Stochastic Minimax Vibration Control for Uncertain Nonlinear Quasi-Hamiltonian Systems with Noisy Observations

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A stochastic minimax control strategy for uncertain nonlinear quasi-Hamiltonian systems with noisy observations under random excitations is proposed based on the extended Kalman filter and minimax stochastic dynamical programming principle. A structure system with smart sensors and actuators is modeled as a controlled, excited and dissipative Hamiltonian system with noisy observations. The differential equations for the uncertain nonlinear quasi-Hamiltonian system with control and observation under random excitation are given first. The estimated nonlinear stochastic control system with uncertain parameters is obtained from the uncertain quasi-Hamiltonian system with noisy observation. In this case, the optimally estimated state is determined by the observation based on the extended Kalman filter. The dual dynamical programming equation for the estimated uncertain system is then obtained based on the minimax stochastic dynamical programming principle. The worst-case disturbances are determined for bounded uncertain parameters and the optimal control law is determined for the worst case by the programming equation. The proposed minimax control strategy is applied to two uncertain nonlinear stochastic systems with controls and noisy observations. The control effectiveness for the stochastic vibration response reductions of the systems is illustrated with numerical results. The proposed minimax control strategy is applicable to general uncertain nonlinear multi-degree-of-freedom structure systems with noisy observations under random excitations.

1. INTRODUCTION

The vibration control for engineering structures subjected to strong random excitations is a significant research subject.^{1–3} Smart materials such as piezoelectric and magnetorheological materials have been applied to the structural vibration mitigation.^{4–20} A structure system with smart sensors, actuators and controllers is called a smart structure. A smart structure system can sense structural response to external excitations and produce action to control structural responses.^{21–23} The control performance of the smart structure system depends strongly on its used control strategy. The system control includes the state estimation based on sensing data and the response control based on estimated states, which are coupled with each other.^{24–28} However, the sensing data, or observations and excitations, inevitably have random noise components. Only linear control strategies for smart structure systems with noisy observations have been proposed presently.

The smart structure, including multi-degree-of-freedom systems, beams and plates in modal vibration, can be modeled as a controlled, excited and dissipative Hamiltonian system or quasi-Hamiltonian system with observation.²⁹ The quasi-Hamiltonian system is nonlinear for strong random excitations. The stochastic optimal control for linear and nonlinear systems has been studied and many control strategies have been pre-

sented.^{29–46} In general, the separation theorem is used to convert the original system with noisy observations into another system with complete observations and then the optimal control law is determined based on the stochastic dynamical programming principle. However, for the nonlinear stochastic system with noisy observations, the separation theorem yields an estimated system with infinite-dimensional probability density, such that the optimal control cannot be determined.²⁷ Only several studies presented the stochastic optimal control for a nonlinear system with noisy observation under very specified condition. A non-trivial solution to the estimation is still difficult to obtain.⁴⁴ Thus, an approximate optimal estimation is an alternative for practice. The extended Kalman filter, as an optimum feasible approximate filter, has been applied to the nonlinear stochastic control system with noisy observation recently.^{24, 28, 45, 46}

The analysis system of an actual engineering structure has usually uncertain parameters and external disturbances, which can make the designed control performance degenerate.^{47, 48} The control problem on uncertain systems can be resolved according to a robust control strategy.⁴⁹ The uncertain optimal control is generally treated as a minimax control problem, in which the minimization is taken over all control strategies and the maximization is taken over possible uncertain-

ties.⁵⁰ The robust control for deterministic linear and nonlinear systems with uncertain disturbances has been studied extensively.^{49–51} The stochastic robustness of linear control systems was analysed.⁵² The minimax control for linear stochastic uncertain systems was proposed by optimizing the worst-case performance under certain uncertainty constraints and separating estimator and controller.^{53–56} The minimax control for nonlinear stochastic uncertain systems was proposed based on the stochastic differential game theory for bounded uncertainties.^{57,58} However, the system state estimation and observation have not been considered in the minimax nonlinear stochastic control strategy. Thus, the minimax vibration control for uncertain smart structure systems, or uncertain nonlinear quasi-Hamiltonian systems, with noisy observations under random excitations needs to be studied further, for example, based on the extended Kalman filter and stochastic minimax control strategy which includes the stochastic differential game theory and stochastic dynamical programming principle.

This paper studied the stochastic minimax control for the vibration response reduction of uncertain nonlinear quasi-Hamiltonian systems with noisy observations. A new stochastic minimax control strategy for the uncertain quasi-Hamiltonian systems was proposed based on the extended Kalman filter and minimax stochastic dynamical programming principle. First, the structure systems with smart sensors and actuators were modeled as controlled, excited and dissipative Hamiltonian systems with noisy observations. The differential equations for the uncertain nonlinear quasi-Hamiltonian systems with controls and observations under random excitation were given. Second, the extended Kalman filter was applied to convert the quasi-Hamiltonian systems with noisy observations into the completely observed nonlinear stochastic control systems with uncertain parameters for the estimated state. Third, the minimax stochastic dynamical programming principle was applied to derive the dual dynamical programming equation. The worst-case disturbances were determined for bounded uncertain parameters, and then the optimal control law was determined for the worst case and estimated state by the programming equation. Finally, the proposed minimax control strategy was applied to two nonlinear stochastic systems with controls and noisy observations. Numerical results were given to illustrate the control effectiveness.

2. VIBRATION EQUATIONS OF UNCERTAIN QUASI-HAMILTONIAN SYSTEM WITH NOISY OBSERVATION

Smart materials such as piezoelectric material have been applied recently to the vibration control of engineering structures under random excitations. A structure with smart devices can sense structural response to external excitations and produce action to control structural response. The feedback control is based on the state observation, or sensing response. The state estimation needs to be considered first for the response control. The basic equations for main structure, actuator structures and sensor structures have been given by.^{45,46,59,60} These equations can be transformed into dynamic equations in the form of second-order differentials, with respect to time, by using structural dominant modes. The equations are commonly expressed as Lagrangian equations. However, for optimal estimation and control, the Lagrangian equations need to be con-

verted into other equations in the form of first-order differentials. Generalized Hamiltonian equations are an optimum option. The Hamiltonian generally represents system energy and the stochastic averaging method can be applied to evaluate system response. Thus, the main structure with actuators was expressed as quasi-Hamiltonian equations with control, excitation and dissipative forces. The sensor structure was expressed as an observation equation, in which the inertia force was small and neglected. Actual engineering structures have inevitably uncertain parameters and external disturbances, which can make the control performance degenerate and need to be considered. The differential equation of the uncertain quasi-Hamiltonian system with n degree-of-freedom was given by:⁴⁵

$$\dot{\mathbf{X}} = \mathbf{J} \frac{\partial H_a}{\partial \mathbf{X}} + \mathbf{C}\mathbf{X} + \Delta(\mathbf{X}, t) + \mathbf{B}\Phi_a + \mathbf{F}\mathbf{W}(t); \quad (1)$$

where H_a was the Hamiltonian. $\mathbf{X}$ was the $2n$ -dimensional state vector. $\mathbf{J}$ was the unit symplectic matrix. $\mathbf{C}$ was the damping coefficient matrix. Δ was the uncertain component vector including uncertain stiffness, damping and excitation. $\mathbf{B}$ was the control coefficient matrix. Φ_a was the n_a -dimensional control vector. $\mathbf{F}$ was the excitation coefficient matrix and $\mathbf{W}$ was the m -dimensional unit intensity excitation vector, which is considered as Gaussian white noise. The equation of the observation system was⁴⁵:

$$\Phi_s = \mathbf{D}\mathbf{X} + \mathbf{E}\mathbf{W}_s(t); \quad (2)$$

where Φ_s was the n_s -dimensional observation vector. $\mathbf{W}_s$ was the m_s -dimensional Gaussian white noise vector with unit intensity. $\mathbf{D}$ and $\mathbf{E}$ were the coefficient matrices. The observation noise was independent of the white noise excitation.

Equations (1) and (2) describe an uncertain nonlinear quasi-Hamiltonian system with control and noisy observation under random excitation. The system state was estimated by using Eq. (2). The estimated state was used to determine the feedback control for system (1). Both the optimal estimation and control needed to be considered for the nonlinear stochastic system. For the system uncertainty, the minimax strategy was used to determine the optimal control for the worst case.

3. STOCHASTIC OPTIMAL ESTIMATION AND CONTROL BASED ON MINIMAX STRATEGY

The optimal control of the nonlinear stochastic system (1) with observation (2) included the optimal state estimation and the optimal response control based on the estimated state. The optimal estimation was to find an accurate state by minimizing the estimated error. The error index for observation (2) was expressed as:

$$J_F(\hat{\mathbf{X}}) = E \left[l_F(\mathbf{X} - \hat{\mathbf{X}}) | \Phi_s \right]; \quad (3)$$

where $\hat{\mathbf{X}}$ was the estimated state. $E[\cdot]$ was the expectation operator and $l_F(\cdot) \geq 0$ was a continuous differentiable function, such as a quadratic function. For the nonlinear stochastic system (1) and observation (2) with index (3), the optimally estimated state probability density with infinite dimensions was

generally difficult to obtain exactly. Following this, the extended Kalman filter,^{24,28} as an optimum feasible approximate filter, was applied to yield the approximate estimation. Based on the extended Kalman filter, the nonlinear stochastic differential equation for the estimated state was obtained from system (1) with observation (2) and index (3) as²⁴:

$$\dot{\hat{\mathbf{X}}} = \mathbf{J} \frac{\partial H_a(\hat{\mathbf{X}})}{\partial \hat{\mathbf{X}}} + \mathbf{C}\hat{\mathbf{X}} + \Delta_s(\hat{\mathbf{X}}) + \mathbf{B}\Phi_a + \mathbf{R}\mathbf{D}^T\mathbf{R}_s^{-1}\mathbf{W}_I(t) + \Delta_e\mathbf{W}_u(t); \quad (4)$$

where $\mathbf{W}_I$ was the Gaussian white noise vector with covariance matrix $\mathbf{R}_s = \mathbf{E}\mathbf{E}^T$, and $\mathbf{R}$ was the covariance matrix of the estimated error $\tilde{\mathbf{X}} = \mathbf{X} - \hat{\mathbf{X}}$, which was determined by:

$$\dot{\mathbf{R}} = \left\{ \mathbf{J} \frac{\partial^2 H_a(\hat{\mathbf{X}})}{\partial \hat{\mathbf{X}}^2} + \mathbf{C} \right\} \mathbf{R} + \mathbf{R} \left\{ \mathbf{J} \frac{\partial^2 H_a(\hat{\mathbf{X}})}{\partial \hat{\mathbf{X}}^2} + \mathbf{C} \right\}^T + \mathbf{F}\mathbf{R}_W\mathbf{F}^T - \mathbf{R}\mathbf{D}^T\mathbf{R}_s^{-1}\mathbf{D}\mathbf{R}; \quad (5)$$

where $\mathbf{R}_W = \mathbf{I}$ was the identity matrix. The uncertain component vector Δ_s was the part of Δ that was independent of excitation disturbances. The uncertain component vector $\Delta_e\mathbf{W}_u$ contained the other Δ part and estimated error effect. $\mathbf{W}_u$ was the Gaussian white noise vector with unit intensity. Therefore, the optimal control of the nonlinear stochastic system (1) with observation (2) had been converted into that of the estimated system (4) with complete observation.

The optimal control was used to find a control law by minimizing certain performance indexes. The performance index for system (4) was expressed as:

$$J_C(\Phi_a) = E \left[\int_0^{t_f} L_C(\hat{\mathbf{X}}(t), \Phi_a(t)) dt + \Psi(\hat{\mathbf{X}}(t_f)) \right]; \quad (6)$$

where $L_C(\cdot) \geq 0$ was a continuous differentiable function. $\Psi(t_f)$ was the terminal cost and t_f was the terminal time. Equations (4) and (6) describe the optimal control problem of the nonlinear stochastic system with the estimated state. However, system (4) has uncertain parameters and excitations. Thus, the minimax control strategy^{50,57} was applied to search for an optimal worst-case control, such that the control was robust. The minimax control was described as a stochastic differential game problem with the objective:

$$\min_{\Phi_a} \max_{\Delta_s, \Delta_e} J_C(\Phi_a). \quad (7)$$

According to the minimax stochastic dynamical programming principle, the dynamical programming equation for system (4) with index (6) and strategy (7) was obtained as⁵⁷:

$$\begin{aligned} \frac{\partial V}{\partial t} + \min_{\Phi_a} \max_{\Delta_s, \Delta_e} \left\{ \frac{1}{2} \text{tr} \left[(\mathbf{R}\mathbf{D}^T\mathbf{R}_s^{-1}\mathbf{D}\mathbf{R} + \Delta_e\Delta_e^T) \frac{\partial^2 V}{\partial \hat{\mathbf{X}}^2} \right] \right. \\ \left. + \left[\mathbf{J} \frac{\partial H_a}{\partial \hat{\mathbf{X}}} + \mathbf{C}\hat{\mathbf{X}} + \Delta_s(\hat{\mathbf{X}}) + \mathbf{B}\Phi_a \right]^T \frac{\partial V}{\partial \hat{\mathbf{X}}} \right. \\ \left. + L_C(\hat{\mathbf{X}}, \Phi_a) \right\} = 0; \quad (8) \end{aligned}$$

where V was the value function and $\text{tr}(\cdot)$ was the trace operator. In this case, the uncertain parameters in Δ_s and Δ_e were

considered as bounded. Following this, the maximization of the left side of Eq. (8) determined the worst-case disturbances:

$$\begin{aligned} \Delta_s^* &= \Delta_s|_{\max\{\Delta_s^T \partial V / \partial \hat{\mathbf{X}}\}}; \\ \Delta_e^* &= \Delta_e|_{\max\{\text{tr}[\Delta_e\Delta_e^T \partial^2 V / \partial \hat{\mathbf{X}}^2]\}}. \end{aligned} \quad (9)$$

By using Eq. (9), the dynamical programming equation (8) for the worst-case disturbances became:

$$\begin{aligned} \frac{\partial V}{\partial t} + \min_{\Phi_a} \left\{ \frac{1}{2} \text{tr} \left[(\mathbf{R}\mathbf{D}^T\mathbf{R}_s^{-1}\mathbf{D}\mathbf{R} + \Delta_e^*\Delta_e^{*T}) \frac{\partial^2 V}{\partial \hat{\mathbf{X}}^2} \right] \right. \\ \left. + \left[\mathbf{J} \frac{\partial H_a}{\partial \hat{\mathbf{X}}} + \mathbf{C}\hat{\mathbf{X}} + \Delta_s^*(\hat{\mathbf{X}}) + \mathbf{B}\Phi_a \right]^T \frac{\partial V}{\partial \hat{\mathbf{X}}} \right. \\ \left. + L_C(\hat{\mathbf{X}}, \Phi_a) \right\} = 0. \quad (10) \end{aligned}$$

The minimization of the left side of Eq. (10) yielded the algebraic equation for the optimal control Φ_a^* as:

$$\frac{\partial L_C(\hat{\mathbf{X}}, \Phi_a^*)}{\partial \Phi_a^*} + \mathbf{B}^T \frac{\partial V}{\partial \hat{\mathbf{X}}} = 0. \quad (11)$$

The optimal control law was obtained by Eq. (11). For example, for the function of quadratic control $L_C = g(\hat{\mathbf{X}}) + \Phi_a^T \mathbf{R}_C \Phi_a$, where $\mathbf{R}_C$ was a positive definite symmetric constant matrix and $g(\hat{\mathbf{X}}) \geq 0$, it was:

$$\Phi_a^* = -\frac{1}{2} \mathbf{R}_C^{-1} \mathbf{B}^T \frac{\partial V}{\partial \hat{\mathbf{X}}}. \quad (12)$$

By substituting Eq. (12) into Eq. (10), the value function equation was obtained as:

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2} \text{tr} \left[(\mathbf{R}\mathbf{D}^T\mathbf{R}_s^{-1}\mathbf{D}\mathbf{R} + \Delta_e^*\Delta_e^{*T}) \frac{\partial^2 V}{\partial \hat{\mathbf{X}}^2} \right] \\ + \left[\mathbf{J} \frac{\partial H_a}{\partial \hat{\mathbf{X}}} + \mathbf{C}\hat{\mathbf{X}} + \Delta_s^*(\hat{\mathbf{X}}) + \mathbf{B}\Phi_a^* \right]^T \frac{\partial V}{\partial \hat{\mathbf{X}}} \\ + L_C(\hat{\mathbf{X}}, \Phi_a^*) = 0. \quad (13) \end{aligned}$$

Equation (13) was solved to obtain V , and then the optimal control Φ_a^* for the worst-case disturbances was determined by Eq. (12), which was based on the estimated state $\hat{\mathbf{X}}$. The optimally controlled system with uncertain parameters was determined by substituting the control Φ_a^* into Eq. (4). The controlled state $\hat{\mathbf{X}}$ was obtained by solving the equation. The statistics of state $\mathbf{X}$ were calculated by using the state $\hat{\mathbf{X}}$ and covariance $\mathbf{R}$, which were used for evaluating control effectiveness. According to the minimax strategy, the stochastic response of the optimally controlled system with non-worst-case disturbances is smaller than that of the optimally controlled system with the worst-case disturbances. Thus, the minimax control for the uncertain nonlinear stochastic system or uncertain quasi-Hamiltonian system with noisy observation was robust.

4. EXAMPLES AND NUMERICAL RESULTS

4.1. Example 1

To illustrate the application and effectiveness of the proposed stochastic minimax control strategy, first consider the

control for a main mode vibration of geometrically nonlinear beam with piezoelectric sensor and actuator under random excitation. The cantilever beam under support excitation has the nonlinear stochastic vibration, which can be expressed by the dominant first mode. The modal vibration has the uncertainty of stiffness, damping and excitation, for example, due to other mode effects. The vibration response was estimated by the amplified signal of the sensor with noise. The control was performed by the actuator which can be combined with the beam system. The uncertain nonlinear stochastic system with control and noisy observation was expressed as:

$$\ddot{q} + (c + \tilde{c})\dot{q} + (k_1 + \tilde{k}_1)q + (k_3 + \tilde{k}_3)q^3 = b\phi_a + e_0W(t) + \tilde{e}_w; \quad (14)$$

$$\phi_s = dq + e_1W_s(t); \quad (15)$$

where q was the generalized modal displacement. ϕ_a was the control. c , k_1 and k_3 were respectively the nominal damping, linear stiffness and nonlinear stiffness coefficients. b was the control coefficient. e_0 was the nominal excitation amplitude. W was the Gaussian white noise with unit intensity. ϕ_s was the observation. d was the observation coefficient. e_1 was the observation noise amplitude. W_s was the unit Gaussian white noise. $\tilde{c}$, $\tilde{k}_1$, $\tilde{k}_3$ and $\tilde{e}_w$ were the uncertain parameters and external disturbance with bounded values $\tilde{c} \in [-c^0, c^0]$, $\tilde{k}_1 \in [-k_1^0, k_1^0]$, $\tilde{k}_3 \in [-k_3^0, k_3^0]$ and $\tilde{e}_w \in [-e_w^0, e_w^0]$. Rewrite Eqs. (14) and (15) as Eqs. (1) and (2), respectively, where $\Phi_a = \phi_a$, $\Phi_s = \phi_s$, $p = \dot{q}$, and:

$$\begin{aligned} H_a &= \frac{1}{2}p^2 + \frac{1}{2}k_1q^2 + \frac{1}{4}k_3q^4; \\ \mathbf{X} &= \begin{Bmatrix} q \\ p \end{Bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & -c \end{bmatrix}; \\ \mathbf{B} &= \begin{Bmatrix} 0 \\ b \end{Bmatrix}; \quad \mathbf{F} = \begin{Bmatrix} 0 \\ e_0 \end{Bmatrix}; \\ \Delta &= [0 \quad -\tilde{c}p - \tilde{k}_1q - \tilde{k}_3q^3 + \tilde{e}_w]^T; \\ \mathbf{D} &= [d \quad 0]; \quad E = e_1. \end{aligned} \quad (16)$$

Applying the extended Kalman filter yielded the differential equation (4) for the estimated state $(\hat{q}, \hat{p})$ with the covariance equation (5). The Hamiltonian system associated with the equation was integrable and non-resonant. To reduce the dimension of the control system and to convert the system state control to system energy control, the stochastic averaging method for integrable Hamiltonian systems was used. Equation (4) was converted into the Itô differential equation for the averaged Hamiltonian²⁹:

$$\begin{aligned} d\hat{H}_a &= [m_h(\hat{H}_a) + \langle (b\phi_a - \tilde{c}\hat{p} - \tilde{k}_1\hat{q} - \tilde{k}_3\hat{q}^3 + \tilde{e}_w) \frac{\partial \hat{H}_a}{\partial \hat{p}} \rangle]dt \\ &\quad + \sigma_h(\hat{H}_a)d\xi(t); \end{aligned} \quad (17)$$

where m_h and σ_h were respectively the drift and diffusion coefficients. ξ was the unit Wiener process and $\langle \cdot \rangle$ was the averaging operator. According to the minimax stochastic dynamical programming principle, the dynamical programming equation like Eq. (8) could be obtained. The worst-case disturbances (9) and optimal control law (12) became:

$$\tilde{c}^* = -c^0 \text{sgn} \left(\hat{p}^2 \frac{dV}{d\hat{H}_a} \right);$$

$$\begin{aligned} \tilde{k}_1^* &= -k_1^0 \text{sgn} \left(\hat{q}\hat{p} \frac{dV}{d\hat{H}_a} \right); \\ \tilde{k}_3^* &= -k_3^0 \text{sgn} \left(\hat{q}^3\hat{p} \frac{dV}{d\hat{H}_a} \right); \\ \tilde{e}_w^* &= -e_w^0 \text{sgn} \left(\hat{p} \frac{dV}{d\hat{H}_a} \right). \end{aligned} \quad (18)$$

$$\phi_a^* = -\frac{b\hat{p}}{2R_C} \frac{dV}{d\hat{H}_a}. \quad (19)$$

The corresponding stationary value function equation was:

$$\begin{aligned} \frac{1}{2}\sigma_h^2(\hat{H}_a) \frac{d^2V}{d\hat{H}_a^2} + m_h(\hat{H}_a) \frac{dV}{d\hat{H}_a} \\ + \left\langle c^0 \left| \hat{p}^2 \frac{dV}{d\hat{H}_a} \right| + k_1^0 \left| \hat{q}\hat{p} \frac{dV}{d\hat{H}_a} \right| + k_3^0 \left| \hat{q}^3\hat{p} \frac{dV}{d\hat{H}_a} \right| \right\rangle \\ + \left\langle e_w^0 \left| \hat{p} \frac{dV}{d\hat{H}_a} \right| \right\rangle - \frac{b^2}{4R_C} \langle \hat{p}^2 \rangle \left(\frac{dV}{d\hat{H}_a} \right)^2 \\ + g(\hat{H}_a) = \gamma_0; \end{aligned} \quad (20)$$

where γ_0 was a constant. The optimal control was determined finally by Eqs. (19) and (20). Then the controlled response and its statistics were obtained by solving Eqs. (4) and (15) numerically. The analytical system responses were evaluated by using the probability density, which was obtained by solving the Fokker-Planck-Kolmogorov equation associated with the Itô equation (17).²⁹ The efficacy of the proposed optimal control for the uncertain nonlinear stochastic system with noisy observation was evaluated based on the response statistics.

Numerical results of the uncertain control system (14) and observation (15) with non-dimensional parameters $c = 1.0$, $k_1 = 10.0$, $k_3 = 3.0$, $e_0 = 1.0$, $b = 1.0$, $d = 100.0$, $e_1 = 0.03$, $c^0 = 0.1c$, $k_1^0 = 0.1k_1$, $k_3^0 = 0.1k_3$, $e_w^0 = 0.1R_{11}d/e_1$, $R_C = 1.0$ and the quadratic control coefficient $S_{c2} = 1.0$ (i.e., coefficient of linear term of function g) were obtained and shown in Figs. 1–7. The control effectiveness of the uncertain system was represented by using the relative response reduction of the proposed stochastic minimax control K_m , and the control effectiveness by using the stochastic nominal control K_n was also given for comparison (the stochastic nominal control was the stochastic optimal control for the corresponding nominal system and the nominal system was the system with nominal parameter values or without considering parametric uncertainty). The probability densities of the controlled and uncontrolled responses of generalized displacement (q) and velocity (p) are shown in Figs. 1(a) and 1(b), respectively. The controlled response probability density near zero is larger than the uncontrolled probability density, and thus the system response is reduced by the proposed control. Figure 2 shows samples of the controlled displacement (q) and velocity (p) responses compared with the uncontrolled responses, which supports the results in Fig. 1. Figure 3 illustrates that the controlled Root-Mean-Square (RMS) displacement response is much smaller than the uncontrolled response, and the controlled RMS response changes slightly with the relative disturbance bound of damping c^0/c . Figure 4 illustrates that the relative RMS response reduction for the proposed control K_m is larger than that for the nominal control K_n , and the proposed relative response reduction $K_m > 80\%$. The stochastic

Figure 1. Controlled and uncontrolled probability densities (solid line: analytical; dot: simulated). (a) Probability densities of displacement q and (b) Probability densities of velocity p .

response of the uncertain system is reduced largely by using the proposed stochastic minimax control for various relative disturbance bounds of damping c^0/c .

Figures 5 and 6 illustrate that the relative RMS response reduction using the proposed stochastic minimax control K_m is larger than that using the stochastic nominal control K_n for the various relative disturbance bounds of linear stiffness k_1^0/k_1 and nonlinear stiffness k_3^0/k_3 , respectively. The proposed relative response reduction K_m changes slightly with the relative disturbance bounds. Figure 7 illustrates that the relative RMS response reduction for the proposed control K_m is larger than that for the nominal control K_n under various relative disturbance bounds of excitation $e_w^0/(R_{11}d/e_1)$. Therefore, the proposed stochastic minimax control can reduce largely the stochastic vibration response of the uncertain nonlinear system (14) with noisy observation (15), and the control effectiveness has good robustness for various uncertain parameters.

4.2. Example 2

To further illustrate the application and effectiveness of the proposed stochastic minimax optimal control strategy, consider the control for a two-main-mode coupling vibration of a geometrically nonlinear beam with piezoelectric sensors and actuators under random excitation. The uncertain nonlinear stochastic system with control and noisy observation was de-

Figure 2. Samples of controlled and uncontrolled responses of displacement q and velocity p (solid line: controlled; dashed line: uncontrolled).

Figure 3. Controlled and uncontrolled RMS responses q for various relative disturbance bounds of damping c^0/c (solid line: analytical; dot: simulated).

scribed by Eqs. (1) and (2), in which

$$\begin{aligned}
 H_a &= \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(k_{11}q_1^2 \\
 &\quad + k_{12}q_2^2) + k_2q_1q_2 + \frac{1}{4}k_3(q_1 - q_2)^4; \\
 \mathbf{X} &= [q_1 \quad q_2 \quad p_1 \quad p_2]^T; \\
 \Phi_a &= [\varphi_{a1}, \varphi_{a2}]^T; \\
 \Phi_s &= [\varphi_{s1}, \varphi_{s2}]^T; \\
 \mathbf{C} &= \text{diag}[0, 0, -c_1, -c_2]; \\
 \mathbf{F} &= [0 \quad 0 \quad e_{01} \quad e_{02}]^T; \\
 \mathbf{B} &= \begin{bmatrix} 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_2 \end{bmatrix}^T; \\
 \Delta &= [0 \quad 0 \\
 &\quad -\tilde{c}_1p_1 - \tilde{k}_{11}q_1 - \tilde{k}_2q_2 - \tilde{k}_3(q_1 - q_2)^3 + \tilde{e}_{w1} \\
 &\quad -\tilde{c}_2p_2 - \tilde{k}_{12}q_2 - \tilde{k}_2q_1 - \tilde{k}_3(q_2 - q_1)^3 + \tilde{e}_{w2}]^T; \\
 \mathbf{D} &= \begin{bmatrix} d_{11} & d_{12} & 0 & 0 \\ d_{12} & d_{22} & 0 & 0 \end{bmatrix}; \\
 \mathbf{E} &= \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix};
 \end{aligned} \tag{21}$$

Figure 4. Relative reductions K_m and K_n of controlled RMS responses q for various relative disturbance bounds of damping c^0/c .

Figure 6. Relative reductions K_m and K_n of controlled RMS responses q for various relative disturbance bounds of nonlinear stiffness k_3^0/k_3 .

Figure 5. Relative reductions K_m and K_n of controlled RMS responses q for various relative disturbance bounds of linear stiffness k_1^0/k_1 .

Figure 7. Relative reductions K_m and K_n of controlled RMS responses q for various relative disturbance bounds of excitation $e_w^0/(R_{11}d/e_1)$.

where q_i and p_i ($i = 1, 2$) were respectively the generalized displacement and momentum, ϕ_{ai} ($i = 1, 2$) was the control and ϕ_{si} ($i = 1, 2$) was the observation. c_i , k_{1i} , k_2 and k_3 ($i = 1, 2$) were respectively the nominal damping, linear stiffness and nonlinear stiffness coefficients. b_i ($i = 1, 2$) was the control coefficient. e_{0i} ($i = 1, 2$) was the excitation amplitude. d_{ij} ($i, j = 1, 2$) was the observation coefficient and e_{1i} ($i = 1, 2$) was the observation noise amplitude. $\tilde{c}_i$, $\tilde{k}_{1i}$, $\tilde{k}_2$, $\tilde{k}_3$ and $\tilde{e}_{wi}$ ($i = 1, 2$) were the uncertain parameters and external disturbances with bounded values $\tilde{c}_i \in [-c_i^0, c_i^0]$, $\tilde{k}_{1i} \in [-k_{1i}^0, k_{1i}^0]$, $\tilde{k}_2 \in [-k_2^0, k_2^0]$, $\tilde{k}_3 \in [-k_3^0, k_3^0]$ and $\tilde{e}_{wi} \in [-e_{wi}^0, e_{wi}^0]$.

According to the proposed procedure, using the extended Kalman filter yielded the differential equation (4) for the estimated state. The Hamiltonian system associated with the equation is non-integrable, and the stochastic averaging method for non-integrable Hamiltonian systems was applied to reduce the dimension of the control system and to convert the system state control to system energy control. Equation (4) was converted into the Itô differential equation, which was similar to Eq. (17), for the averaged Hamiltonian. Using the minimax stochastic dynamical programming principle yielded a dynamical programming equation similar to example 1. The worst-case disturbances and optimal control law were obtained as:

$$\tilde{c}_i^* = -c_i^0 \operatorname{sgn} \left(\hat{p}_i^2 \frac{dV}{d\hat{H}_a} \right);$$

$$\begin{aligned} \tilde{k}_{1i}^* &= -k_{1i}^0 \operatorname{sgn} \left(\hat{q}_i \hat{p}_i \frac{dV}{d\hat{H}_a} \right); \\ \tilde{k}_2^* &= -k_2^0 \operatorname{sgn} \left((\hat{q}_1 \hat{p}_2 + \hat{q}_2 \hat{p}_1) \frac{dV}{d\hat{H}_a} \right); \\ \tilde{k}_3^* &= -k_3^0 \operatorname{sgn} \left(((\hat{q}_1 - \hat{q}_2)^3 \hat{p}_1 + (\hat{q}_2 - \hat{q}_1)^3 \hat{p}_2) \frac{dV}{d\hat{H}_a} \right); \\ \tilde{e}_{wi}^* &= -e_{wi}^0 \operatorname{sgn} \left(\hat{p}_i \frac{dV}{d\hat{H}_a} \right); \end{aligned} \quad (22)$$

$$\phi_{ai}^* = -\frac{1}{2} R_{C,ij}^{-1} b_j \hat{p}_j \frac{dV}{d\hat{H}_a}. \quad (23)$$

The corresponding stationary value function equation was:

$$\begin{aligned} \frac{1}{2} \sigma_h^2(\hat{H}_a) \frac{d^2 V}{d\hat{H}_h^2} + m_h(\hat{H}_a) \frac{dV}{d\hat{H}_a} \\ + \left\langle c_i^0 \left| \hat{p}_i^2 \frac{dV}{d\hat{H}_a} \right| + k_{1i}^0 \left| \hat{q}_i \hat{p}_i \frac{dV}{d\hat{H}_a} \right| \right\rangle + \\ \left\langle k_2^0 \left| (\hat{q}_1 \hat{p}_2 + \hat{q}_2 \hat{p}_1) \frac{dV}{d\hat{H}_a} \right| \right. \\ \left. + k_3^0 \left| ((\hat{q}_1 - \hat{q}_2)^3 \hat{p}_1 + (\hat{q}_2 - \hat{q}_1)^3 \hat{p}_2) \frac{dV}{d\hat{H}_a} \right| \right\rangle + \end{aligned}$$

Figure 8. Controlled and uncontrolled probability densities (solid line: analytical; dot: simulated). (a) Probability densities of displacement q_1 and (b) Probability densities of displacement q_2 .

$$\left\langle e_{wi}^0 \left| \hat{p}_i \frac{dV}{d\hat{H}_a} \right| \right\rangle - \frac{1}{4} \left\langle b_i p_i R_{C,ij}^{-1} b_j \hat{p}_j \right\rangle \left(\frac{dV}{d\hat{H}_a} \right)^2 + g(\hat{H}_a) = \gamma_0. \quad (24)$$

The optimal control was finally determined by Eqs. (23) and (24). Then, the controlled response and its statistics were obtained by either solving Eqs. (4) and (2) or solving the Fokker-Planck-Kolmogorov equation associated with the Itô equation (17). The efficacy of the proposed optimal control for the uncertain nonlinear stochastic system with noisy observation was evaluated based on the response statistics.

Numerical results of the control system and observation with non-dimensional parameters $c_1 = 1.0$, $c_2 = 5.0$, $k_{11} = 1.0$, $k_{12} = 4.0$, $k_2 = 0.01$, $k_3 = 1.0$, $e_{01} = 1.0$, $e_{02} = 0.3$, $b_1 = 1.0$, $b_2 = 0.5$, $d_{11} = 100.0$, $d_{12} = 10.0$, $d_{22} = 80.0$, $e_{11} = 0.03$, $e_{12} = 0.03$, $c_i^0 = 0.1c_i$, $k_{1i}^0 = 0.1k_{1i}$, $k_2^0 = 0.1k_2$, $k_3^0 = 0.1k_3$, $e_{wi}^0 = 0.0$, $\mathbf{R}_C = \text{diag}[1.0, 1.0]$ and the quadratic control coefficient $S_{c2} = 1.0$ (i.e., coefficient of linear term of function g) were obtained and shown in Figs. 8–13. The control effectiveness for the proposed stochastic minimax control K_m and stochastic nominal control K_n was defined as in example 1. The probability densities of the controlled and uncontrolled responses of generalized displacements q_1 and q_2 are shown in Figs. 8(a) and 8(b), respectively. The controlled response probability densities near zero are larger than the corresponding uncontrolled probability densities, and thus the system responses are reduced by the proposed con-

Figure 9. Samples of controlled and uncontrolled responses of displacements (q_1 , q_2) and velocities (p_1 , p_2) (solid line: controlled; dashed line: uncontrolled).

trol. Figure 9 shows samples of the controlled displacement (q_1 and q_2) and velocity (p_1 and p_2) responses compared with the uncontrolled responses, in which the response reduction was observed. Figure 10 illustrates that the controlled RMS displacement response (q_1) is much smaller than the uncontrolled response, and the controlled RMS response changes slightly with the relative disturbance bound of damping c_1^0/c_1 . Figure 11 illustrates that the relative RMS response reduction using the proposed stochastic minimax control K_m is larger than that using the stochastic nominal control K_n for various relative disturbance bounds of damping c_1^0/c_1 . Figures 12 and 13 illustrate that the relative RMS response reduction, using the proposed control K_m , is larger than that using the nominal control K_n for various relative disturbance bounds of linear stiffness k_{11}^0/k_{11} and nonlinear stiffness k_3^0/k_3 , respectively. The proposed relative response reduction K_m varying with the relative disturbance bound of nonlinear stiffness is larger than the others. It is obtained again that the proposed stochastic minimax control can reduce remarkably the stochastic vibration response of the uncertain nonlinear system with noisy observation (21), and the control effectiveness is robust for uncertain parameters except nonlinear stiffness.

5. CONCLUSIONS

The stochastic minimax control strategy for the uncertain nonlinear quasi-Hamiltonian systems with noisy observations under random excitations has been proposed based on the extended Kalman filter and minimax stochastic dynamical programming principle. The control strategy includes three parts as follows: the uncertain nonlinear stochastic quasi-Hamiltonian system with noisy observation is first converted into the completely observed nonlinear stochastic system with the uncertain parameters for the estimated state based on the extended Kalman filter. The dual dynamical programming equation is derived based on the minimax stochastic dynamical programming principle. The worst-case disturbances are determined for bounded uncertain parameters. Lastly, the optimal control law is determined for the worst case and estimated state by the programming equation. The proposed minimax control

Figure 10. Controlled and uncontrolled RMS responses q_1 for various relative disturbance bounds of damping c_1^0/c_1 (solid line: analytical; dot: simulated).

Figure 12. Relative reductions K_m and K_n of controlled RMS responses q_1 for various relative disturbance bounds of linear stiffness k_{11}^0/k_{11} .

Figure 11. Relative reductions K_m and K_n of controlled RMS responses q_1 for various relative disturbance bounds of damping c_1^0/c_1 .

Figure 13. Relative reductions K_m and K_n of controlled RMS responses q_1 for various relative disturbance bounds of nonlinear stiffness k_3^0/k_3 .

strategy is applicable to various uncertain nonlinear stochastic multi-degree-of-freedom systems with noisy observations.

The proposed minimax control strategy has been applied to the vibration control of two uncertain nonlinear stochastic systems with noisy observations. Numerical results have illustrated that the proposed minimax control can reduce largely the stochastic vibration response of uncertain nonlinear systems with noisy observations, and the control effectiveness is insensitive to varying system parameters such as observation and nonlinear coefficients. Thus, the proposed stochastic minimax control strategy is feasible and effective for the vibration response reduction of uncertain nonlinear structure systems with noisy observations under random excitations.

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