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Lie groupoids in digital images

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Abstract: In this paper, we introduce Lie groupoids associated with a digital k -manifold in digital images. Moreover, we give some results related to digital Lie groupoid morphisms. We prove that the composition of any two digital Lie groupoid morphisms is a digital Lie groupoid morphism.

Keywords: Lie groupoid, groupoid, digital image, digital manifold, generalized manifold.

1 Introduction

The theory of Lie groupoids has been used in many parts of mathematics such as differential geometry, geometric topology and algebraic topology with significant properties. The concept of Lie groupoids was first introduced by Ehresmann [8] in 1959. A Lie groupoid can be seen as a generalization of a Lie group. The notions of topological manifold and generalized manifold play important role in the work of Lie groupoids. For other related works on Lie groupoids, see [7, 9].

On the other hand, digital topology has been a growing area in various part of sciences such as mathematics, computer science, algebraic topology and image processing. Researchers get important results on digital images using Algebraic Topology and Geometric Topology. Boxer [1, 2, 3, 4, 5] introduce the digital versions of some notions in topology and then many works have been done up to now.

This paper is organized as follows: In the first section, we give required definitions and theorems about Lie groupoids, digital images and digital manifolds. Next, we introduce digital Lie groupoids and deal with their some properties. Finally, we give some comments on our work.

2 Preliminaries

In this section, we recall some notions related to groupoids and Lie groupoids. We begin with the definition of a groupoid.

Definition 1. [8]. A groupoid is a small category where each morphism is invertible. A groupoid (G, B) consists of two sets G and B , the set of morphisms and the set of objects, respectively, together with two maps $s, t : G \rightarrow B$ called source and target maps. In a groupoid, there are three maps such that a composition map $m : G_1 \rightarrow G$ defined on

$$G_1 = \{(g, h) \in G \times G \mid s(g) = t(h)\},$$

an inverse map $i : G \rightarrow G$ and an inclusion map of objects $e : B \rightarrow G$. Every groupoid satisfies the following properties:

- (1) $s(gh) = s(h)$ and $t(gh) = t(g)$, for all $gh \in G_1$;
- (2) $g(hf) = (gh)f$ such that $s(h) = t(f)$ and $s(g) = t(h)$, for all $g, h, f \in G$;
- (3) $s(1_x) = t(1_x) = x$, for all $x \in B$ where $e(x) = 1_x$;
- (4) $h1_{s(h)} = h$ and $1_{t(h)}h = h$, for all $h \in G$ and
- (5) $s(h^{-1}) = t(h)$ and $t(h^{-1}) = s(h)$, $h^{-1}h = 1_{s(h)}$ and $hh^{-1} = 1_{t(h)}$.

Definition 2. [13]. A Lie groupoid is a groupoid G on B together with G and B are both manifolds and s, t, e, i, m are all smooth maps and s, t are surjective submersions.

Definition 3. [13]. A morphism between two Lie groupoids H and G is a groupoid morphism (f, g) such that f and g are smooth maps.

We now would like to other required definitions from digital topology.

A digital image is a pair (X, κ) , where $X \subseteq \mathbb{Z}^n$ for some positive integer n and κ is an adjacency relation for the members of X . For various adjacency relations, see [11].

Definition 4. [4]. For a positive integer l with $1 \leq l \leq n$ and two distinct points $p = (p_1, p_2, \dots, p_n)$, $q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$, p and q are c_l -adjacent, if

- (1) there are at most l indices i such that $|p_i - q_i| = 1$, and
- (2) for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

The notation c_l denotes the number of points $q \in \mathbb{Z}^n$ that are adjacent to a given point $p \in \mathbb{Z}^n$. As a result, in $\mathbb{Z}$, we have $c_1 = 2$ -adjacency; in $\mathbb{Z}^2$, we have $c_1 = 4$ -adjacency and $c_2 = 8$ -adjacency; in $\mathbb{Z}^3$, we have $c_1 = 6$ -adjacency, $c_2 = 18$ -adjacency, and $c_3 = 26$ -adjacency [4]. Given a natural number l in conditions (1) and (2) with $1 \leq l \leq n$, l determines each of the κ -adjacency relations of $\mathbb{Z}^n$ in terms of (1) and (2) [10] as follows:

$$\kappa \in \left\{ 2n \ (n \geq 1), 3^n - 1 \ (n \geq 2), 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1 \ (2 \leq r \leq n-1, n \geq 3) \right\}$$

where $C_t^n = \frac{n!}{(n-t)!t!}$.

Definition 5. [1]. The set $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ is called a digital interval where $a, b \in \mathbb{Z}$ and $a < b$.

A κ -neighbor of $p \in \mathbb{Z}^n$ [2] is a point of $\mathbb{Z}^n$ that is κ -adjacent to p . A digital image $X \subset \mathbb{Z}^n$ is κ -connected [11] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \dots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ -neighbors where $i = 0, 1, \dots, r-1$.

Definition 6. [2]. Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 -adjacency and κ_1 -adjacency, respectively. A function $f : X \rightarrow Y$ is said to be (κ_0, κ_1) -continuous if for every κ_0 -connected subset U of X , $f(U)$ is a κ_1 -connected subset of Y . We say that such a function is digitally continuous.

Proposition 1. [2]. If $(X, \kappa_0) \rightarrow (Y, \kappa_1)$ and $g : (Y, \kappa_1) \rightarrow (Z, \kappa_2)$ are, respectively, (κ_0, κ_1) -continuous and (κ_1, κ_2) -continuous functions, then the composite function $g \circ f : X \rightarrow Z$ is (κ_0, κ_2) -continuous.

Proposition 2. [2]. Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 -adjacency and κ_1 -adjacency, respectively. The function $f : X \rightarrow Y$ is (κ_0, κ_1) -continuous if and only if for every κ_0 -adjacent points $\{x_0, x_1\}$ of X , either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1 -adjacent in Y .

Definition 7. [5]. Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. A function $f : X \rightarrow Y$ is a (κ_0, κ_1) -isomorphism, if f is (κ_0, κ_1) -continuous and bijective and further f^{-1} is (κ_1, κ_0) -continuous.

Definition 8. [15]. Let S be a set of nonempty subset a digital image (X, κ) . Let the following statements hold:

- (i) If p and q are distinct points of $s \in S$, then p and q are κ -adjacent,
- (ii) If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.

Then the members of S are called *simplexes* of (X, κ) . A k -simplex is a simplex S such that $|S| = k + 1$.

We recall some notions in [6] because they will help us to define digital Lie groupoids. The digital n -manifold was studied in 1993 and its definition was given by Chen and Zhang. Before giving definition of digital manifold, we will give some notations. A one-dimensional digital space is the integer set $\mathbb{Z}$ and an m -dimensional digital space is $\mathbb{Z}^m = \mathbb{Z} \times \dots \times \mathbb{Z}$. The notations $\Sigma_1 = \{1, 2, \dots, N\}$ and Σ_m indicate a one-dimensional digital space and an m -dimensional digital space, respectively.

A digital k -cell is a digital k -simplex. For a subset (S, κ) of Σ_m , we denote $\Gamma_{(0)}^\kappa(S)$ as the set of 0-cells in S , $\Gamma_{(1)}^\kappa(S)$ as the set of 1-cells in S and inductively, $\Gamma_{(k)}^\kappa(S)$ as the set of k -cells of S (see [6]).

Two k -cells are called k' -adjacent ($0 \leq k' < k$) if they share a k' -cell. A k -cell path with k' -adjacency is a sequence of k -cells $v_0, v_1, \dots, v_n$, where v_i and v_{i+1} are k' -adjacent and $v_0, v_1, \dots, v_n$ are different. Two k -cells u and v are said to be k' -connected if they are two end elements of a k -cells path with k' -adjacency.

Definition 9. [6]. Let (A, κ) be a subset of Σ_m . (A', κ') is called a *parallel-move* of (A, κ) if

- (1) $|A| = |A'|$;
- (2) $A \cap A' = \emptyset$;
- (3) there exists a biconjunction map $f : A \rightarrow A'$ such that $d(a, f(a)) = 1$ for all $a \in A$.

Definition 10. [6]. A k -connected subset M of Σ_m is an n -dimensional digital manifold if $M = \cup \{a \mid a \in \Gamma_n^k(M)\}$ and

- (1) $\Gamma_n^k(M)$ is $(n-1)$ -connected;
- (2) Each element in $\Gamma_{(n-1)}^k(M)$ has only one or two parallel-moves;
- (3) $\Gamma_{(n+1)}^k(M)$ is empty.

The concept of digital category is defined as follows:

Definition 11. [14]. A digital category is a quintuple $\mathcal{C} = (\mathcal{O}, \mathcal{M}, \text{dom}, \text{cod}, \circ)$ where

- (1) $\mathcal{O}$ is a class consists of $\mathcal{C}$ -objects ($\mathcal{C}$ -objects are digital images);
- (2) $\mathcal{M}$ is a class whose members are called $\mathcal{C}$ -morphism ($\mathcal{C}$ -morphisms are digital functions between digital images);
- (3) dom and cod are digital functions from $\mathcal{M}$ to $\mathcal{O}$;
- (4) $\circ$ is a function from

$$D = \{(f, g) \mid f, g \in \mathcal{M} \text{ and } \text{dom}(f) = \text{cod}(g)\}$$

into $\mathcal{M}$. It satisfies the following conditions:

- (i) If $f \circ g$ is defined, then $\text{dom}(f \circ g) = \text{dom}(g)$ and $\text{cod}(f \circ g) = \text{cod}(f)$;
- (ii) If $f \circ g$ and $h \circ f$ are defined, $h \circ (f \circ g) = (h \circ f) \circ g$;
- (iii) For each $\mathcal{C}$ -object A , there exists a $\mathcal{C}$ -morphism e such that $\text{dom}(e) = A = \text{cod}(e)$ and
 - (a) $f \circ e = f$ when $f \circ e$ is defined,
 - (b) $e \circ g = g$ when $e \circ g$ is defined;

(iv) For any pair (A, B) of $\mathcal{C}$ -objects, the following class

$$\{f \mid f \in \mathcal{M}, \text{dom}(f) = A \text{ and } \text{cod}(f) = B\}$$

is a set.

Definition 12. [14]. A digital category $\mathcal{C}$ is called small if $\mathcal{C}$ is a digital set.

3 Digital Lie groupoids

In this section, we present Lie groupoids in digital images. Firstly we give the definition of a digital groupoid.

Definition 13. A digital groupoid is a digital small category where every digital map is invertible. A digital groupoid (G, B) consists of two sets G and B , the set of digital continuous maps and the set of digital images, respectively, together with five maps:

- (1) The digital maps $s, t : G \rightarrow B$ called source and target maps.
- (2) The composition map $m : G_1 \rightarrow G$ defined on

$$G_1 = \{(g, h) \in G \times G \mid s(g) = t(h)\}.$$

- (3) An inclusion map $e : B \rightarrow G$ mapping each $x \in B$ to a two-sided unit $e(x) = 1_x$ for the composition.
- (4) An inverse map $i : G \rightarrow G$ mapping each $g \in G$ to a two-sided inverse $i(g) = g^{-1}$ for the composition.

Remark. Five digital maps in Definition 13 satisfy the followings:

- (I) $s(uv) = s(u)$ and $t(uv) = t(v)$ for each $(u, v) \in G_1$.
- (II) $u(vw) = (uv)w$ for all $u, v, w \in G$ such that $s(v) = t(u)$ and $s(w) = t(v)$.
- (III) $s(e(u)) = t(e(u)) = u$ for each $u \in B$.
- (IV) $ue(t(u)) = u$ and $e(s(u))u = u$ for all $u \in G$.
- (V) $s(u^{-1}) = t(u)$, $t(u^{-1}) = s(u)$, $uu^{-1} = e(s(u))$ and $u^{-1}u = e(t(u))$.

Definition 14. A digital Lie groupoid is a digital groupoid G on B with G and B are both digital k -manifolds and s, t, m, e, i are all digital continuous maps and s, t are surjective maps.

Definition 15. Let (G, B) and (G', B') be two digital groupoids. A digital groupoid morphism is a digital map $(f, g) : (G, B) \rightarrow (G', B')$ where $f : G \rightarrow G'$ and $g : B \rightarrow B'$ such that

- (i) $f(m(a, b)) = m'(f(a), f(b))$ for each $(a, b) \in G_1$,
- (ii) $s' \circ f = g \circ s$ and $t' \circ f = g \circ t$.

Definition 16. A digital Lie groupoid morphism between two digital Lie groupoids (G, B) and (G', B') is a digital groupoid morphism (f, g) such that f and g are digital continuous maps.

Theorem 1. Let (G, B) , (G', B') and (G'', B'') be digital Lie groupoids and $(f_1, g_1) : (G, B) \rightarrow (G', B')$ and $(f_2, g_2) : (G', B') \rightarrow (G'', B'')$ be digital Lie groupoid morphisms. Then the composition of these morphisms

$$(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1) : (G, B) \rightarrow (G'', B'')$$

is a digital Lie groupoid morphism.

Proof. Since (f_1, g_1) and (f_2, g_2) are digital Lie groupoid morphisms, we have the following equations:

$$\begin{aligned} f_1(m(a, b)) &= m'(f_1(a), f_1(b)), \\ s' \circ f_1 &= g_1 \circ s \\ t' \circ f_1 &= g_1 \circ t \end{aligned} \tag{1}$$

for all $(a, b) \in G_1$ and

$$\begin{aligned} f_2(m'(u, v)) &= m''(f_2(u), f_2(v)), \\ s'' \circ f_2 &= g_2 \circ s' \\ t'' \circ f_2 &= g_2 \circ t' \end{aligned} \tag{2}$$

for all $(u, v) \in G'_1$. Let's use the following notations

$$p = f_2 \circ f_1 \quad \text{and} \quad q = g_2 \circ g_1.$$

We shall show that

$$\begin{aligned} (i) \quad & p(m(a, b)) = m''(p(a), p(b)), \\ (ii) \quad & s'' \circ p = q \circ s \\ (iii) \quad & t'' \circ p = q \circ t \end{aligned}$$

for all $(a, b) \in G_1$.

(i) If we use equations in (1) and (2), we conclude that

$$\begin{aligned} p(m(a, b)) &= f_2 \circ f_1(m(a, b)) \\ &= f_2(m'(f_1(a), f_1(b))) \\ &= m''(f_2 \circ f_1(a), f_2 \circ f_1(b)) \\ &= m''(p(a), p(b)). \end{aligned}$$

(ii) Using again equations in (1) and (2),

$$\begin{aligned} s'' \circ p &= s'' \circ f_2 \circ f_1 \\ &= g_2 \circ s' \circ f_1 \\ &= g_2 \circ g_1 \circ s \\ &= q \circ s. \end{aligned}$$

(iii) Similarly, we obtain

$$\begin{aligned} t'' \circ p &= t'' \circ f_2 \circ f_1 \\ &= g_2 \circ t' \circ f_1 \\ &= g_2 \circ g_1 \circ t \\ &= q \circ t. \end{aligned}$$

On the other hand, since f_1, f_2, g_1, g_2 are digital continuous maps, the composition maps of $p = f_2 \circ f_1$ and $q = g_2 \circ g_1$ are digital continuous maps by the Proposition 1. As a result, $(f_2, g_2) \circ (f_1, g_1)$ is a digital Lie groupoid morphism.

Theorem 2. Consider a digital continuous function $f : G \rightarrow G'$. A digital map $(f, f_0) : (G, B) \rightarrow (G', B')$ is a digital Lie groupoid morphism if and only if

$$(f(u), f(v)) \in G'_1 \text{ and } f(m(u, v)) = m'(f(u), f(v)) \text{ for all } (u, v) \in G_1. \quad (3)$$

Proof. ($\Rightarrow$) It is clear from the Definition 15 and Definition 16.

($\Leftarrow$) Let $f : G \rightarrow G'$ satisfy the condition in (3). Consider the digital continuous function $f_0 : B \rightarrow B'$ defined by $f_0(b) = s'(f(e(b)))$ for every $b \in B$. It is obvious that f_0 is a digital continuous function from the Proposition 1.

We should prove that $s' \circ f = f_0 \circ s$ and $t' \circ f = f_0 \circ t$. If $(u, e \circ t(u)) \in G_1$, then we can say that $(f(u), f(e \circ t(u))) \in G'_1$. Therefore we obtain

$$f(u) \circ f(e \circ t(u)) = f(u \circ e \circ t(u)) = f(u) \quad (4)$$

and

$$f(u) \circ e' \circ (t'(f(u))) = f(u). \quad (5)$$

By the equations (4) and (5), we get the following:

$$e' \circ (t' \circ f(u)) = f(e \circ t(u)) \Rightarrow s'(e' \circ (t' \circ f(u))) = s'(f(e \circ t(u))).$$

If we use the Remark 3, we conclude that $t' \circ f(u) = f_0 \circ t(u)$ for all $u \in G$. Writing s instead of t in the above statements, we get $s' \circ f = f_0 \circ s$ in a similar way. This completes the proof.

Lemma 1. Let $(f, g) : (G, B) \rightarrow (G', B')$ be a digital Lie groupoid morphism. Define $I(G) = \{u \mid s(u) = t(u)\}$ for $u \in G$. The following statements hold:

- (a) If f is injective, then g is injective.
- (b) If f is surjective, then g is surjective.
- (c) $f(I(G)) \subseteq I(G')$.
- (d) If f is surjective and g is injective, then $f(I(G)) = I(G')$.

Proof.

- (a) Since f is injective digital map, we have

$$f(u) = f(v) \Rightarrow u = v \text{ for all } u, v \in G.$$

From the equality $f(u) = f(v)$, we obtain

$$s'(f(u)) = s'(f(v)) \text{ and } t'(f(u)) = t'(f(v)).$$

If $u = v$, then

$$s(u) = s(v) \text{ and } t(u) = t(v).$$

Consider the equality $g(t(u)) = g(t(v))$. By the Definition 15 (ii) and Definition 16, we get

$$t' \circ f(u) = t' \circ f(v) \text{ and } t(u) = t(v),$$

since $u = v$. Similarly, as the same result can be obtained for s , we conclude that g is injective digital map.

- (b) Let f be a surjective digital map. Then for every $b \in B'$ there exists $a \in G$ such that $f(a) = b$. From the Definition 15 (ii), we have $s' \circ f = g \circ s$. Then we get the following:

$$g \circ s(a) = s' \circ f(a) = s'(b) \Rightarrow g(s(a)) = s'(b),$$

where $s(a) \in B$ and $s'(b) \in B'$. This concludes that g is a surjective digital map.

- (c) If $u' \in f(I(G))$, then there exists $u \in I(G)$ such that $f(u) = u'$. Moreover,

$$\begin{aligned} s'(u') &= s'(f(u)) \\ &= g(s(u)) \quad \text{from Definition 15 (ii)} \\ &= g(t(u)) \\ &= t'(f(u)) \quad \text{from Definition 15 (ii)} \\ &= t'(u'). \end{aligned}$$

The fact that $s(u) = t(u)$ shows $u' \in I(G')$, i.e., $f(I(G)) \subseteq I(G')$.

- (d) By the part (c), we only show that $I(G') \subseteq f(I(G))$. Considering $u' \in I(G')$, we get $s'(u') = t'(u')$. Using the surjectivity of f , for $u' \in G'$, we have $u \in G$ such that $u' = f(u)$. Since g is an injective digital map,

$$s'(f(u)) = t'(f(u)) \Rightarrow g(s(u)) = g(t(u)) \Rightarrow s(u) = t(u).$$

So $u \in I(G)$ and $u' \in f(I(G))$. This completes the proof.

We remark that G_x denotes the set of digital morphisms started at any digital object $x \in B$ in any digital Lie groupoids.

Definition 17. Let (G, B) , (G', B') be two digital Lie groupoids and $(f, g) : (G, B) \rightarrow (G', B')$ be a digital Lie groupoid morphism.

- (a) If g is injective digital map (respectively, surjective, bijective), then (f, g) is called digital base injective (resp., surjective, bijective).
- (b) If $f_u : G_u \rightarrow G'_{g(u)}$ is injective digital map (resp. surjective, bijective), for each $u \in B$, then (f, g) is called digital fibrewise injective (resp., surjective, bijective).

Definition 18. Consider two digital Lie groupoids (G, B) , (G', B') and a digital Lie groupoid morphism $(f, g) : (G, B) \rightarrow (G', B')$. The set

$$\text{Ker}(f) = \{u \in G \mid f(u) \in e'(B')\}$$

is called the kernel of f . If $\text{Ker}(f) = e(B)$, then (f, g) has discrete kernel.

Proposition 3. Let (G, B) , (G', B') be two digital Lie groupoids and $(f, g) : (G, B) \rightarrow (G', B')$ be a digital Lie groupoid morphism. (f, g) is digital fibrewise injective if and only if $\text{Ker}(f)$ is discrete.

Proof. $(\Rightarrow)$ Let (f, g) be digital fibrewise injective. Then $f_u : G_u \rightarrow G'_{g(u)}$ is injective map. Since $e(B) \subseteq \text{Ker}(f)$, we have to show $\text{Ker}(f) \subseteq e(B)$. If $b \in \text{Ker}(f)$, then $f(b) \in e'(B')$, i.e., $f(b) = e'(u')$ where $u' \in B'$. Since f_u is injective,

$$\begin{aligned} f \circ e(u) &= e' \circ g(u) = e'(u') \Rightarrow f(b) = f \circ e(u) \\ &\Rightarrow f_u(b) = f_u \circ e(u) \\ &\Rightarrow b = e(u), \end{aligned}$$

we get $b \in e(B)$. So $\text{Ker}(f) = e(B)$ and $\text{Ker}(f)$ is discrete.

($\Leftarrow$) Suppose that $\text{Ker}(f)$ is discrete. Taking $v, w \in G_u$ such that $f_u(v) = f_u(w)$, then we have

$$s(v) = s(w) = u \text{ and } f(v) = f(w).$$

So $f(v)(f(w))^{-1} = e'(u')$ where $u' \in B'$. From the following statements,

$$\begin{aligned} f(vw^{-1}) = e'(u') &\Rightarrow vw^{-1} \in \text{Ker}(f) = e(B) \\ &\Rightarrow vw^{-1} = e(u), \text{ where } u \in B \\ &\Rightarrow v = e(u)w \\ &\Rightarrow e(u)v = e(u)e(u)w \\ &\Rightarrow e(u)v = e(u)w \\ &\Rightarrow v = w, \text{ } e \text{ injective} \end{aligned}$$

we conclude that (f, g) is a digital fibrewise injective.

4 Conclusion

The essential goal of this paper is to present digital Lie groupoids and conclude some results. Since both digital topology and Lie groupoids are very popular areas in mathematics, we hope that this topic with significant image processing and computer graphics applications will draw great interest from researchers in digital topology.

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