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Co-Hopf space structure of generalized manifolds in digital images

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Abstract: In this study, we present co-Hopf space structure of a generalized manifold in digital images. We also prove that a pointed generalized digital k -manifold having digital co-Hopf space structure is digitally contractible. We show that a κ -retract of a digital co-Hopf space is a digital co-Hopf space. Finally, we get some results on digital co-Hopf groups.

Keywords: Generalized manifold, digital image, digital co-Hopf space, digital manifold.

1 Introduction

Topological manifolds can be viewed as a substantial class of topological spaces with various applications in mathematics. Its properties are generally used in algebraic topology, differential topology, computer science and differential geometry. The generalized manifolds were introduced in the 1930s. They play important role in geometric topology. For more detailed knowledge, see [7, 8].

The theory of digital topology has been developing since the 1980s. It has been described by methods from topology, geometric topology and especially algebraic topology. Since it is used in many fields such as image analysis, computer graphics, geometric design, etc., this topic has managed to attract the attention of many researchers. Significant and useful results have been obtained by using some notions from classical topology, algebraic topology and pure mathematics.

In recent years, there have been many progresses in digital topology. In [2] and [4], the digital versions of some notions in topology and digital continuous functions are studied. Some results on the digital homology groups are given in [10] and [16]. Digital H -spaces are introduced [11] and their some properties are given [13]. Moreover, the relative homology groups of digital images are presented in [12].

A co-Hopf space (X, ϕ) consists of a topological space X and a continuous function $\phi : X \rightarrow X \vee X$, called comultiplication, such that

$$\pi_1 \circ \phi \simeq 1_X \simeq \pi_2 \circ \phi$$

where π_1, π_2 are the projections $X \vee X \rightarrow X$ onto the first and second summands of the wedge product. Since a binary operation can be defined via co-Hopf spaces on $[(X, x_0), (Y, y_0)]$ which is a homotopy class, co-Hopf spaces have great importance in algebraic topology.

This paper is organized as follows: The Section 2 provides required definitions and theorems about digital images, digital homotopy, digital relative homology and digital manifolds. In Section 3, we introduce a pointed generalized digital k -manifolds with digital co-Hopf space structure and prove some significant theorems on topic. Lastly, we state some conclusions about our study.

2 Preliminaries

A digital image consists of (X, κ) , where $X \subseteq \mathbb{Z}^n$ for some positive integer n and κ indicates some adjacency relation for the elements of X . A pointed digital image with κ -adjacency is a triple (X, p, κ) , where p is the base point of X .

Definition 1. [5]. For a positive integer l with $1 \leq l \leq n$ and two distinct points $p = (p_1, p_2, \dots, p_n)$, $q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$, p and q are c_l -adjacent, if

- (1) there are at most l indices i such that $|p_i - q_i| = 1$, and
- (2) for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

We note that c_l represents the number of points $q \in \mathbb{Z}^n$ that are adjacent to a given point $p \in \mathbb{Z}^n$. So we have $c_1 = 2$ -adjacency in $\mathbb{Z}$; $c_1 = 4$ -adjacency and $c_2 = 8$ -adjacency in $\mathbb{Z}^2$; $c_1 = 6$ -adjacency, $c_2 = 18$ -adjacency, and $c_3 = 26$ -adjacency in $\mathbb{Z}^3$ [5]. For a natural number l in conditions (1) and (2) with $1 \leq l \leq n$, l determines each of the κ -adjacency relations of $\mathbb{Z}^n$ in terms of (1) and (2) [14] as follows:

$$\kappa \in \left\{ 2n \ (n \geq 1), 3^n - 1 \ (n \geq 2), 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1 \ (2 \leq r \leq n-1, n \geq 3) \right\}$$

where $C_t^n = \frac{n!}{(n-t)!t!}$.

Definition 2. [2]. The set $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ is said to be a digital interval where $a, b \in \mathbb{Z}$ and $a < b$.

A κ -neighbor of $p \in \mathbb{Z}^n$ [3] is a point of $\mathbb{Z}^n$ that is κ -adjacent to p . A digital image $X \subset \mathbb{Z}^n$ is κ -connected [15] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \dots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ -neighbors where $i = 0, 1, \dots, r-1$.

Definition 3. [3]. Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. A function $f : X \rightarrow Y$ is said to be (κ_0, κ_1) -continuous if for every κ_0 -connected subset U of X , $f(U)$ is a κ_1 -connected subset of Y . We say that such a function is digitally continuous.

Proposition 1. [3]. Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 -adjacency and κ_1 -adjacency, respectively. The function $f : X \rightarrow Y$ is (κ_0, κ_1) -continuous if and only if for every κ_0 -adjacent points $\{x_0, x_1\}$ of X , either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1 -adjacent in Y .

Definition 4. [6]. Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. A function $f : X \rightarrow Y$ is a (κ_0, κ_1) -isomorphism, if f is (κ_0, κ_1) -continuous and bijective and further f^{-1} is (κ_1, κ_0) -continuous.

A $(2, \kappa)$ -continuous function $f : [0, m]_{\mathbb{Z}} \rightarrow X$ such that $f(0) = x$ and $f(m) = y$ is called a digital κ -path [3] from x to y in a digital image X . A digital image (X, κ) is called κ -path connected if there is a κ -path between for every two points in X .

Definition 5. [3]. Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. Two (κ_0, κ_1) -continuous functions $f, g : X \rightarrow Y$ are digitally (κ_0, κ_1) -homotopic in Y if there is a positive integer m and a function $H : X \times [0, m]_{\mathbb{Z}} \rightarrow Y$ such that

- for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$;
- for all $x \in X$, the induced function $H_x : [0, m]_{\mathbb{Z}} \rightarrow Y$ defined by $H_x(t) = H(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$, is $(2, \kappa_1)$ -continuous; and
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $H_t : X \rightarrow Y$ defined by $H_t(x) = H(x, t)$ for all $x \in X$, is (κ_0, κ_1) -continuous.

The function H is called a digital (κ_0, κ_1) -homotopy between f and g . If these functions are digitally (κ_0, κ_1) -homotopic, it is denoted $f \simeq_{(\kappa_0, \kappa_1)} g$.

Definition 6. [3].

- A digital image (X, κ) is said to be κ -contractible if its identity map is (κ, κ) -homotopic to a constant function $\bar{c}$ for some $c \in X$ where the constant function $\bar{c} : X \rightarrow X$ is defined by $\bar{c}(x) = c$ for all $x \in X$.
- Let $\emptyset \neq A \subset X$. Assume κ -adjacency for A and X .

We say that A is a κ -retract of X if and only if there is a κ -continuous function $r : X \rightarrow A$ such that $r(a) = a$ for all $a \in A$. The function r is called a κ -retraction of X onto A .

Let $f : X \rightarrow Y$ be a (κ, λ) -continuous function and $g : Y \rightarrow X$ be a (λ, κ) -continuous function such that

$$f \circ g \simeq_{(\lambda, \lambda)} 1_X \quad \text{and} \quad g \circ f \simeq_{(\kappa, \kappa)} 1_Y.$$

Then X and Y have the same (κ, λ) -homotopy type [4] and f is called a digital homotopy equivalence.

Definition 7. [6]. Let (X, x_0) and (Y, y_0) be the pointed digital images with κ -adjacency relations in $\mathbb{Z}^n$. The digital wedge product $X \vee Y$ is defined by

$$X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$$

which is the pointed digital image with κ_v -adjacency and the base point (x_0, y_0) in $\mathbb{Z}^{2n}$.

Definition 8. [17]. Let S be a set of nonempty subset a digital image (X, κ) . Let the following statements hold:

- If p and q are distinct points of $s \in S$, then p and q are κ -adjacent,
- If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.

Then the members of S are called *simplexes* of (X, κ) .

An m -simplex is a simplex S such that $|S| = m + 1$. Let P be a digital m -simplex. If P' is a nonempty proper subset of P , then P' is called a face of P . $\text{Vert}(P)$ is used to denote the vertex set of P . A digital subcomplex A of a digital simplicial complex X with κ -adjacency is a digital simplicial complex [17] contained in X with $\text{Vert}(A) \subset \text{Vert}(X)$.

Let (X, κ) be a finite collection of digital m -simplices, $0 \leq m \leq d$ for some non-negative integer d . If the following statements hold, then (X, κ) is called a finite digital simplicial complex [1]:

- If P belongs to X , then every face of P also belongs to X .
- If $P, Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q .

The dimension of a digital simplicial complex X is the largest integer m such that X has an m -simplex. $C_q^\kappa(X)$ is a free abelian group [1] with basis all digital (κ, q) -simplices in X .

Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m . The homomorphism $\partial_q : C_q^\kappa(X) \rightarrow C_{q-1}^\kappa(X)$ defined (see [1]) by

$$\partial_q(u) = \begin{cases} \sum_{i=0}^q (-1)^i \langle p_0, p_1, \dots, \hat{p}_i, \dots, p_q \rangle, & q \leq m \\ 0, & q > m \end{cases}$$

is called a boundary homomorphism, where $\hat{p}_i$ means delete the point p_i and $u = \langle p_0, p_1, \dots, p_q \rangle$. For all $1 \leq q \leq m$, $\partial_{q-1} \circ \partial_q = 0$.

Definition 9. [1]. Let (X, κ) be a digital simplicial complex.

- (1) $Z_q^\kappa(X) = \text{Ker } \partial_q$ is called the group of digital simplicial q -cycles.
- (2) $B_q^\kappa(X) = \text{Im } \partial_{q+1}$ is called the group of digital simplicial q -boundaries.
- (3) $H_q^\kappa(X) = Z_q^\kappa(X)/B_q^\kappa(X)$ is called the q th digital simplicial homology group.

We need the definition of relative homology groups of digital images [12] to define digital generalized k -manifold. Let (A, κ) be a digital subcomplex of the digital simplicial complex (X, κ) . The chain group $C_q^\kappa(A)$ is a subgroup of the chain group $C_q^\kappa(X)$. The quotient group

$$C_q^\kappa(X, A) = C_q^\kappa(X)/C_q^\kappa(A)$$

is called the group of relative chains of X modulo A . The boundary operator

$$\partial_q : C_q^\kappa(A) \longrightarrow C_{q-1}^\kappa(A)$$

is the restriction of the boundary operator on $C_q^\kappa(X)$. It induces a homomorphism

$$C_q^\kappa(X, A) \longrightarrow C_{q-1}^\kappa(X, A)$$

of the relative chain groups and this is also denoted by ∂_q .

Definition 10. [12]. Let (A, κ) be a digital subcomplex of the digital simplicial complex (X, κ) .

- $Z_q^\kappa(X, A) = \text{Ker } \partial_q$ is called the group of digital relative simplicial q -cycles.
- $B_q^\kappa(X, A) = \text{Im } \partial_{q+1}$ is called the group of digital relative simplicial q -boundaries.
- $H_q^\kappa(X, A) = Z_q^\kappa(X, A)/B_q^\kappa(X, A)$ is called the q th digital relative simplicial homology group.

We now mention some notions in [9]. The notion of digital manifold was first studied in 1993. A definition of the digital n -manifold was given by Chen and Zhang. We shall give some notations. A one-dimensional digital space is the integer set $\mathbb{Z}$ and an m -dimensional digital space is $\mathbb{Z}^m = \mathbb{Z} \times \dots \times \mathbb{Z}$. We will use Σ_m to represent an m -dimensional digital space.

A digital k -cell is a digital k -simplex. Two k -cells are called k' -adjacent ($0 \leq k' < k$) if they share a k' -cell. A k -cell path with k' -adjacency is a sequence of k -cells $v_0, v_1, \dots, v_n$, where v_i and v_{i+1} are k' -adjacent and $v_0, v_1, \dots, v_n$ are different. Two k -cells u and v are said to be k' -connected if they are two end elements of a k -cells path with k' -adjacency.

Definition 11. [9]. Let (A, κ) be a digital image in Σ_m . (A', κ') is called a parallel-move of (A, κ) if

- (1) $|A| = |A'|$;
- (2) $A \cap A' = \emptyset$;
- (3) there exists a biconjunction map $f : A \rightarrow A'$ such that $d(a, f(a)) = 1$ for all $a \in A$.

Definition 12. [9]. A k -connected subset X of Σ_m is a digital n -manifold if $X = \cup \{a \mid a \in C_n^k(X)\}$ and

- (1) $C_n^k(X)$ is $(n-1)$ -connected;
- (2) Each element in $C_{(n-1)}^k(X)$ has only one or two parallel-moves;
- (3) $C_{(n+1)}^k(X)$ is empty.

3 Co-Hopf space structure of generalized k -manifold in digital images

In this section, we first define the generalized k -manifold in digital images.

Definition 13. Let (X, κ) be a digital k -manifold in $\mathbb{Z}^k$ with finite digital homology dimension over a group G such that for any $x \in X$,

$$H_q^{\kappa}(X, X - \{x\}, G) = \begin{cases} G, & q = k \\ 0, & q \neq k. \end{cases}$$

Then X is called a generalized digital k -manifold.

Example 1. Since every topological manifold is a generalized manifold in Algebraic Topology, any digital k -manifold can be seen as a generalized digital k -manifold.

We remark that every pointed digital image in this section will be considered as a pointed generalized digital k -manifold. We now would like to define co-Hopf space structure for pointed generalized digital k -manifolds.

Definition 14. Let (X, p, κ) be a pointed generalized digital k -manifold where $p \in X$. If there is a (κ, κ_v) -continuous function $\phi : X \rightarrow X \vee X$ (digital comultiplication) such that the following diagrams

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \vee X \\ & \searrow 1_X & \downarrow \pi_1 \\ & & X \end{array}$$

and

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \vee X \\ & \searrow 1_X & \downarrow \pi_2 \\ & & X \end{array}$$

are commutative up to digital homotopy, i.e.,

$$\pi_1 \circ \phi \simeq_{(\kappa, \kappa)} 1_X \simeq_{(\kappa, \kappa)} \pi_2 \circ \phi,$$

where 1_X is the identity map on X and $\pi_1, \pi_2 : X \vee X \rightarrow X$ are the first and second projections, respectively, then (X, p, κ) is called a digital co-Hopf space.

Definition 15. Consider the pointed generalized digital k -manifolds (A, a, κ_1) , (B, b, κ_2) , (X, x, κ_3) and digital continuous functions $f : A \rightarrow X$ and $g : B \rightarrow X$. The $(\kappa_v, \kappa_{v'})$ -continuous function

$$f \vee g : A \vee B \rightarrow X \vee X$$

is defined by $(f \vee g)(a, b) = (f(a), g(b))$ where κ_v and $\kappa_{v'}$ are adjacency relations on $A \vee B$ and $X \vee X$, respectively.

Theorem 1. Let (X, p, κ) be a pointed generalized digital k -manifold. Every digital co-Hopf space of (X, p, κ) is digitally contractible.

Proof. Let (X, p, κ) be a digital co-Hopf space with a digital comultiplication $\phi : X \rightarrow X \vee X$. For an element x in X , consider $\phi(x) = (a, b)$ where a or b is the base point p of X . If we use the digital comultiplication ϕ on X , we get the

following results for each $x \in X$:

$$\begin{aligned} x &= 1_X(x) \\ &\simeq_{(\kappa, \kappa)} \pi_1 \circ \phi(x) \\ &= \pi_1(a, b) \\ &= a \end{aligned}$$

and

$$\begin{aligned} x &= 1_X(x) \\ &\simeq_{(\kappa, \kappa)} \pi_2 \circ \phi(x) \\ &= \pi_2(a, b) \\ &= b. \end{aligned}$$

We must examine the two cases seperately.

- If $a = p$, then we get

$$x = 1_X(x) \simeq_{(\kappa, \kappa)} e_p(x) = p$$

for each $x \in X$, where $e_p : X \rightarrow X$ is the constant map at p .

- If $b = p$, then

$$x = 1_X(x) \simeq_{(\kappa, \kappa)} e_p(x) = p$$

for all $x \in X$.

This shows that (X, p, κ) is a digitally contractible image.

Theorem 2. A pointed generalized digital k -manifold having the same digital homotopy type as a digital co-Hopf space is itself a digital co-Hopf space.

Proof. Let a pointed generalized digital k -manifold (X, p, κ) be a digital co-Hopf space with the digital continuous comultiplication ϕ and (Y, q, κ') has the same (κ, κ') -homotopy type as (X, p, κ) . Then there are two digital homotopy functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that

$$f \circ g \simeq_{(\kappa', \kappa')} 1_Y \text{ and } g \circ f \simeq_{(\kappa, \kappa)} 1_X.$$

Since (X, p, κ) is a digital co-Hopf space, the following equivalences exist:

$$\pi_1 \circ \phi \simeq_{(\kappa, \kappa)} 1_X \simeq_{(\kappa, \kappa)} \pi_2 \circ \phi,$$

where $\pi_1, \pi_2 : X \vee X \rightarrow X$. Consider the following maps:

$$\phi' : Y \rightarrow Y \vee Y \text{ and } \pi'_1, \pi'_2 : Y \vee Y \rightarrow Y.$$

We will show that

$$\pi'_1 \circ \phi' \simeq_{(\kappa', \kappa')} 1_Y \simeq_{(\kappa', \kappa')} \pi'_2 \circ \phi'.$$

If we define $\phi' = (f \vee f) \circ \phi \circ g$ and $\pi'_1 = f \circ \phi^{-1} \circ (g \vee g)$, then we get

$$\begin{aligned}
 \pi'_1 \circ \phi' &= f \circ \phi^{-1} \circ (g \vee g) \circ (f \vee f) \circ \phi \circ g \\
 &= f \circ \phi^{-1} \circ ((g \circ f) \vee (g \circ f)) \circ \phi \circ g \\
 &\simeq_{(\kappa', \kappa')} f \circ \phi^{-1} \circ 1_{Y \vee Y} \circ \phi \circ g \\
 &= f \circ \phi^{-1} \circ \phi \circ g \\
 &= f \circ 1_X \circ g \\
 &= f \circ g \\
 &\simeq_{(\kappa', \kappa')} 1_Y.
 \end{aligned}$$

Similarly, it can be easily shown that $\pi'_2 \circ \phi' \simeq_{(\kappa', \kappa')} 1_Y$. As a result, we have the required result.

Theorem 3. Let (X, p, κ) be a pointed generalized digital k -manifold having a digital co-Hopf space structure. A κ -retract of (X, p, κ) is a digital co-Hopf space.

Proof. Let $A \subset X$ be a κ -retract with a κ -retraction $r : X \rightarrow A$. So we have $r \circ i \simeq_{(\kappa, \kappa)} 1_A$ and $i \circ r \simeq_{(\kappa, \kappa)} 1_X$, where $i : A \rightarrow X$ is an inclusion map. Since (X, p, κ) is a digital co-Hopf space, it has a digital comultiplication map $\phi : X \rightarrow X \vee X$. Defining $\phi_A = (r \vee r) \circ \phi \circ i$ and $\pi_{1_A} = \pi_{2_A} = r \circ \phi^{-1} \circ (i \vee i)$, we obtain the following:

$$\begin{aligned}
 \pi_{1_A} \circ \phi_A &= r \circ \phi^{-1} \circ (i \vee i) \circ (r \vee r) \circ \phi \circ i \\
 &= r \circ \phi^{-1} \circ ((i \circ r) \vee (i \circ r)) \circ \phi \circ i \\
 &\simeq_{(\kappa, \kappa)} r \circ \phi^{-1} \circ (1_{X \vee X}) \circ \phi \circ i \\
 &= r \circ \phi^{-1} \circ \phi \circ i \\
 &= r \circ i \\
 &\simeq_{(\kappa, \kappa)} 1_A.
 \end{aligned}$$

In a similar way, we have $\pi_{2_A} \circ \phi_A \simeq_{(\kappa, \kappa)} 1_A$. So we conclude that (A, q, κ) is a digital co-Hopf space where $r(p) = q \in A$.

Definition 16. A pointed generalized digital k -manifold (X, p, κ) is called a digital co-Hopf group if there exists a (κ, κ_v) -continuous function $\phi : X \rightarrow X \vee X$ such that $\pi_1 \circ \phi = 1_X$ and $\pi_2 \circ \phi = 1_X$.

Lemma 1. Let (X, p, κ) be a pointed generalized digital k -manifold. Every digital co-Hopf group of (X, p, κ) is a set which consists of a single digital image as the base point.

Proof. By the Definition 16, (X, p, κ) has a (κ, κ_v) -continuous function ϕ such that

$$\pi_1 \circ \phi = 1_X \text{ and } \pi_2 \circ \phi = 1_X.$$

The last statement requires the following digital homotopy equivalences:

$$\pi_1 \circ \phi \simeq_{(\kappa, \kappa)} 1_X \text{ and } \pi_2 \circ \phi \simeq_{(\kappa, \kappa)} 1_X.$$

From the Theorem 1, we get that any digital co-Hopf space is contractible. Thus we have the required result.

Definition 17. Let (X, κ) and (Y, κ') be digital co-Hopf spaces with the digital continuous comultiplications γ and γ' , respectively. A function $f : X \rightarrow Y$ is called digital co-Hopf homomorphism if

$$(f \vee f) \circ \gamma \simeq_{(\kappa, \kappa')} \gamma' \circ f$$

where κ'_v is the adjacency in $Y \vee Y$.

Theorem 4. *Let the pointed generalized digital k -manifolds (X, κ) and (Y, κ') be digital co-Hopf spaces with the same (κ, κ') -homotopy type. Then the digital equivalences are digital co-Hopf homomorphism.*

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be digital homotopy equivalences, i.e.,

$$f \circ g \simeq_{(\kappa', \kappa')} 1_Y \text{ and } g \circ f \simeq_{(\kappa, \kappa)} 1_X.$$

Let $\gamma = (g \vee g) \circ \gamma' \circ f$ and γ' be digital continuous co-multiplications of X and Y , respectively. Since $f \circ g \simeq_{(\kappa', \kappa')} 1_Y$, we get

$$\begin{aligned} \gamma \circ g &= (g \vee g) \circ \gamma' \circ f \circ g \\ &\simeq_{(\kappa', \kappa_v)} (g \vee g) \circ \gamma' \circ 1_Y \\ &= (g \vee g) \circ \gamma' \end{aligned}$$

where κ_v is the adjacency in $X \vee X$. So g is a digital co-Hopf homomorphism. On the other hand, we can say the following:

$$f \circ g \simeq_{(\kappa', \kappa')} 1_Y \Rightarrow (f \circ g) \vee (f \circ g) \simeq_{(\kappa'_v, \kappa'_v)} 1_{Y \vee Y},$$

where κ'_v is the adjacency in $Y \vee Y$. Then we obtain

$$\begin{aligned} (f \vee f) \circ \gamma &= (f \vee f) \circ (g \vee g) \circ \gamma' \circ f \\ &= ((f \circ g) \vee (f \circ g)) \circ \gamma' \circ f \\ &\simeq_{(\kappa, \kappa'_v)} 1_{Y \vee Y} \circ \gamma' \circ f \\ &= \gamma' \circ f. \end{aligned}$$

Therefore f is a digital co-Hopf homomorphism.

4 Conclusion

The principal aim of this work is to introduce co-Hopf space structure of generalized digital k -manifold and to show some important results. The idea of associating of digital topology and generalized manifolds is very interesting. We believe that acquired all results with nice image processing applications will draw significant interest from the community of digital topology.

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