \infty$, and again allow $\alpha \rightarrow 0$. The resulting $F - F$ beam will now be vibrating at its k_1 -mode. More generally, as $\alpha \rightarrow 0$, the k_j -mode of the $F - R$ beam “goes to” the k_{j-1} -mode of the $F - F$ beam, $j = 1, 2, \dots$. This behavior is consistent with the curves in Figure 5. We also note that we have been unable to explain the odd size and shape of the first curve, and we hope that future research will lead to an explanation.

We have found that the same explanation is valid for all 16 cases and is consistent with Figures 3–10. We should note that, as α varies from zero to infinity or vice versa, each mode “goes to” a mode with the same number of turning points.

Next, we look at the damping rates for each of the eight

cases considered above. Thus, in Figures 13–20, we plot $\text{Re}(-ik^2)$ (i.e., the real part of the coefficient in the exponent of e^{-ik^2t} ; see Eq. (9)) versus the damping constant α . In each Figure, the top curve is the damping rate for the first mode, the next one below for the second mode, etc. We note that these plots give an idea of the value of the damping parameter α that maximizes the damping rate for each mode and, of course the value of $\frac{1}{\alpha}$ that does the same for the corresponding paired case not appearing. Each curve is zero at the left end and approaches zero at the right end, as expected. All of the curves are smooth except for three obvious exceptions, namely, the topmost (starting from $\alpha = 0$) curve in each of Figures 15, 16 and 20, each of which corresponds, of course, to the oddly shaped wave number curves in Figures 5, 6 and 10, respectively. In particular, we see that each of these curves appears to have a cusp. To determine whether it truly is a cusp, we have included in Figure 21 close-up plots of the topmost curve from Figure 15. On the left we have plotted the whole curve and, on the right, we have zoomed in on the cusp-like part of the curve. It does, indeed, seem to be a cusp. Again, we do not know the reasons behind this interesting and anomalous behavior, and we hope that future research will shed some light on the issue.

5. CONCLUSION

The exact equation for the complex wave numbers of the Euler-Bernoulli beam, subject to standard linear boundary damping, has been derived. In addition, we have presented 16 examples/special cases and have plotted the curves for the first five wave numbers, as well as those for the corresponding (real) exponential damping rates, for each case. We have discovered some interesting behavior in six of these cases. A “shifting” in the wave number curves in these cases has been explained by looking at the corresponding vibration modes. However, we have been unable to explain the anomalous behavior of the first wave number curve in each of these cases, along with the presence of a cusp in the corresponding damping rate curve. We hope that future research will shed some light on this interesting behavior.

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Figure 11. First four modes, including the 0^{th} -mode, for the $F-S$ beam (top) and for the $F-F$ beam (bottom). Ends marked “•” are simply supported, while those marked “o” are free.

Figure 12. First four modes, including the 0^{th} -mode, for the $F-R$ beam (top) and for the $F-F$ beam (bottom). Ends marked “•” are roller-supported, while those marked “o” are free.

Figure 13. Plots of $\text{Re}(-ik^2) = \text{Im}(k^2)$ versus α for the first five wave numbers, k (top to bottom, respectively), for Case 1A, see Eq. (36).

Figure 15. Plots of $\text{Re}(-ik^2) = \text{Im}(k^2)$ versus α for the first five wave numbers, k (top to bottom, respectively), for Case 1C, see Eq. (38).

Figure 14. Plots of $\text{Re}(-ik^2) = \text{Im}(k^2)$ versus α for the first five wave numbers, k (top to bottom, respectively), for Case 1B, see Eq. (37).

Figure 16. Plots of $\text{Re}(-ik^2) = \text{Im}(k^2)$ versus α for the first five wave numbers, k (top to bottom, respectively), for Case 1D, see Eq. (39).

Figure 17. Plots of $\text{Re}(-ik^2) = \text{Im}(k^2)$ versus α for the first five wave numbers, k (top to bottom, respectively), for Case 3A, see Eq. (44).

Figure 18. Plots of $\text{Re}(-ik^2) = \text{Im}(k^2)$ versus α for the first five wave numbers, k (top to bottom, respectively), for Case 3B, see Eq. (45).

Figure 19. Plots of $\text{Re}(-ik^2) = \text{Im}(k^2)$ versus α for the first five wave numbers, k (top to bottom, respectively), for Case 4A, see Eq. (48).

Figure 20. Plots of $\text{Re}(-ik^2) = \text{Im}(k^2)$ versus α for the first five wave numbers, k (top to bottom, respectively), for Case 4B, see Eq. (49).

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Figure 21. On the left, a plot of the topmost curve from Figure 15, Case 1C and, on the right, a close-up of the cusp-like part of the same curve.

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