Free Vibration of Axially Inhomogeneous Beams that are Made of Functionally Graded Materials

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A simple approach to deal with an engineering problem of free vibration of functionally graded beams with non-uniform cross-sections where the material properties arbitrarily vary along the axial direction is presented. Instead of directly solving the fourth-order governing differential equation with variable coefficients, we transform the governing equation to a system of linear algebraic equations based on the polynomial expansion and integral technique. Then, a characteristic equation in natural frequencies will be obtained. Several examples of estimating natural frequencies for axially graded beams and non-uniform beams are presented, which show that our method has a fast convergence and the obtained numerical results are accurate. The proposed method is important to investigate because of the dynamic behavior of axially non-uniform beams in engineering applications.

1. INTRODUCTION

Non-homogeneous structural components with axially varying material properties are common in buildings, bridges, machine parts. There has been a great amount of research dedicated to investigating the dynamic behaviors of axially nonuniform beams. In this field, many approximate methods or numerical techniques have been formulated, which include Rayleigh's quotient method, the finite element method, the finite difference method, the boundary element method, and the differential quadrature method. By assuming that the beam's cross sectional area and the moment of inertia is arbitrary to polynomial functions of the position, Abrate¹ used the Rayleigh-Ritz approach and Lagrange multiplier method to analyze the free vibration of nonuniform beams with the general shape and arbitrary boundary conditions. Analytical solutions of the vibration for the nonuniform beam with cross section geometries with exponentially variable widths have been obtained for three different types of boundary conditions associated with simply supported, clamped and free ends.² Nachum and Altus used the functional perturbation method to determine the natural frequencies and mode shapes of non-homogeneous rods and beams.³ By applying the Adomian modified decomposition method, Hsu et al. converted the governing differential equation to a recursive algebraic equation and kept the boundary conditions within simple algebraic frequency equations which were suitable for symbolic computation.⁴ Based on the fact that a nonuniform beam can be partitioned into multi homogeneous uniform sub-beams, Singh et al.⁵ developed a numerical method for determining the natural frequencies of a nonuniform beam. By assuming the material constituents are varied throughout the thickness or longitudinal directions according to a simple power law, Alshorbagy et al. used the numerical finite element method to investigate the dynamic behaviors of functionally graded beams.⁶ Based on the homotopy perturbation method and variational iteration method, Coskun and co-workers investigate the free vibration and buckling behaviors of beam.⁷⁻¹⁰

Clearly, most of the aforementioned methods do not yield a closed-form solution, but rather numerical or approximate results. It is well-known that when the graded elasticity and/or cross section of the beam varies along the axial direction, a problem of solving a differential equation with nonuniform

coefficients based on the Euler-Bernoulli assumption arises. Therefore, closed-form solutions for free vibration of nonuniform beams are often difficult to find. So far, few analytical solutions are found in arbitrary gradient change due to the difficulty of mathematical treatment. Fortunately, by making use of the semi-inverse method, Elishakoff and co-workers treated a large class of problems involving graded beams of special forms such as polynomials and obtained explicit fundamental frequency of free vibration.¹¹⁻¹⁴ Aydogdu¹⁵ also used the semi-inverse method to study the vibration and buckling of simply supported beams. However, the semi-inverse method cannot apply for graded beams of any axial inhomogeneity. By using the differential transform method, Catal investigated the problem of free vibration of one end fixed and the other simply supported beam resting on elastic soil.¹⁶ The free vibration and stability of axially functionally graded tapered Euler-Bernoulli beams have been studied by Shahba and Rajasekaran¹⁷ through solving the governing differential equations based on the differential transform element method. Based on the Euler-Bernoulli theory and Haar wavelet approach, Hein and Feklistova¹⁸ investigated the free vibration nonuniform and functionally graded beams with various boundary conditions and varying cross sections. The free vibration and stability analysis of axially functionally graded tapered Timoshenko beams were carried out using the finite element approach.¹⁹ Recently, Li et al.²⁰ investigated the free vibration of exponential functionally graded beams, where the characteristic equations in closed form with different boundary conditions have been derived.

The objective of this paper is to present a simple approach for analyzing the dynamic behavior of axially functionally graded and nonuniform beams, where the material properties are varying arbitrarily along the axial direction. By expanding the mode shapes as power series, we transform the governing differential equation with variable coefficients to a system of algebraic equations with unknown coefficients. The natural frequencies can be easily determined from the existing condition of a nontrivial solution in the resulting system. We will apply the method to evaluate the natural frequencies of uniform homogeneous and nonuniform beams with the kinds of boundary conditions. Moreover, the effects of the gradient parameter and cross section on the natural frequencies under various end supports will be discussed.

2. FORMULATION AND METHOD

A functionally graded and/or nonuniform beam of length L , which is subject to the action of transverse loading, is considered. Based on the Euler-Bernoulli beam theory, the governing differential equation for free vibration of axially graded and nonuniform beams can be written as:¹¹

$$\frac{\partial^2}{\partial x^2} \left[D(x) \frac{\partial^2 w}{\partial x^2} \right] + m(x) \frac{\partial^2 w}{\partial t^2} = q(x, t); \quad (1)$$

where x , w , and q are the axial coordinate, deflection, and distributed transverse loading, respectively; $D(x) = E(x)I(x)$ is the flexural rigidity of the beam, which is a function of the axial coordinate x , and depends upon both Young's modulus $E(x)$ and the second moment of cross sectional area $I(x)$; and $m(x) = \rho(x)A(x)$ is the mass of the beam per unit length, which depends upon the cross sectional area $A(x)$ and mass density $\rho(x)$. We introduce a non-dimensional coordinate $\xi = x/L$ and consider harmonic vibration so that w is represented as:

$$w(x, t) = W(x)e^{i\omega t}. \quad (2)$$

By placing Eq. (2) into Eq. (1) and then assuming $q(x, t) = 0$, one can get:

$$\frac{d^2}{d\xi^2} \left[D(\xi) \frac{d^2 W}{d\xi^2} \right] - km(\xi) W = 0, \quad 0 \leq \xi \leq 1; \quad (3)$$

with $D(\xi) = E(\xi L)I(\xi L)$, $k = \omega^2 L^4$. By avoiding to solve the differential Eq. (3) directly, we introduce a new and simpler method²¹ to investigate the free vibration of axially inhomogeneous beams. Now, we take an axially graded and nonuniform clamped-pinned (C-P) beam with clamped end at $\xi = 0$ and a simply supported end at $\xi = 1$ for example. Therefore, the corresponding boundary conditions read:

$$W = 0, \quad \frac{dW}{d\xi} = 0, \quad \xi = 0; \quad (4)$$

$$W = 0, \quad \frac{d^2 W}{d\xi^2} = 0, \quad \xi = 1. \quad (5)$$

Actually, the problem becomes solving a set of a fourth-order ordinary differential governing Eq. (3) combining with the boundary conditions given by Eqs. (4) and (5). First of all, we expand $W(\xi)$ as a power series. Or rather, if it neglected a sufficiently small error, the unknown $W(\xi)$ can be approximately expanded as:

$$W(\xi) = \sum_{i=0}^N \frac{c_i \xi^i}{i!}, \quad 0 \leq \xi \leq 1; \quad (6)$$

where c_i are unknown coefficients and N is a certain positive integer, which is chosen and should be large enough so that the rest have a negligible error. By using the clamped conditions at $\xi = 0$, then substituting Eq. (6) into Eq. (4), one gets:

$$c_0 = 0; \quad (7)$$

$$c_1 = 0. \quad (8)$$

Bearing Eq. (5) in mind, the application of the pinned condition $\xi = 1$ to Eq. (6) leads to:

$$c_0 + c_1 + \frac{c_2}{2!} + \dots + \frac{c_N}{N!} = 0; \quad (9)$$

$$c_2 + c_3 + \frac{c_4}{2!} + \dots + \frac{c_N}{(N-2)!} = 0. \quad (10)$$

It is easily found that in order to solve the linear Eqs. (7) to (10), other $N - 3$ independent equations are needed for uniquely determining c_i . This can be achieved by using the following method.²¹ Inserting Eq. (6) into the resulting differential Eq. (3) for each case leads to:

$$\sum_{i=0}^N \left[C_i^4 D(\xi) + 2C_i^3 \xi \frac{dD(\xi)}{d\xi} + C_i^2 \xi^2 \frac{d^2 D(\xi)}{d\xi^2} \right] \frac{\xi^{i-4}}{i!} c_i - k \sum_{i=0}^N \frac{m(\xi) \xi^i}{i!} c_i = 0; \quad (11)$$

where $C_i^4 = i(i-1)(i-2)(i-3)$, $C_i^3 = i(i-1)(i-2)$, $C_i^2 = i(i-1)$. We multiply both sides of Eq. (11) by ξ^t ($t = 0, 1, 2, \dots, N-4$) and then integrate it with respect to ξ between 0 and 1, which yields another $N - 3$ linear algebraic equations of unknown coefficients c_i :

$$\sum_{i=0}^N (f_{ji} - k g_{ji}) c_i = 0, \quad j = 4, 5, 6, \dots, N; \quad (12)$$

with

$$f_{ji} = \int_0^1 \left[C_i^4 D(\xi) + 2C_i^3 \xi \frac{dD(\xi)}{d\xi} + C_i^2 \xi^2 \frac{d^2 D(\xi)}{d\xi^2} \right] \frac{\xi^{j+i-8}}{i!} d\xi;$$

$$g_{ji} = \int_0^1 \frac{\xi^{j+i-4}}{i!} m(\xi) d\xi. \quad (13)$$

Consequently, making use of Eqs. (7) to (10) and (12), we can obtain a system of linear algebraic equations for unknown coefficient c_i , which can be rewritten in a compact form as $(F_{nn} - kG_{nn})C_n = 0$, $F_{nn} = (f_{ij})_{(N+1) \times (N+1)}$, $G_{nn} = (g_{ij})_{(N+1) \times (N+1)}$, $C_n = (c_i)_{(N+1) \times 1}$. To obtain a nontrivial solution of the resulting system, the determinant of the coefficient matrix of the system has to vanish, so we immediately get a characteristic equation in eigenvalues k :

$$\det(F_{nn} - kG_{nn}) = 0. \quad (14)$$

By inspecting the above procedure, if $D(\xi)$ and $m(\xi)$ are given functions, one can readily find that the obtained corresponding characteristic Eq. (14) has multi-roots, since it is a polynomial in natural frequency k . Therefore, a numerical procedure to look for its roots is simple with the aid of commercial software. As a result, both the lower and higher natural frequencies of axially nonuniform beams can be determined simultaneously from the multi-roots.

By using this method, we can easily deal with the problem for all kinds of boundary conditions. In the case of cantilever or clamped-free (C-F) beams, substituted Eq. (6) into the corresponding boundary conditions can aid in obtaining the first four linear equations for c_i . It is only needed to replace Eqs. (7) to (10) for the newly obtained linear equations while keeping the last $N - 3$ equations as the same form of Eq. (12). There is no doubt that the introduced method can give rise to great simplification of dealing with the free vibration of axially functionally graded nonuniform beams for different boundary conditions. It should be noted that besides the expansion of Eq. (6) of $W(\xi)$, other expansion approximations of the unknown W such as Chebyshev polynomial, trigonometric functions, etc., are also applicable. It is well known that the first

Table 1. The first three non-dimensional natural frequencies Ω_n for uniform beams.

	n	Case A			Chebyshev polynomial $N = 14$	Case B $N = 14$	Exact ²²
		$N = 6$	$N = 10$	$N = 14$			
C-F	1	3.517076	3.516015	3.516015	3.516015	3.516015	3.5160
	2	22.233362	22.034492	22.034492	22.034492	22.034492	22.0345
	3	118.144436	61.715119	61.697215	61.697215	61.697260	61.6972
C-G	1	5.604273	5.593321	5.593321	5.593321	5.593321	5.5933
	2	30.996978	30.225861	30.225848	30.225848	30.225848	30.2258
	3	132.954050	74.692768	74.638884	74.638884	74.639024	74.6389
P-P	1	9.872527	9.869604	9.869604	9.869604	9.869604	9.8696
	2	50.199602	39.479138	39.478418	39.478418	39.478453	39.4784
	3	131.812493	88.848180	88.826440	88.826440	88.822660	88.8264
C-P	1	15.547849	15.418206	15.418206	15.418206	15.418206	15.4182
	2	66.389739	49.968013	49.964862	49.964862	49.964871	49.9649
	3	179.002422	104.382769	104.247702	104.247702	104.247325	104.2477
C-C	1	22.564206	22.373286	22.373285	22.373285	22.373285	22.3733
	2	91.651514	61.688324	61.672823	61.672823	61.672866	61.6728
	3	223.362612	121.106105	120.903400	120.903400	120.899064	120.9034

Table 2. The first three non-dimensional natural frequencies k_n for axially graded P-P beams.

E_{ratio}	n	$\alpha = 0$		$\alpha = 0.2$		$\alpha = 1$		$\alpha = 5$	
		Alshorbagy et al. ⁶	Present	Alshorbagy et al. ⁶	Present	Alshorbagy et al. ⁶	Present	Alshorbagy et al. ⁶	Present
0.25	1	2.2214 (2.2203)	2.221441	2.4118 (2.4106)	2.411854	2.7546 (2.7533)	2.754715	3.0849 (3.0834)	3.085017
	2	4.4425 (4.4338)	4.442883	4.847 (4.8374)	4.847367	5.484 (5.473)	5.484458	6.0756 (6.0636)	6.076306
	3	6.6631 (6.6338)	6.664324	7.2792 (7.247)	7.280589	8.2123 (8.1753)	8.213893	9.0664 (9.0262)	9.068501
0.5	1	2.6417	2.641754	2.7271	2.727195	2.9119	2.911949	3.1067	3.106839
	2	5.2831	5.283508	5.4681	5.468500	5.8162	5.816727	6.158	6.158589
	3	7.9238	7.925262	8.2076	8.209143	8.7199	8.721549	9.2114	9.213255
1	1	3.1415	3.141593	3.1415	3.141593	3.1415	3.141593	3.1415	3.141593
	2	6.2827	6.283185	6.2827	6.283185	6.2827	6.283185	6.2827	6.283185
	3	9.423	9.424778	9.423	9.424778	9.423	9.424778	9.423	9.424778
2	1	3.7359	3.736004	3.6671	3.667188	3.4628	3.462911	3.1939	3.193902
	2	7.4714	7.472008	7.3183	7.318925	6.9167	6.917292	6.4611	6.461511
	3	11.206	11.208013	10.968	10.970467	10.3087	10.371728	9.7163	9.717998
4	1	4.4428	4.442882	4.3166	4.316696	3.8957	3.895756	3.2684	3.268399
	2	8.885	8.885765	8.6022	8.603071	7.7555	7.756194	6.7036	6.704037
	3	13.326	13.328649	12.886	12.888748	11.614	11.616200	10.105	10.106382

Table 3. The first three non-dimensional natural frequencies Ω_n for P-P and C-C graded beams.

	n	Present	Case A Exact ¹³	Present	Case B Exact ¹³	Present	Case C Exact ¹³
P-P	1	18.973666	18.973666 ($\Omega_1^2 = 360$)	28.982753	28.982753 ($\Omega_1^2 = 840$)	67.970582	67.970582 ($\Omega_1^2 = 4620$)
	2	74.540233	—	114.560860	—	273.106343	—
	3	167.167790	—	257.124973	—	612.707077	—
C-C	1	41.270740	—	63.894272	—	147.358557	—
	2	114.760318	—	177.140992	—	416.762224	—
	3	225.830912	—	348.131592	—	824.427049	—

kind of Chebyshev polynomials can be obtained by the following recurrence relations:

$$\begin{aligned} T_0(z) &= 1, & T_1(z) &= z, \\ T_{i+1}(z) &= 2zT_i(z) - T_{i-1}(z); \end{aligned} \quad (15)$$

where z is over the interval $[-1, 1]$. By introducing variable substitution $\xi = (z + 1)/2$, we can easily get the Chebyshev polynomials over the interval $[0, 1]$:

$$\begin{aligned} T_0^*(\xi) &= 1, & T_1^*(\xi) &= 2\xi - 1, \\ T_{i+1}^*(\xi) &= 2(2\xi - 1)T_i^*(\xi) - T_{i-1}^*(\xi). \end{aligned} \quad (16)$$

As a result, the unknown $W(\xi)$ can be expanded into the shifted Chebyshev polynomials in general over the interval $[0, 1]$ as:

$$W(\xi) = \sum_{i=0}^N c_i T_i^*(\xi), \quad 0 \leq \xi \leq 1. \quad (17)$$

By substituting Eq. (17) to the governing equation and corresponding boundary conditions, we can easily determine the natural frequency based on the Chebyshev polynomial. And the procedures are omitted here.

3. NUMERICAL RESULTS AND DISCUSSION

In order to examine the effectiveness of the proposed method, let us first consider the case of a homogeneous beam with uniform cross sections. In this case, $D(\xi) = EI$ and $m(\xi) = \rho A$ are unchanged, and the non-dimensional natural frequencies $\Omega = \omega L^2 \sqrt{\rho A / EI}$ can be exactly calculated from the following frequency equations:²² P-P, $\sin \sqrt{\Omega} = 0$; C-C, $\cos \sqrt{\Omega} \cosh \sqrt{\Omega} - 1 = 0$; C-F, $\cos \sqrt{\Omega} \cosh \sqrt{\Omega} + 1 = 0$; C-P, $\tan \sqrt{\Omega} - \tanh \sqrt{\Omega} = 0$; and C-G, $\tan \sqrt{\Omega} + \tanh \sqrt{\Omega} = 0$. It should be noted that the above frequency equations are highly nonlinear except for the P-P beams and that it is not easy to find an analytical solution for any assumed stiffness parameter. However, based on the aforementioned approach, the numerical results of natural frequencies for any modes can be easily calculated. For checking the convergence of the suggested method, we have calculated non-dimensional natural frequencies Ω of a homogeneous beam with different boundary conditions by taking different N values in Eq. (6). Evaluated results of first three non-dimensional natural frequencies Ω_n , and the exact ones are tabulated in Table 1. By comparing our numerical results with the exact ones, it is clear from Table 1 that the

numerical results have a rapid convergence. With N increasing from 6 to 14, the errors between the numerical and exact results dramatically decrease, which indicates that the present approach is very efficient. As a result, a higher accuracy can be achieved through increasing N . Furthermore, we have calculated the non-dimensional natural frequencies by using Chebyshev expansions in Eq. (17) with $N = 14$. Seen from Table 1, the results based on two different expansions are identical to each other up to 6 decimal digits. Through observing the linear equations in Eq. (12), one may find that those equations can be obtained in another manner, such as multiplying both sides of Eq. (11) by ξ^{2j} and changing j from 2 to $N - 2$, while keeping the first four linear Eqs. (7) to (10) the same. For clarity, we denote the former way as Case A and the latter as Case B. From the numerical results, we can discover that treatment has little influence on the natural frequencies although the last $N - 4$ linear equations are changed, which indicates that the behaviors of free vibration are mostly determined by the end supports in nature. Therefore, we will choose the procedure of Case A with $N = 14$ in the following computation.

Next, we consider a functionally graded beam with axial inhomogeneity, where the cross sectional area A and moment of inertia I are unchanged. This is of benefit to design of non-homogeneous beams for certain purposes. To show the effect of the inhomogeneity on the natural frequencies, the material characteristics (such as Young's modulus, mass density, etc.) can be assumed to be varied a power law distribution as:⁶

$$P(\xi) = P_R + (P_L - P_R)(1 - \xi)^\alpha; \quad (18)$$

where P_L and P_R are the corresponding material properties of the left ($\xi = 0$) and the right side ($\xi = 1$) of the beam, respectively, and α is the non-negative power-law exponent parameter describing the volume fraction change of both constituents involved through the axis of the beam. Based on the introduced method, the effects of Young modulus ratio $E_{\text{ratio}} = E_L/E_R$ and the gradient parameter α on first three dimensionless natural frequencies $k_n = \sqrt{\omega L^2 \sqrt{\frac{\rho_R A}{E_R I}}}$ are displayed in Table 2 with $N = 14$ for simply supported boundary conditions. The free vibration characteristics and dynamic behavior of a functionally graded beam for axially material distributions as a power law form Eq. (18) have been analyzed numerically by Alshorbagy et al. through using the finite element method.⁶ We should note that the results in parentheses cited from the work of Alshorbagy et al.⁶ were calculated from slenderness ratio $L/h = 20$, while others were obtained with $L/h = 100$. By comparing the proposed model with published work,⁶ we can find that the present numerical results agree reasonably well with existing results with $L/h = 100$. However, the accuracy between the present and existing results drops with the ratio $L/h = 20$.⁶ This phenomenon indicates that the Euler-Bernoulli theory is suited for slender beams. In addition, the fundamental frequencies are also sensitive to the Young modulus ratio and the gradient parameter α , as expected. From Table 2, the natural frequencies increase with an increase in the power exponent (when $E_{\text{ratio}} < 1$) and decrease with an increase in the power exponent (when $E_{\text{ratio}} > 1$). For a constant power exponent, an increase in E_{ratio} causes the increase in fundamental frequencies. For comparison, here we also list numerical results of the well-known first three natural frequencies k_n of homogeneous beams ($E_{\text{ratio}} = 1$) with simply end conditions: 3.1415926, 6.2831853, 9.4247780. Our results are

identical to the exact ones up to 7 decimal digits for homogeneous beams.

For axially graded beams with a uniform cross section, Elishakoff and Guede presented closed-form solutions to vibrating non-homogeneous beams by the semi-inverse method.¹³ For comparison, we take the mass density ρ and Young's modulus E as polynomial functions for three cases: Case A, $\rho(\xi) = \rho_0$, $E(\xi) = E_0(3 + 3\xi - 2\xi^2 - 2\xi^3 + \xi^4)$; Case B, $\rho(\xi) = \rho_0$, $E(\xi) = E_0(\frac{31}{3} - 6\xi^2 + \xi^4)$; and Case C, $\rho(\xi) = \rho_0(\frac{26}{165} + \xi^2)$, $E(\xi) = E_0(\frac{7337}{270} - \frac{52}{3}\xi^2 + \frac{13}{54}\xi^4 + \xi^6)$. Elishakoff and Guede¹³ first preselected polynomial functions serve as an exact mode shape of the inhomogeneous beam then solved the semi-inverse problem of finding the distributions of the stiffness and the inertial term. For simply supported beams, they determined the results of exact first-order non-dimensional natural frequency $\Omega_1^2 = \omega_1^2 L^4 \rho_0 A / E_0 I$, which are identical to 360, 840, 4620 for Case A, B and C, respectively. Based on the present approach, the numerical results of $\Omega_n = \omega_n L^2 \sqrt{\rho_0 A / E_0 I}$ for P-P and C-C graded beams and those of the first three modes are listed in Table 3. From Table 3, we find that the results of first-order of non-dimensional natural frequency for P-P beams are the same as the exact one.¹³ Other natural frequencies for the arbitrary order vibration mode for C-C beams are also obtained, but they seem not to be derived in closed form by the inverse method.¹³

In what follows, we discuss the free vibration of Euler-Bernoulli beams with a nonuniform cross section along the length direction, where the material properties E and ρ of the beams keep constants. Firstly, we take the cross sectional area A and moment of inertia I as $A = A_0(1 + \beta\xi)^4$, $I = I_0(1 + \beta\xi)^4$, where β is a geometrical parameter, and A_0 and I_0 are the cross sectional area and the second moment of cross sectional area of the beam at $\xi = 0$, respectively. For this case, Abrate¹ derived the exact solution of Eq. (3) as

$$W = \frac{1}{(1 + \beta\xi)^2} \left[\gamma_1 \sin(\sqrt{\Omega_n} \xi) + \gamma_2 \cos(\sqrt{\Omega_n} \xi) + \gamma_3 e^{\sqrt{\Omega_n} \xi} + \gamma_4 e^{-\sqrt{\Omega_n} \xi} \right]; \quad (19)$$

where $\Omega_n = \omega_n L^2 \sqrt{\rho A_0 / E I_0}$ are the non-dimensional natural frequencies, and the constants γ_i ($i = 1, \dots, 4$) are to be determined from the boundary conditions. Therefore, we can easily obtain the characteristic equations for kinds of boundary conditions applying the support ends to Eq. (19). Based on the introduced method, the first four non-dimensional natural frequencies $\Omega_n = \omega_n L^2 \sqrt{\rho A_0 / E I_0}$ for tapered beams of C-P, P-P, and C-C ends are evaluated in Table 4 with different values for parameter β . The exact results derived previously by Abrate¹ are also presented in Table 4. It should be noted that the exact results for C-P end are cited from Abrate's publication,¹ and other results for P-P and C-C boundary conditions are calculated based on the exact corresponding characteristic equations. From Table 4, it is seen that our numerical results show excellent consistency with the above-mentioned results. In particular, our numerical results of the first three natural frequencies are nearly identical to the results acquired by Abrate. However, even for the fourth-order natural frequency, the errors between two results are less than 0.004%. Moreover, we can find that the geometrical parameter β plays an important role in determining the natural frequencies for C-P and P-P beams. However, the natural frequencies of clamped-clamped

Table 4. Non-dimensional natural frequencies Ω_n for tapered beams $A/A_0 = I/I_0 = (1 + \beta\xi)^4$.

Support ends	n	$\beta = 0$		$\beta = 1$		$\beta = 2$	
		Abrate ¹	Present	Abrate ¹	Present	Abrate ¹	Present
C-P	1	15.4182	15.418206	12.3635	12.363516	10.5984	10.598379
	2	49.9649	49.964862	47.6265	47.626496	46.6678	46.667829
	3	104.248	104.247702	102.025	102.024580	101.174	101.173899
	4	178.270	178.277198	176.105	176.107010	175.304	175.304062
P-P	1	9.8696	9.869604	9.1757	9.175715	8.2900	8.290058
	2	39.4784	39.478418	40.1156	40.115623	40.8774	40.877363
	3	88.8264	88.826439	89.8680	89.868021	91.2205	91.220566
	4	157.9137	157.916169	159.1757	159.175845	160.9028	160.902625
C-C	1	22.3733	22.373285	22.3733	22.373285	22.3733	22.373291
	2	61.6728	61.672823	61.6728	61.672823	61.6728	61.672857
	3	120.9034	120.903400	120.9034	120.903479	120.9034	120.903638
	4	199.8585	199.886189	199.8585	199.861921	199.8585	199.857348

Table 5. The first mode of non-dimensional natural frequencies Ω_1 for C-C beams.

$c_h \setminus c_b$		0	0.2	0.4	0.6	0.8
0	Shahba et al. ²³	20.4721	20.4152	20.2883	20.0186	19.3845
	Present	20.472088	20.415143	20.288273	20.0185778	19.384355
0.2	Shahba et al. ²³	18.2170	18.1996	18.1286	17.9437	17.4566
	Present	18.216992	18.199537	18.128557	17.943601	17.456370
0.4	Shahba et al. ²³	15.8282	15.8498	15.8350	15.7367	15.4025
	Present	15.828146	15.849733	15.834917	15.736555	15.402140
0.6	Shahba et al. ²³	13.2293	13.2896	13.3319	13.3238	13.1529
	Present	13.229150	13.289418	13.331634	13.323402	13.152117
0.8	Shahba et al. ²³	10.2235	10.3229	10.4255	10.5168	10.5339
	Present	10.221839	10.321203	10.423581	10.514532	10.530924

beams are the same regardless of the parameter β . This is due to the fact that the tapered beams for clamped-clamped beams with properties varying as in this example will have the same governing differential equation and the same boundary conditions as the uniform beam.¹

Finally, free vibration of axially graded tapered beams with a nonuniform cross section is concerned. Not only the effective material properties of mass density and Young's modulus, but also the cross section area and moment of inertia vary continuously along the axial direction. Here, the distribution of mass density $\rho(\xi)$ and Young's modulus $E(\xi)$ are assumed to be polynomial functions:

$$\rho(\xi) = \rho_0 (1 + \xi + \xi^2), \quad E(\xi) = E_0 (1 + \xi), \quad \xi \in [0, 1]; \quad (20)$$

where ρ_0 and E_0 are the Young's modulus and mass density of the beam at $\xi = 0$, respectively; and the cross sectional area A and moment of inertia I can be expressed as:

$$A(\xi) = A_0(1 - c_b\xi)(1 - c_h\xi), \\ I(\xi) = A_0(1 - c_b\xi)(1 - c_h\xi)^3, \quad \xi \in [0, 1]; \quad (21)$$

where c_b is the breadth and c_h is the height taper ratios, respectively, which must satisfy $0 \leq c_b, c_h \leq 1$ to insure that $A(\xi)$ and $I(\xi)$ are positive. Taking advantage of the shape functions of homogeneous uniform beam elements, Shahba et al. have investigated the free vibration of such axially functionally graded tapered Euler-Bernoulli beams through exploiting finite element method.²³ The numerical results of non-dimensional natural frequency parameter $\Omega_n = \omega L^2 \sqrt{\rho_0 A_0 / E_0 I_0}$ of first mode are evaluated for C-C beams and those together with other existing results are listed in Table 5. From Table 5, it is seen that our results clearly agree with the existing results by using the finite element method.²³ We can also find that the results of the first mode natural frequencies decrease with an increase in the cross sectional parameters c_b (or c_h). However, the effect of height tapering on non-dimensional natural frequencies is greater than that of breadth tapering. This is due

to the fact that the effect of the former on moment of inertia is greater than the latter.

4. CONCLUSIONS

A new and simple approach has been presented to exactly determine the natural frequencies of Euler-Bernoulli beams with axially varying flexural rigidity and mass density. By expanding the mode shapes as a power series, we directly get corresponding characteristic equations in natural frequencies for all kinds of boundary conditions. An advantage of the suggested approach is that the derived characteristic equation is a polynomial equation, where the lower and higher-order natural frequencies can be determined simultaneously from the multi-roots by using commercial software. The effectiveness of the method has been confirmed by comparing our numerical results with the exact ones and other numerical results available for homogeneous and nonhomogeneous beams. It has been seen that our numerical results show excellent consistency with the existing results. Furthermore, we have investigated the free vibration of axially graded and nonuniform beams to illustrate the effects of the gradient parameter and cross section on the natural frequencies under various end supports. This method is still efficient for treating similar stability problems of axially graded tapered Euler-Bernoulli beams.

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