Free Vibration Analysis of Rotating Plates in High Thermal Environments Using the Finite Element Method

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The aim of the work is to study the vibration characteristics of rotating functionally graded material (FGM) plates using the finite element method. The mechanical properties of the ordinary and temperature dependent FGM plates are assumed to be varying in the thickness direction according to a power law distribution in terms of the volume fractions of the constituents. The temperature field is ascertained using a nonlinear distribution in the thickness direction only. The present method results are compared with numerical results available in the literature. The frequency characteristics of the rotating ordinary FGM and temperature dependent FGM plates are examined. The effect of parameters such as hub radius, rotating speed, temperature change, and power law index value on the natural frequencies of rotating FGM plates are investigated through this study. The plots reveal that the temperature field, rotational speed, and the gradient index in the material properties have a significant effect on the vibration characteristics of the rotating ordinary FGM and temperature dependent FGM plates.

1. INTRODUCTION

FGMs are advanced composite materials made of two or more constituent's phases with continuous and smoothly varying composition. Many rotating structures such as turbine blades, wind turbine blades, and helicopter rotor blades can be modelled as rotating plates. Due to the extensive use of rotating plates, FGM plates are of significant interest to many researchers in the field of modelling, analysis, and design. Acquaintance with the natural frequencies of these rotating plate structures is essential in the design stages for studying their dynamical behaviour under resonance. The frequency analysis of rotating plates often varies considerably from those of non-rotating plates due to the influence of centrifugal inertia force. Due to this, there is a considerable variation of vibration characteristics that result from rotation.

The free vibration characteristics of rotating cantilever isotropic plates have been studied in several investigations. The Coriolis effect on the first bending and first torsional frequencies of flat rotating low aspect ratio cantilever plates using the finite element method has been investigated by Ramamurti and Kielb.¹ Murthy and Ramamurti presented the eigen frequencies of twisted rotating plates by using two different shape functions.² Here, the method of stress smoothing has been used for reducing numerical errors when calculating the improved stiffness matrix. The free vibration analysis of rotating rectangular plates has been studied by Wang et al.³ Yoo et al. derived the equation of motion for the vibration analysis of a rotating pre-twisted blades.⁴ They have used a modelling method that employs hybrid deformation variables. The linear equations of motion for the flap wise bending vibration analysis of rotating plates was developed by Yoo and Kim.⁵ They used two in-plane stretch variables by which the exact in-plane strain energy of a plate can be expressed in a quadratic form.

Yoo and Pierre have investigated the model characteristics of a rotating cantilever plate by dynamic modelling method.⁶ In this method, the parameters have been identified through a dimensional analysis and the effects of the parameters on the variations of model characteristics have also been studied. A modelling method for the modal analysis of cantilever plates undergoing accelerated in-plane motion has been presented by Lim and Yoo.⁷ In their analysis, Von Karman strain measures were used to derive the in-plane and the lateral equations of motion. Finite element formulation for vibration analysis of rotating thick plates has been reported by Hashemi et al.⁸ They used the Kane dynamic method for the derivation of nonlinear governing equations of motion, which included Coriolis effects and the couplings between in-plane and out of plane deformations.

A theoretical analysis of the FGM thin plates based on the physical neutral surface has been studied by Zhang and Zhou.⁹ They assumed changeable parameters such as Young's modulus and the mass density along the thickness of the plate, the physical neutral surface of an FGM plate is determined by the theory of thin plate, when Poisson's ratio is assumed to be constant in the whole plate. The effects of loading conditions and the aspect ratio on the mechanical behaviour of an FGM plate have been discussed by Chi and Chung.¹⁰ They gave a comparison of the results of Polynomial-FGM, Sigmoidal-FGM, and Exponential-FGM. Croce and Venini formulated the governing equations for rectangular plates made of FGMs using the variational approach.¹¹ The free vibration and static analysis of FGM plates using higher order shear deformation theory and with a special modification in the transverse displacement in conjunction with finite element models have been proposed by Talha and Singh.¹² A new exact closed form approach was presented by Hashemi et al. for free vibration analysis of thick rectangular FG plates using the third-order shear deformation

Figure 1. A rotating cantilever FGM plate.

theory of Reddy.¹³

An analytical solution for nonlinear free and forced vibration of FGM plates in thermal environment has been developed by Huang and Shen.¹⁴ Here the temperature field was assumed to be constant in the plane and only varies in the thickness direction of the plate. Yang and Shen¹⁵ have proposed a theoretical method for vibration characteristics study of initially stressed functionally graded rectangular plates made up of metal and ceramic in the thermal environment. The thermo-mechanical induced vibration characteristics of FGM plates have been studied by Talha and Singh.¹⁶ In this work, their objective was to develop a higher order shear deformation theory with an appropriate modification in transverse displacement that provides an additional freedom to the displacements through the thickness and thus eliminates the over prediction. Vibration analysis of temperature dependent functionally graded rectangular plates in thermal environment has been explained by Young-Wann Kim.¹⁷ Dynamic stability of FGM plates in high temperature environment has been studied by Ramu and Mohanty.¹⁸ Exact solution for free vibration analysis of variable thickness thick annular FGM plates was studied by Efraim and Eisenberger.¹⁹

According to our assessment of the published literature, free vibration analysis of a rotating FGM plate in high thermal environment needs of further exploration. The proposed work studies the effects of parameters such as the angular speed, hub ratio, index values, and temperature change on the free vibration characteristics of the rotating FGM plate in high thermal surroundings.

2. MATHEMATICAL FORMULATION

2.1. Theoretical Background

Consider a cantilever FGM plate of a length L , width W , and thickness h , which is rigidly attached to the hub with a radius R , as shown in Fig. 1. The plate is modelled assuming the first order shear deformation theory and it has material properties varying along the thickness direction. Thus, the transverse shear effects are considered in the formulation.

Figure 2. Variation of the modulus of elasticity through the thickness.

2.2. Functionally Graded Material Plate Constitutive Law

An FGM plate is made of metal and ceramic mixtures, the material composition varies from the bottom to the top surface. The top surface ($z = h/2$) of the plate is ceramic-rich, whereas the bottom surface ($z = -h/2$) is metal-rich. The effective material properties are assumed to vary through the thickness direction of the plate according to the simple power law. The effective material properties can be expressed as:

$$P = P_c V_c + P_m V_m; \quad (1)$$

where P_c and P_m are the material properties of the ceramic and metal phase respectively. V_c and V_m are the volume fraction and metal phase and they are related as:

$$V_C(Z) + V_M(Z) = 1. \quad (2)$$

The volume fraction of the constituent ceramic material as per a power law can be expressed as

$$V_c = \left(\frac{2 \times z + h}{2 \times h} \right)^k, \quad 0 \leq k \leq \infty; \quad (3)$$

where k is the power law index, which prescribes the constituent material variation along the thickness direction of the plate. The effective material properties such as Young's modulus, Poisson's ratio, and mass density of an ordinary FGM plate can be expressed based on the volume fraction of the constituent materials, according to a simple power law function as:

$$\begin{aligned} E(z) &= E_m + (E_c - E_m) \left(\frac{1}{2} + \frac{z}{h} \right)^k \\ \rho(z) &= \rho_m + (\rho_c - \rho_m) \left(\frac{1}{2} + \frac{z}{h} \right)^k \\ v(z) &= v_m + (v_c - v_m) \left(\frac{1}{2} + \frac{z}{h} \right)^k \end{aligned} \quad (4)$$

For an FGM plate made up of steel and alumina, Fig. 2 shows the variation of the modulus of elasticity through the dimensionless thickness.

2.3. Temperature Dependent FGM Properties

The elastic modulus E , the mass density ρ , Poisson's ratio ν , thermal conductivity K , and the coefficient of thermal expansion α are temperature-dependent material properties. The

Figure 3. Geometry of the FGM plate.

temperature-dependent material properties are obtained using the following expression by Praveen and Reddy.²⁰

$$R(T) = P_0 (P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + P_3T^3) \quad (5)$$

where P_0 , P_1 , P_2 , and P_3 are the coefficients of temperature T in Kelvin and are unique to each constituent. $T = T_0 + T(z)$, where $T(z)$ is temperature rise through the thickness direction and T_0 is room temperature. From the above equations, the effective material properties with two constituents for FGM plates can be expressed as follows:

$$\begin{aligned} E(z, T) &= E_m(T) + [E_c(T) - E_m(T)] \left(\frac{2z+h}{2h} \right)^k; \\ \alpha(z, T) &= \alpha_m(T) + [\alpha_c(T) - \alpha_m(T)] \left(\frac{2z+h}{2h} \right)^k; \\ \nu(z, T) &= \nu_m(T) + [\nu_c(T) - \nu_m(T)] \left(\frac{2z+h}{2h} \right)^k; \\ \rho(z) &= \rho_m + [\rho_c - \rho_m] \left(\frac{2z+h}{2h} \right)^k; \\ K(z) &= K_m + [K_c - K_m] \left(\frac{2z+h}{2h} \right)^k. \end{aligned} \quad (6)$$

During the analysis, the temperature field was applied in the thickness direction only and the temperature field was assumed to be constant in the XY plane of the plate.

2.4. Physical Neutral Surface of the FGM Plate

For plate made of FGM (Fig. 3), the neutral surface does not coincide with its geometric mid-surface. The distance (d) of the neutral surface from the geometric mid-surface may be expressed as:

$$d = \frac{\int_{-h/2}^{h/2} z E(z, T) dz}{\int_{-h/2}^{h/2} E(z, T) dz}. \quad (7)$$

2.5. Nonlinear Temperature Distribution

A steady-state heat transfer equation used to obtain the temperature distribution along the thickness, can be represented as:

$$-\frac{d}{dz} \left[K(z') \frac{dT}{dz} \right] = 0. \quad (8)$$

Figure 4. Plate structure before and after deformation.

This equation is solved by prescribing temperature load at top and bottom surfaces such as $T = T_c$ at $z' = h/2 - d$ and $T = T_m$ at $z' = h/2 + d$. The temperature distribution along the thickness direction can be obtained as:

$$T(z') = T_m + (T_c - T_m) \frac{\int_{-h/2-d}^{z'} \frac{1}{k(z')} dz'}{\int_{-h/2-d}^{h/2+d} \frac{1}{k(z')} dz'}. \quad (9)$$

2.6. Kinematics

Plate structures made of graded materials are characterized by non-negligible shear deformation so that the extension of the classical model suggested by Croce and Venini¹¹ to the case of graded material plates provides a good compromise between numerical accuracy and computational load. The displacements u, v, w at a point (x, y, z') from the neutral plane of an FGM plate (Fig. 4) can be expressed as:

$$\begin{aligned} u(x, y, z) &= z' \theta_x(x, y), \\ v(x, y, z) &= z' \theta_y(x, y), \\ w(x, y, z) &= w(x, y); \end{aligned} \quad (10)$$

where θ_x and θ_y the rotations of the normal in xz and yz planes, respectively.

In-plane strain and transverse shear strains γ_x and γ_y are conversely constant through the thickness and are defined as:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = z' \begin{bmatrix} \theta_{x,x} \\ \theta_{y,y} \\ \theta_{x,y} + \theta_{y,x} \end{bmatrix}, \quad \begin{bmatrix} \gamma_x \\ \gamma_y \end{bmatrix} = \begin{bmatrix} w_{,x} + \theta_x \\ w_{,y} + \theta_y \end{bmatrix} \quad (11)$$

The plane stress and transverse shear stress relationship are given as:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = z' D_b \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}, \quad \begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} = D_s \begin{bmatrix} \gamma_x \\ \gamma_y \end{bmatrix} \quad (12)$$

$$\text{where } D_b = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad D_s = \frac{Ehk_{sh}}{2(1-\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Bending moment and shear stress relationships are:

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-h/2-d}^{h/2-d} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} z' dz' = \begin{bmatrix} \theta_{x,x} \\ \theta_{y,y} \\ \theta_{x,y} + \theta_{y,x} \end{bmatrix} \int_{-h/2-d}^{h/2-d} D z'^2 dz'. \quad (13)$$

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \int_{-h/2-d}^{h/2-d} \begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} dz' = \begin{bmatrix} w_{,x} + \theta_x \\ w_{,y} + \theta_y \end{bmatrix} \int_{-h/2-d}^{h/2-d} D_s dz'. \quad (14)$$

2.6.1. Constitutive Relations

The linear thermo-mechanical constitutive relations are:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \alpha(z', T) \right); \quad (15)$$

where $Q_{11} = \frac{E(z')}{1-\nu^2}$, $Q_{12} = Q_{21} = \frac{\nu E(z')}{1-\nu^2}$, $Q_{66} = \frac{E(z')}{2(1+\nu)}$.

The total in-plane force resultants and moments produced due to temperature rise are defined as:

$$N_t = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2-d}^{h/2-d} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz', \quad M_t = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2-d}^{h/2-d} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} z' dz'. \quad (16)$$

The thermal thrust and moment can also be represented as:

$$\{N_{tx}\} = \{N_{ty}\} = \int_{-h/2-d}^{h/2-d} \{\beta\} \Delta T dz', \quad \{M_t\} = \int_{-h/2-d}^{h/2-d} \{\beta\} \Delta T z' dz'; \quad (17)$$

where $\{\beta\} = Q \{\alpha\} = \begin{Bmatrix} (Q_{11} + Q_{12}) \alpha \\ (Q_{12} + Q_{22}) \alpha \\ 0 \end{Bmatrix}$.

3. ENERGY EQUATIONS

The potential strain energy U^e of the plate element can be written in terms of the stress resultants and strain components through integrating by parts with respect to coordinates as follows

$$U^e = U_{ps}^e + U_{ts}^e. \quad (18)$$

U_{ps}^e Strain energy contribution from plane stress, U_{ts}^e strain energy contribution from transverse shear stress $U^e = \frac{1}{2} \iiint_V \varepsilon^T \sigma dv$.

$$U^e = \frac{1}{2} \int_a \int_{-h/2-d}^{h/2-d} \begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_{xy} \end{bmatrix} D \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} dz' da + \frac{k_{sh}}{2} \int_a \int_{-h/2-d}^{h/2-d} \begin{bmatrix} \gamma_x & \gamma_y \end{bmatrix} S \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} dz' da. \quad (19)$$

Here k_{sh} is shear correction factor, $k_{sh} = \frac{5}{6 - (\nu_m V_m + \nu_c V_c)}$, ν_m , and ν_c represents the Poisson's ratio of metal and ceramic, V_m and V_c stands for the volume fraction of metal and ceramic, respectively.¹⁹

As the plate is exposed to thermal environment, it is subjected to in-plane stress resultants. Therefore, the work done by the in-plane forces produced due to temperature change produces out-of-plane displacement ' w '.

Work done due to thermal loading of plate element is as follows:

$$W_t^e = \frac{1}{2} \left(\int_A N_{tx} \left(\frac{\partial w}{\partial x} \right)^2 dx + \int_A N_{ty} \left(\frac{\partial w}{\partial y} \right)^2 dy \right); \quad (20)$$

where N_{tx} and N_{ty} is the thermal thrust.

When the plate is rotating, it subjected to centrifugal forces. Therefore, the work is done by the plate centrifugal forces produced due to rotating speed. The centrifugal force on an element of the plate can be expressed as:

$$F_c^e = \int_{x_i}^{x_i+l} \int_{-h/2-d}^{h/2-d} b \rho(z') \Omega^2 (R + x) dz' dx; \quad (21)$$

where x_i is the distance of i^{th} node from axis of rotation, Ω (rad/s) is angular velocity of plate element and R is the radius of the hub.

Work done by the centrifugal force is given by:

$$W_c^e = \frac{1}{2} F_c^e w^2 dx = \frac{1}{2} \{q\}^T [k_c] \{q\}. \quad (22)$$

The kinetic energy of the plate element can be expressed as follows

$$T^e = \frac{1}{2} \iint_A \rho h \dot{w}^2 dA. \quad (23)$$

Figure 5. The finite element model of the rectangular element.

3.1. Finite Element Model Formulation

In the present FEM formulation, the derivatives are assumed as independent field variables by considering w , θ_x , and θ_y as separate DOFs. A four-noded rectangular element is employed for finite-element modelling as shown in Fig. 5. Over each of the elements, the displacement vector is expressed by:

$$\begin{aligned} w &= \sum_{i=1}^n N_i w_i \\ \theta_x &= \sum_{i=1}^n N_i \theta_x^i ; \\ \theta_y &= \sum_{i=1}^n N_i \theta_y^i \end{aligned} \quad (24)$$

where N_i are the appropriate shape functions, n represents number of nodes in the element.

3.2. Strain energy of the plate element

The strain energy of the FGM plate element is given by

$$U^e = \frac{1}{2} \{q^e\}^T [K_s^e] \{q^e\} ; \quad (25)$$

where $[K_s^e]$ and $\{q^e\}$ are defined as element stiffness matrix and displacement vector of the element, respectively. The element stiffness matrix is derived as:

$$K_s^e = \int_v [B_b]^T [D_b] [B_b] dv + \int_v [B_s]^T [D_s] [B_s] dv ; \quad (26)$$

$$\text{where } [B_b] = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} [N_i], \quad [B_s] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & N_i & 0 \\ \frac{\partial N_i}{\partial y} & 0 & N_i \end{bmatrix}.$$

$$W_T^e = \int q^{(e)T} \lambda_T [K_t^e] q^{(e)} da ; \quad (27)$$

where λ_T and $[K_t^e]$ are defined as thermal buckling load parameters and the element geometric stiffness matrix for the element, respectively. The element geometric stiffness matrix is derived as:

$$[K_t^e] = N_{tx} \int \int \left(\frac{\partial^2 w}{\partial x^2} \right) dx + N_{ty} \int \int \left(\frac{\partial^2 w}{\partial y^2} \right) dy.$$

Here, N_{tx} and N_{ty} are thermal thrust along x and y axis. The Element centrifugal stiffness matrix is defined as:

$$\begin{aligned} W_c^e &= \frac{1}{2} \{q\}^{(e)T} [K_c^e] \{q\}^{(e)} ; \\ [K_c^e] &= F_c \int \int [N]^T [N] dxdy. \end{aligned} \quad (28)$$

3.3. Kinetic energy of the plate

The kinetic energy of the vibrating plate may be written as

$$\begin{aligned} T^e &= \frac{1}{2} \int_v \{\dot{q}^e\}^T [M^e] \{\dot{q}^e\} ; \\ \delta &= L/W, \zeta = R/L, \lambda = \Omega T, \eta = h/L ; \\ \varpi &= \omega L^2 \sqrt{\rho h/D} ; \\ \{\beta\} &= Q \{\alpha\} = \begin{Bmatrix} (Q_{11} + Q_{12}) \alpha \\ (Q_{12} + Q_{22}) \alpha \\ 0 \end{Bmatrix}. \end{aligned} \quad (29)$$

Here, $[M^e]$ is the mass matrix of the element.

The Element mass matrix is derived as $[M^e] = \int N^T \rho N dv$ in which N is the shape function matrix given as: $N_{(3 \times 12)} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix}$, $N_1 = \frac{(1-\xi)(1-\eta)}{4}$, $N_2 = \frac{(1+\xi)(1-\eta)}{4}$, $N_3 = \frac{(1+\xi)(1+\eta)}{4}$, $N_4 = \frac{(1-\xi)(1+\eta)}{4}$, where $\xi = x/a$, $\eta = y/b$, a and b are the element length and width.

4. GOVERNING EQUATIONS OF MOTION

The element equation of motion for a rotating FGM plate in thermal environment is obtained by using Hamilton's principle:

$$\delta \int_{t_1}^{t_2} (U^e - T^e) dt = 0 \quad (30)$$

By dividing the plate into a number of elements, the potential energy and the kinetic energy for an element can be written in terms of element nodal displacement vector as:

$$\begin{aligned} U^e &= \frac{1}{2} \{q^e\}^T ([K_s^e] + [K_c^e]) \{q^e\} - \frac{1}{2} \{q^e\}^T [K_t^e] \{q^e\} = \\ &= \frac{1}{2} \{q^e\}^T [K_e^{ef}] \{q^e\} ; \end{aligned} \quad (31)$$

$$T^e = \frac{1}{2} \{\dot{q}^e\}^T [M^e] \{\dot{q}^e\}. \quad (32)$$

Substituting Eq. (31) and Eq. (32) in Eq. (30) and carrying out the variation the equation of motion for the element can be written as

$$[M^e] \{\ddot{q}^e\} + [K_e^{ef}] \{q^e\} = 0 ; \quad (33)$$

where $[K_e^{ef}] = [K_s^e] + [K_c^e] - [K_t^e]$ is the effective element stiffness matrix. Assembling the element matrices the equation of motion in terms of global matrices and global displacement vector can be written as follows:

$$[M] \{\ddot{q}\} + [K_e^{ef}] \{q\} = 0 ; \quad (34)$$

Table 1. Compression of the first five natural frequencies parameter by the present and Yoo and Kim.⁵

Dimensionless angular speed	Mode	Yoo and Kim ⁵	Yoo and Kim ⁵ Southwell method	Present method
$\lambda = 1$	1	3.65280	3.65427	3.5567
	2	8.64590	8.64350	8.6205
	3	21.53372	21.55690	21.4360
	4	27.38478	27.47469	27.2818
	5	31.21853	31.38484	31.0489
$\lambda = 2$	1	4.11312	4.112807	4.1190
	2	9.00317	8.98438	9.4010
	3	21.96654	22.01202	22.1913
	4	27.62312	27.86609	28.1300
	5	31.58544	32.17962	31.6746

Table 2. Compression of the first five natural frequencies parameter by the present and Yoo and Kim.⁵ $\delta = 1$, $\varsigma = 1$.

Dimensionless angular speed	Mode	Yoo and Kim ⁵	Yoo and Kim ⁵ Southwell method	Present method
$\lambda = 1$	1	3.86185	3.80871	3.9703
	2	8.73585	8.72801	9.1854
	3	21.71968	21.74132	21.9558
	4	27.42572	27.62624	27.9096
	5	31.36240	31.77682	31.4956
$\lambda = 2$	1	4.81385	4.63956	5.1079
	2	9.34355	9.30544	10.9537
	3	22.79012	22.72604	23.7636
	4	27.79012	28.45908	29.9878
	5	32.14933	33.68323	33.0389

where $[K^{ef}]$ is effective global effective stiffness matrix, and $[M]$ is the global mass matrix. Eigen values of Eq. (34) gives the natural frequencies of the plate.

5. RESULTS AND DISCUSSION

The numerical results of thermo-mechanical vibration analysis of the rotating FGM plates are calculated using the proposed mathematical model using FEM. A four node rectangular element, with 12 DOFs per element has been used to discretize the plate. A computer program has been developed in an environment created using MATLAB. The validation and efficiency of the proposed algorithm is examined by comparing the results with those available in the literature.

5.1. Validations of the Results

5.1.1. Rotating Plate Frequency

The rotating isotropic material plate frequency parameters obtained by using the present method are compared with those obtained by Yoo and Kim⁵ to show the reliability of method of calculation presented in this paper. The rotating plate frequency parameters obtained are shown in Tables 1 and 2. As shown in these tables, the results obtained by using the present method are in reasonable agreement with those obtained by Southwell's method.⁵

Here, in all the presented tables and figures, δ , ς , λ , η , and ϖ denote the aspect ratio, hub radius ratio, dimensionless rotation speed thickness ratio, and natural frequency parameter respectively. They are defined as: $\delta = L/W$, $\varsigma = R/L$, $\lambda = \Omega T$, $\eta = h/L$, $\varpi = \omega L^2 \sqrt{\rho h/D}$ where $D = Eh^3/12(1 - \nu^2)$.

5.1.2. Comparison of Natural Frequencies of FGM Plates

Table 3 shows the SUS304, Si_3N_4 , and Al_2O_3 temperature-dependent properties of the FGM constituents that have been used for the computation of the present analysis.

Table 4 shows the comparison of the first five natural frequency parameter of FGM plate obtained by applying first order shear deformation theory in this study and those obtained by Huang and Shen using the higher-order shear deformation plate theory.¹⁴ The results show good agreement. The difference is due to the higher order displacement functions that are used in the reference results.

Table 4 shows the effect of volume fraction index k on the natural frequency parameter of FGM ($\text{Si}_3\text{N}_4/\text{SUS304}$) plates under thermal loading condition, $T_m = 300$ K, $T_c = 400$ K, and $\nu = 0.28$. Temperature-dependent and temperature-independent material properties are considered for this analysis. It can be seen that the natural frequency parameter of the FGM plate decreases with the increase of volume fraction index k .

5.2. The Rotating Ordinary FGM Plate Natural Frequencies

Now we study the FGM ($\text{Al}_2\text{O}_3/\text{SUS304}$) plate for this analysis. For this numerical study, the following material properties are consider as:

SUS304, $\rho = 8.166$ kg/m³, $E = 207.78 \times 10^9$ N/m², $\nu = 0.3$
 Al_2O_3 , $\rho = 3800$ kg/m³, $E = 320.024 \times 10^9$ N/m², $\nu = 0.3$.

Figures 6(a) and 6(b) show the variations of the first five natural frequencies of rotating FGM plates with an aspect ratio of 1, a thickness ratio of 0.01, and the a hub radius ratio of 0 and 1. According to these figures, it can be seen that the first five natural frequencies decrease as the volume fraction index increases at rotating speed 100 RPM. Figures 7 shows the first five natural frequencies variation with hub radius ratio. The increasing hub radius increases the first five natural frequencies. The variation of natural frequency versus rotating speed for FGM plates is plotted in Figs. 8(a) to 8(d). These figures are plotted for 1st, 2nd, 3rd, and 4th mode frequencies of FGM plate, for different power law index values ($k = 0, 0.5, 2$) at rotating speed 100 RPM. As expected, the natural frequencies increase as the angular speed increases. This results from the centrifugal inertia force which increases as the hub radius and the angular speed increase.

5.3. Natural Frequencies of the Rotating Temperature Dependent FGM Plate

Here, the upper surface of the temperature dependent FGM plate is maintained at a maximum of 800 K, which is ceramic-rich, whereas the lower one is metal-rich at 300 K. The aspect ratio (L/W) is taken as 1. The temperature eld is assumed to vary in the thickness direction only and is determined by the steady-state heat conduction equation across the thickness of the plate.

Figures 9(a) to 9(d) show the variation of first four frequencies of the FGM plate in a thermal environment with a temper-

Table 3. Temperature-dependent coefficients of the elastic modulus E [GPa] and thermal expansion coefficient [1/K] for ceramics and metals.¹⁴

Materials	Proprieties	P_0	P_{-1}	P_1	P_2	P_3
SUS304	E_m	201.04×10^9	0	3.079×10^{-4}	-6.534×10^{-7}	0.0
	α	12.33×10^{-6}	0	8.086×10^{-4}	0.0	0.0
Si_3N_4	E_c	348.43×10^9	0	-3.079×10^{-4}	2.160×10^{-7}	-8.946×10^{-11}
	α	5.8723×10^{-6}	0	9.095×10^{-4}	0.0	0.0
Al_2O_3	E_c	349.55×10^9	0	-3.853×10^{-4}	4.027×10^{-7}	-1.673×10^{-11}
	α	6.826×10^{-6}	0	1.838×10^{-4}	0.0	0.0

Table 4. Comparison of the first five natural frequency parameter $\Omega = \omega (L^2/h) [\rho_m (1 - \nu^2) / E_m]^{1/2}$ for FGM ($\text{Si}_3\text{N}_4/\text{SUS304}$) plates in thermal environments.

	Index value	First five natural frequencies parameter					
			1	2	3	4	5
Temperature independent	Si_3N_4	Huang and Shen ¹⁴	12.382	29.243	29.243	44.072	53.105
		Present	12.633	29.717	29.717	44.801	54.668
	0.5	Huang and Shen ¹⁴	8.641	20.316	20.316	30.682	37.007
		Present	8.798	20.698	20.698	31.205	38.078
	1	Huang and Shen ¹⁴	7.514	17.694	17.694	26.717	32.242
		Present	7.676	18.060	18.060	27.227	33.225
	2	Huang and Shen ¹⁴	6.728	15.836	15.836	23.893	28.816
		Present	6.795	15.987	15.987	24.103	29.412
	Si_3N_4	Huang and Shen ¹⁴	12.397	29.083	29.083	43.835	52.822
		Present	12.319	29.316	29.316	44.320	54.138
Temperature dependent	0.5	Huang and Shen ¹⁴	8.615	20.215	20.215	30.530	36.824
		Present	8.522	20.370	20.370	30.827	37.670
	1	Huang and Shen ¹⁴	7.474	17.607	17.607	26.590	32.088
		Present	7.412	17.755	17.755	26.883	32.857
	2	Huang and Shen ¹⁴	6.693	15.762	15.762	23.786	28.686
		Present	6.540	15.702	15.702	23.788	29.080

Figure 6. (a) The rotating ordinary FGM plate for the first five natural frequencies variation with a power law index value at speed 100 RPM and $R/L = 0$. (b) The rotating ordinary FGM plate for the first five frequencies parameter variation with a power law index value at speed 100 RPM and $R/L = 1$ **Figure 7.** Rotating ordinary FGM plate for the first five natural frequencies variation with the hub radius ratio (R/L) at speed 150 RPM and $k = 0.5$.

ature change (0-500 K); the frequencies reduce as the temperature change increases. It is shown that the effect of temperature change on the values of first four frequencies decrease with an increase of the temperature difference from 0 to 500 K. This tendency is observed because Young's modulus decreases as temperature goes up and consequently, the lesser Young's modulus lowers the natural frequencies.

Figures 10(a) to 10(d) show the variation of first four mode frequencies of FGM plate in thermal environment with hub radius ratio, for different power law indices $k = 0, 0.5$ and 2 at rotating speed 150 RPM. From Figs. 10(a) to 10(d), one can observe that the first four frequencies increase with increase in hub radius ratio. This happens due to centrifugal stiffening with an increased hub radius.

Figures 11(a) to 11(d) depict the first four frequencies versus the rotating speed for different power law index values ($k = 0, 0.5, 2$), respectively. The increasing rotating speed increases the

Figure 8. (a) Variation of first mode frequency with the rotational speed at the hub radius ($R/L = 0.5$). (b) Variation of second mode frequency with the rotational speed at the hub radius ($R/L = 0.5$). (c) Variation of third mode frequency with the rotational speed at the hub radius ($R/L = 0.5$). (d) Variation of fourth mode frequency with the rotational speed at the hub radius ($R/L = 0.5$).

Figure 9. (a) Variation of the first natural frequency of the rotating temperature dependent FGM plate with a temperature change (speed 150 RPM and $R/L = 0.4$). (b) Variation of the second natural frequency of the rotating temperature dependent FGM plate with a temperature change (speed 150 RPM and $R/L = 0.4$). (c) Variation of the third natural frequency of the rotating temperature dependent FGM plate with a temperature change (speed 150 RPM and $R/L = 0.4$). (d) Variation of the fourth natural frequency of the rotating temperature dependent FGM plate with a temperature change (speed 150 RPM and $R/L = 0.4$).

Figure 10. (a) Variation of the rotating temperature dependent FGM plate first natural frequency with a hub radius at speed 150 RPM and temperature difference 100 K. (b) Variation of the rotating temperature dependent FGM plate second natural frequency with a hub radius at speed 150 RPM and temperature difference 100 K. (c) Variation of the rotating temperature dependent FGM plate third natural frequency with a hub radius at speed 150 RPM and temperature difference 100 K. (d) Variation of the rotating temperature dependent FGM plate fourth natural frequency with a hub radius at speed 150 RPM and temperature difference 100 K.

Figure 11. (a) Variation of the first natural frequency of rotating temperature dependent FGM plate with a rotating speed (temperature change 200 K and $R/L = 0.4$). (b) Variation of the second natural frequency of rotating temperature dependent FGM plate with a rotating speed (temperature change 200 K and $R/L = 0.4$). (c) Variation of the third natural frequency of rotating temperature dependent FGM plate with a rotating speed (temperature change 200 K and $R/L = 0.4$). (d) Variation of the first fourth natural frequency of rotating temperature dependent FGM plate with a rotating speed (temperature change 200 K and $R/L = 0.4$).

inertia force, it causes increase in natural frequencies. Thus, from these Figs. 11(a) to 11(d) it is concluded that increase in rotational speed increases the natural frequencies irrespective of the power law index value.

6. CONCLUSIONS

Flapwise bending vibration characteristics of rotating functionally graded plates have been presented. The analysis was carried out using the FSDT along with FEM. The validation studies have been carried out to establish the correctness of the present formulation and calculation. Four parameters are identified through the proposed numerical experiments: power law index, the hub radius, the angular speed, and temperature rise. The effect of these parameters on the natural frequency of rotating FGM plates are studied in detail. Observations from the analysis: the rotating FGM plate natural frequencies increase with the increase in the angular rotating speed and hub radius. The natural frequencies reduces with the increase in power law index value and temperature rise.

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