Batch arrival retrial queue with delay time and additional multi-optional repair

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Abstract

This paper deals with $M^X/G/1$ retrial queue with unreliable server and multi-optional repair. It is assumed that the server is subject to active breakdowns and the repair of the failed server starts after a random amount of time. Restoration of the failed server is done with two phases, first essential and second multi optional. After completion of the repair the server continues the interrupted service and it is not allowed to accept new customers until the interrupted customer leaves the system. The retrial, service, delay and repair times are arbitrarily distributed. The model is analysed using the supplementary variable technique and the probability generating function of system size at random epoch and stochastic decomposition are obtained. Performance measures and reliability indices to predict the behaviour of the system are derived.

Keywords

Bulk arrival, Retrial queue, Delay, Multi optional repair.

1.Introduction

Retrial queueing systems are characterized with the feature that arrivals who find the server unavailable are obliged to leave the service area and repeat their requests for service after some random time. This feature plays a special role in several computer and communication networks. Extensive surveys of retrial queues can be seen in Artalejo [1, 2]. Most of the recent studies have been devoted to batch arrival retrial queueing model. Applicability of batch arrival retrial queue is connected with the performance evaluation of Local Area Network, where messages of variable length arrive at the stations and then they are divided into a number of packets in order to be transmitted to the destination station. If the transmission media are idle, then one packet is selected for transmission automatically and the rest is stored in a buffer. If the transmission media are busy then the entire packet must be stored in the buffer and the station will retry the transmission later on. Server breakdowns have been realized in many service channels. Retrial queues that take into account server failures and repairs were studied by Kulkarni and Choi [3].

Yang and Li [4] analysed $M/G/1$ retrial queue with server subject to starting failures. Wang et al. [5] studied a repairable $M/G/1$ retrial queueing model from the viewpoint of reliability. In most of the papers it is assumed that whenever the system breaks down, the repair process starts immediately. However, that is not the case in many real life situations. It is usually a common phenomenon that after the sudden breakdown, the system has to wait for the repair to start. This waiting time is termed as 'delay time' or 'setup' time. Several researchers have studied queueing systems with delay times [6-9]. The concept of multi-optional repair is more realistic to model the unreliable server queue.

When the server is not restored with first essential repair, then the provision of subsequent repairs may be helpful. Jain et al. [10] studied a system with service interruption and multi-optional repair for the classical queueing model. In this paper, a batch arrival retrial queue with delay time, reserved time and second multi-optional repair is considered.

2.Model description

Consider a queueing system with single service facility. Customers arrive in batches according to a Poisson process with rate $\lambda > 0$. The batch size Y is a

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random variable with $P(Y = k) = C_k$, $k = 1, 2, \dots$. If an arriving batch finds the server idle, then one of the customers in the batch joins the system for service and others join the orbit. If the server is busy or down, all the arriving customers join the orbit. The retrial time is generally distributed with distribution function $A(x)$, density function $a(x)$, Laplace transform $A^*(s)$ and conditional completion rate

$$\eta(x) = \frac{a(x)}{1 - A(x)}.$$

Service times are generally distributed with common distribution function $B(x)$, density function $b(x)$, Laplace-Stieltjes transform $B^*(s)$, the first two moments μ_1 and μ_2 and the hazard rate function

$$\mu(x) = \frac{b(x)}{1 - B(x)}.$$

The server is subject to active breakdown. The life time is exponentially distributed with rate α . As soon as breakdown occurs the server stops providing service to the customer and waits for the repair to start. The waiting time is referred as delay time. The interrupted customer waits in the service facility with probability $1 - \theta$ or joins retrial queue with probability θ and keeps returning at times exponentially distributed with rate τ . The delay times are generally distributed with distribution function $D(y)$, density function $d(y)$, Laplace-Stieltjes transform $D^*(s)$, first two moments d_1 and d_2 and the hazard rate function

$$\gamma(y) = \frac{d(y)}{1 - D(y)}.$$

There are two phases of repair essential and optional. After the completion of first essential repair, the server may opt any one of i ($1 \leq i \leq M$) optional repairs with probability p_i or join the system with probability q ($= 1 - \sum p_i$).

First essential repair times are generally distributed with distribution function $F(y)$, density function $f(y)$, Laplace-Stieltjes transform $F^*(s)$, first two moments f_1 and f_2 and the hazard rate function $\beta(y) = \frac{f(y)}{1 - F(y)}$.

Second multi-optional repair times are also generally distributed with distribution function $F_i(y)$, density function $f_i(y)$, Laplace-Stieltjes transform $F_i^*(s)$, first two moments f_{i1} and f_{i2} and the hazard rate function $\beta_i(y) = \frac{f_i(y)}{1 - F_i(y)}$, ($1 \leq i \leq M$).

After the completion of the repair, the server continues the service of the interrupted customer or waits for the same customer. This time is referred as reserved time. The server is not allowed to accept new customers until the interrupted customer leaves the system.

Define the following probabilities

$I_0(t)$ is the probability that the server is idle at time t , and there is no customer in the retrial orbit.

- $I_n(x, t) dx$ is the joint probability that at time t , there are n customers in the retrial orbit, the server is idle and the elapsed retrial time of a customer in the orbit is between x and $x + dx$, $n \geq 1$.
- $P_n(x, t) dx$ is the joint probability that at time t , there are n customers in the orbit, the server is providing service with the elapsed service time between x and $x + dx$, $n \geq 0$.
- $D_{jn}(x, y, t) dx dy$ is the joint probability that at time t , there are n customers in the retrial orbit, the server is in delay time, the elapsed service time of the interrupted customer is x and the elapsed delay time is between y and $y + dy$, $n \geq 0$, $j = 0, 1$.
- $F_{0jn}(x, y, t) dx dy$ is joint probability that at time t , there are n customers in the orbit, the server is in essential repair and the elapsed service time is x and the elapsed essential repair time is between y and $y + dy$, $n \geq 0$, $j = 0, 1$.
- $F_{ijn}(x, y, t) dx dy$ is joint probability that at time t , there are n customers in the orbit, the server is in i^{th} optional repair, the elapsed service time is x and the elapsed i^{th} optional repair time is between y and $y + dy$, $n \geq 0$, $1 \leq i \leq M$, $j = 0, 1$.
- $R_n(x, y, t) dx dy$ is the joint probability that at time t , there are n customers in the retrial orbit, the elapsed service time is x and the elapsed reserved time is between y and $y + dy$, $n \geq 0$.

In the above definitions, $j = 0$ means the interrupted customer is in service position and $j = 1$ means the customer is not in service position.

3.Stability condition

Let t_n , $n \in N$ be the sequence of epochs at which a service completion occurs. The sequence of random vectors $X_n = \{C(t_n+), N(t_n+)\}$ form a Markov chain, which is embedded in our queueing system.

Theorem 1

The inequality $\lambda m_1 \mu_1 (1 + \alpha (\sum_{i=1}^M P_i f_{i1} + \frac{\theta}{\tau} + f_1 + d_1)) - m_1 (1 - A^*(\lambda)) < 1$ is a necessary and sufficient condition for the system to be stable.

Proof

The theorem can be proved by using Foster's criterion and Kaplan's condition.

4. Governing equations

Using the supplementary variable technique, we obtain the following equations at equilibrium state

$$\lambda I_0 = \int_0^\infty P_0(x) \mu(x) dx \quad (1)$$

$$\frac{dI_n(x)}{dx} = -(\lambda + \eta(x)) I_n(x), \quad n \geq 1 \quad (2)$$

$$\frac{dP_0(x)}{dx} = -(\lambda + \mu(x) + \alpha) P_0(x) + q \int_0^\infty F_{000}(x, y) dy$$

$$\beta(y) dy + \sum_{i=1}^M \int_0^\infty F_{i00}(x, y) \beta_i(y) dy + \int_0^\infty R_0(x, y) dy, \quad (3)$$

$$\begin{aligned} \frac{dP_n(x)}{dx} = & -(\lambda + \mu(x) + \alpha) P_n(x) + q \int_0^\infty F_{00n}(x, y) \beta(y) dy \\ & + \sum_{i=1}^M \int_0^\infty F_{i0n}(x, y) \beta_i(y) dy + \tau \int_0^\infty R_n(x, y) dy \\ & + \lambda \sum_{k=1}^n C_k P_{n-k}(x), \quad n \geq 1 \end{aligned} \quad (4)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) D_{i0}(x, y) = -(\lambda + \gamma(y)) D_{i0}(x, y), \quad i = 0, 1 \quad (5)$$

$$\begin{aligned} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) D_{in}(x, y) = & -(\lambda + \gamma(y)) D_{in}(x, y) \\ & + \lambda \sum_{k=1}^n C_k D_{in-k}(x, y), \quad i = 0, 1, \quad n \geq 1 \end{aligned} \quad (6)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F_{0j0}(x, y) = -(\lambda + \beta(y)) F_{0j0}(x, y), \quad j = 0, 1 \quad (7)$$

$$\begin{aligned} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F_{0jn}(x, y) = & -(\lambda + \beta(y)) F_{0jn}(x, y) \\ & + \lambda \sum_{k=1}^n C_k F_{0jn-k}(x, y), \quad j = 0, 1, \quad n \geq 1 \end{aligned} \quad (8)$$

$$\begin{aligned} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F_{ij0}(x, y) = & -(\lambda + \beta_i(y)) F_{ij0}(x, y), \\ & i = 1 \text{ to } M; j = 0, 1 \end{aligned} \quad (9)$$

$$\begin{aligned} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F_{ijn}(x, y) = & -(\lambda + \beta_i(y)) F_{ijn}(x, y) \\ & + \lambda \sum_{k=1}^n C_k F_{ijn-k}(x, y), \quad n \geq 1, i = 1 \text{ to } M, j = 0, 1 \end{aligned} \quad (10)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) R_0(x, y) = -(\lambda + \tau) R_0(x, y) \quad (11)$$

$$\begin{aligned} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) R_n(x, y) = & -(\lambda + \tau) R_n(x, y) \\ & + \lambda \sum_{k=1}^n C_k R_{n-k}(x, y), \quad n \geq 1 \end{aligned} \quad (12)$$

The boundary conditions are

$$I_n(0) = \int_0^\infty P_n(x) \mu(x) dx, \quad n \geq 1 \quad (13)$$

$$P_0(0) = \lambda C_1 I_0 + \int_0^\infty I_1(x) \eta(x) dx \quad (14)$$

$$\begin{aligned} P_n(0) = & \lambda C_{n+1} I_0 + \int_0^\infty I_{n+1}(x) \eta(x) dx \\ & + \lambda \sum_{k=1}^n C_k \int_0^\infty I_{n-k+1}(x) dx, \quad n \geq 1 \end{aligned} \quad (15)$$

$$D_{0n}(x, 0) = \alpha (1 - \theta) P_n(x), \quad n \geq 0 \quad (16)$$

$$D_{1n}(x, 0) = \alpha \theta P_n(x), \quad n \geq 0 \quad (17)$$

$$F_{0jn}(x, 0) = \int_0^\infty D_{jn}(x, y) \gamma(y) dy, \quad n \geq 0, j = 0, 1 \quad (18)$$

$$F_{ijn}(x, 0) = p_i \int_0^\infty F_{0jn}(x, y) \beta(y) dy, \quad n \geq 0, i = 1 \text{ to } M, j = 0, 1 \quad (19)$$

$$\begin{aligned} R_n(x, 0) = & q \int_0^\infty F_{01n}(x, y) \beta(y) dy \\ & + \sum_{i=1}^M \int_0^\infty F_{i1n}(x, y) \beta_i(y) dy, \quad n \geq 0, \end{aligned} \quad (20)$$

The normalizing condition is given by

$$\begin{aligned} I_0 + \sum_{n=1}^\infty \int_0^\infty I_n(x) dx + \sum_{n=0}^\infty \int_0^\infty P_n(x) dx \\ + \sum_{j=0}^1 \sum_{n=0}^\infty \int_0^\infty \int_0^\infty D_{jn}(x, y) dx dy \\ + \sum_{i=1}^M \sum_{j=0}^1 \sum_{n=0}^\infty \int_0^\infty \int_0^\infty F_{ijn}(x, y) dx dy \\ + \sum_{n=0}^\infty \int_0^\infty \int_0^\infty R_n(x, y) dx dy = 1 \end{aligned} \quad (21)$$

Define the following partial generating functions

$$I(x, z) = \sum_{n=1}^{\infty} I_n(x) z^n; P(x, z) = \sum_{n=0}^{\infty} P_n(x) z^n;$$

$$D_j(z, x, y) = \sum_{n=0}^{\infty} D_{jn}(x, y) z^n; j = 0, 1.$$

$$F_{i,j}(z, x, y) = \sum_{n=0}^{\infty} F_{ijn}(x, y) z^n; j = 0, 1, i = 0 \text{ to } M.$$

$$R(z, x, y) = \sum_{n=0}^{\infty} R_n(x, y) z^n,$$

Theorem 2

In equilibrium state

$$I(z) = I_0 [z - c(z) B^*(G(\lambda - \lambda c(z)))] (1 - A^*(\lambda)) / D(z) \quad (22)$$

$$P(z) = I_0 \lambda (1 - c(z)) A^*(\lambda) (1 - B^*(G(\lambda - \lambda c(z)))) / D(z) G(\lambda - \lambda c(z)) \quad (23)$$

$$D_0(z) = I_0 \alpha (1 - \theta) A^*(\lambda) [1 - B^*(G(\lambda - \lambda c(z)))] [1 - D^*(\lambda - \lambda c(z))] / D(z) G(\lambda - \lambda c(z)) \quad (24)$$

$$D_1(z) = I_0 \alpha \theta A^*(\lambda) [1 - B^*(G(\lambda - \lambda c(z)))] [1 - D^*(\lambda - \lambda c(z))] / D(z) G(\lambda - \lambda c(z)) \quad (25)$$

$$F_{0,0}(z) = I_0 \alpha (1 - \theta) A^*(\lambda) D^*(\lambda - \lambda c(z)) [1 - B^*(G(\lambda - \lambda c(z)))] [1 - F^*(\lambda - \lambda c(z))] / D(z) G(\lambda - \lambda c(z)) \quad (26)$$

$$F_{0,1}(z) = I_0 \alpha \theta A^*(\lambda) D^*(\lambda - \lambda c(z)) [1 - B^*(G(\lambda - \lambda c(z)))] [1 - F^*(\lambda - \lambda c(z))] / D(z) G(\lambda - \lambda c(z)) \quad (27)$$

$$F_{i,0}(z) = I_0 p_i \alpha (1 - \theta) A^*(\lambda) D^*(\lambda - \lambda c(z)) F^*(\lambda - \lambda c(z)) [1 - B^*(G(\lambda - \lambda c(z)))] [1 - F^*(\lambda - \lambda c(z))] / D(z) G(\lambda - \lambda c(z)), i = 1 \text{ to } M \quad (28)$$

$$F_{i,1}(z) = I_0 p_i \alpha \theta A^*(\lambda) D^*(\lambda - \lambda c(z)) F^*(\lambda - \lambda c(z)) [1 - B^*(G(\lambda - \lambda c(z)))] [1 - F^*(\lambda - \lambda c(z))] / D(z) G(\lambda - \lambda c(z)), i = 1 \text{ to } M \quad (29)$$

$$R(z) = I_0 \alpha \theta \lambda (1 - c(z)) A^*(\lambda) D^*(\lambda - \lambda c(z)) F^*(\lambda - \lambda c(z)) (q + \sum_{i=1}^M p_i F_i^*(\lambda - \lambda c(z))) [1 - B^*(G(\lambda - \lambda c(z)))] / D(z) G(\lambda - \lambda c(z)) (\lambda - \lambda c(z) + \tau) \quad (30)$$

$$I_0 = [1 - \lambda m_1 \mu_1 (1 + \alpha (\sum_{i=1}^M P_i f_{i1} + \frac{\theta}{\tau} + f_1 + d_1)) - m_1 (1 - A^*(\lambda))] / A^*(\lambda) \quad (31)$$

where

$$D(z) = [c(z) + (1 - c(z)) A^*(\lambda)] B^*(G(\lambda - \lambda c(z))) - z \text{ and } G(x) = x + \alpha - \alpha D^*(x) F^*(x) (q + (\sum_{i=1}^M P_i F_i^*(x)) \frac{(1 - \theta) x + \tau}{x + \tau})$$

Proof

Proceeding with usual manner, equations (2)-(20) yield

$$I(x, z) = I(0, z) e^{-\lambda x} (1 - A(x)) \quad (32)$$

$$P(x, z) = P(0, z) e^{-G(\lambda - \lambda c(z)) x} (1 - B(x)) \quad (33)$$

$$D_0(x, y, z) = D_0(x, 0, z) e^{-(\lambda(1 - c(z))) y} (1 - D(y)) \quad (34)$$

$$D_1(x, y, z) = D_1(x, 0, z) e^{-(\lambda(1 - c(z))) y} (1 - D(y)) \quad (35)$$

$$F_{0,0}(x, y, z) = F_{0,0}(x, 0, z) e^{-(\lambda(1 - c(z))) y} (1 - F(y)) \quad (36)$$

$$F_{i,0}(x, y, z) = F_{i,0}(x, 0, z) e^{-(\lambda(1 - c(z))) y} (1 - F_i(y)), i = 1 \text{ to } M \quad (37)$$

$$F_{0,1}(x, y, z) = F_{0,1}(x, 0, z) e^{-(\lambda(1 - c(z))) y} (1 - F(y)) \quad (38)$$

$$F_{i,1}(x, y, z) = F_{i,1}(x, 0, z) e^{-(\lambda(1 - c(z))) y} (1 - F_i(y)), i = 1 \text{ to } M \quad (39)$$

$$R(x, y, z) = R(x, 0, z) e^{-(\lambda - \lambda c(z) + \tau) y} \quad (40)$$

$$I(0, z) = I_0 \lambda [z - c(z) B^*(G(\lambda - \lambda c(z)))] / D(z) \quad (41)$$

$$P(0, z) = I_0 \lambda (1 - c(z)) A^*(\lambda) / D(z) \quad (42)$$

$$D_0(x, 0, z) = I_0 \alpha (1 - \theta) \lambda (1 - c(z)) A^*(\lambda) e^{-G(\lambda - \lambda c(z)) x} (1 - B(x)) / D(z) \quad (43)$$

$$D_1(x, 0, z) = I_0 \alpha \theta \lambda (1 - c(z)) A^*(\lambda) e^{-G(\lambda - \lambda c(z)) x} (1 - B(x)) / D(z) \quad (44)$$

$$F_{0,0}(x, 0, z) = I_0 \alpha (1 - \theta) \lambda (1 - c(z)) A^*(\lambda) D^*(\lambda - \lambda c(z)) e^{-G(\lambda - \lambda c(z)) x} (1 - B(x)) / D(z) \quad (45)$$

$$F_{i,1}(x, 0, z) = I_0 \alpha \theta \lambda (1 - c(z)) A^*(\lambda) D^*(\lambda - \lambda c(z)) e^{-G(\lambda - \lambda c(z)) x} (1 - B(x)) / D(z) \quad (46)$$

$$F_{i,0}(x, 0, z) = I_0 \alpha (1 - \theta) p_i \lambda (1 - c(z)) A^*(\lambda) \frac{D^*(\lambda - \lambda c(z)) F^*(\lambda - \lambda c(z))}{e^{-G(\lambda - \lambda c(z))x} (1 - B(x)) / D(z)}, \quad i = 1 \text{ to } M \quad (47)$$

$$F_{i,1}(x, 0, z) = I_0 \alpha \theta p_i \lambda (1 - c(z)) \frac{A^*(\lambda) D^*(\lambda - \lambda c(z)) F^*(\lambda - \lambda c(z))}{e^{-G(\lambda - \lambda c(z))x} (1 - B(x)) / D(z)}, \quad i = 1 \text{ to } M \quad (48)$$

$$R(x, 0, z) = I_0 \alpha \theta \lambda (1 - c(z)) A^*(\lambda) \left(q + \sum_{i=1}^M p_i F_i^*(\lambda - \lambda c(z)) \right) \frac{D^*(\lambda - \lambda c(z)) F^*(\lambda - \lambda c(z))}{e^{-G(\lambda - \lambda c(z))x} (1 - B(x)) / D(z)} \quad (49)$$

Define the generating function of the orbit size as $\psi(z) = \int \psi(x, z) dx$ or $\psi(z) = \int \int \psi(x, y, z) dx dy$

Using equations (41) to (49) in equations (32) to (40) and integrating from 0 to ∞ we get the results in equations (22) to (30).

The normalizing equation (21) is equivalent to

$$I_0 + I(1) + P(1) + \sum_{i=0}^1 D_i(1) + \sum_{i=0}^M \sum_{j=0}^1 F_{i,j}(1) + R(1) = 1 \quad (50)$$

Taking limit as $z \rightarrow 1$ on both sides of equations (22) to (30) and substituting the results so obtained in equation (50), we get I_0 as in (31).

Corollary 1

The partial generating function of the number of customers in the orbit is

$$H(z) = \frac{I_0 A^*(\lambda) (1 - z)}{[c(z) + (1 - c(z)) A^*(\lambda)] B^*(G(\lambda - \lambda c(z))) - z} \quad (51)$$

and the mean queue length is given by

$$L_q = \frac{N_1 D''}{2D'^2} \quad (52)$$

where $N_1 = I_0 A^*(\lambda)$

$$D' = \lambda m_1 \left(1 + \alpha \left(\sum_{i=1}^M p_i f_{i1} + \frac{\theta}{\tau} + f_1 + d_1 \right) \right) \mu_1 + m_1 (1 - A^*(\lambda)) - 1$$

$$\begin{aligned} D'' = & \lambda m_2 + \alpha \left[\sum_{i=1}^M p_i (\lambda m_2 f_{i1} + \lambda^2 m_1^2 f_{i2}) \right. \\ & + (\lambda m_2 f_1 + \lambda^2 m_1^2 f_2) + (\lambda m_2 d_1 + \lambda^2 m_1^2 d_2) \\ & + \sum_{i=1}^M 2\lambda^2 m_1^2 [(f_1 p_i f_{i1} + d_1 p_i f_{i1} + d_1 f_1)] \\ & + \sum_{i=1}^M \frac{\theta}{\tau} \left(\frac{1}{\tau} + p_i f_{i1} + f_1 + d_1 \right) + \frac{\theta \lambda m_2}{\tau} \left. \right] \mu_1 \\ & + 2(1 - A^*(\lambda)) m_1 [\lambda m_1 \mu_1 (1 + \alpha \left(\sum_{i=1}^M p_i f_{i1} \right. \\ & \left. \left. + \frac{\theta}{\tau} + f_1 + d_1 \right)) + (1 - A^*(\lambda)) m_2 \end{aligned}$$

Corollary 2

The partial generating functions of the number of customers in the system is

$$K(z) = \frac{I_0 A^*(\lambda) (1 - z) B^*(G(\lambda - \lambda c(z)))}{[c(z) + (1 - c(z)) A^*(\lambda)] B^*(G(\lambda - \lambda c(z))) - z} \quad (53)$$

and the mean system length is

$$L_s = \frac{N'' D' - N' D''}{2D'^2} \quad (54)$$

where $N' = -I_0 A^*(\lambda)$

$$N'' = -2 I_0 A^*(\lambda)$$

$$[\lambda m_1 \mu_1 (1 + \alpha \left(\sum_{i=1}^M p_i f_{i1} + \frac{\theta}{\tau} + f_1 + d_1 \right))]$$

5. Performance measures

The probability that the server is idle during the retrial time is

$$I = (1 - A^*(\lambda)) [\lambda m_1 \mu_1 (1 + \alpha \left(\sum_{i=1}^M p_i f_{i1} + \frac{\theta}{\tau} + f_1 + d_1 \right)) + m_1 - 1] / A^*(\lambda)$$

The probability that the server is busy is

$$P = \lambda m_1 \mu_1$$

The probability that the server is waiting for repair is

$$D = \alpha \lambda m_1 \mu_1 d_1$$

The probability that the server is under essential repair

$$F_E = \alpha \lambda m_1 \mu_1 f_1$$

The probability that the server is under optional repair is

$$F_0 = \sum_{i=1}^M \alpha \lambda m_1 \mu_1 p_i f_{i1}$$

The probability that the server is under reserved time is

$$R = \alpha \theta \lambda m_1 \mu_1 / \tau$$

6. Reliability indexes

Theorem 3

The steady state availability of the server is given by

$$A = 1 - \lambda m_1 \mu_1 \alpha \left(\sum_{i=1}^M p_i f_{i1} + f_1 + d_1 \right) \quad (55)$$

Proof

$$\begin{aligned} A &= I_0 + \sum_{n=1}^{\infty} \int_0^{\infty} I_n(x) dx + \sum_{n=0}^{\infty} \int_0^{\infty} P_n(x) dx \\ &\quad + \sum_{n=0}^{\infty} \int_0^{\infty} R_n(x) dx \\ &= I_0 + \lim_{z \rightarrow 1} I(z) + \lim_{z \rightarrow 1} P(z) + \lim_{z \rightarrow 1} R(z) \end{aligned}$$

Substitution of the expressions for $I(z)$, $P(z)$ and $R(z)$ yields the result in (55).

Theorem 4

The steady state failure frequency of the server is

$$F = \alpha \lambda m_1 \mu_1 \quad (56)$$

Proof

The steady state failure frequency of the server is

$$\begin{aligned} F &= \alpha \sum_{n=0}^{\infty} \int_0^{\infty} P_n(x) dx \\ &= \alpha \lim_{z \rightarrow 1} P(z) \end{aligned}$$

7. Stochastic decomposition

The number of customers in the system (L_S) can be expressed as the sum of two independent random variables one of which is the number of customers in batch arrival queue with server breakdown, delay time and second multioptional repair (L_M) and the other is the number of customers in the orbit given that server is idle (L_I) i.e., $L_S = L_M + L_I$.

The generating function of the system size distribution can be decomposed as follows:

$P_S(z) = \pi(z) \psi(z)$, where $\pi(z)$ is the probability generating function of the number of customers in batch arrival queue with server breakdown, delay time and second multioptional repair.

$$\begin{aligned} &[1 - \lambda m_1 \mu_1 (1 + \alpha (\sum_{i=1}^M p_i f_{i1} + \frac{\theta}{\tau} + f_1 + d_1))] \\ \pi(z) &= \frac{(1-z) B^*(G(\lambda - \lambda c(z)))}{B^*(G(\lambda - \lambda c(z))) - z} \quad (57) \end{aligned}$$

and $\psi(z)$ is the probability generating function of the number of customers in the orbit when the system is idle.

$$\begin{aligned} \psi(z) &= \frac{I_0 + I(z)}{I_0 + I} \\ &= \frac{I_0 A^*(\lambda) [(B^*(G(\lambda - \lambda c(z))) - z]}{D(z) [1 - \lambda m_1 \mu_1 (1 + \alpha (\sum_{i=1}^M p_i f_{i1} + \frac{\theta}{\tau} + f_1 + d_1))] } \quad (58) \end{aligned}$$

Mean of system size distribution due to retrials

The expected number of customers in the orbit during retrial time is

$$\begin{aligned} L_\psi &= \psi'(1) \\ &= L_q - \frac{K_2}{2(1 - K_1)} \end{aligned}$$

where

$$\begin{aligned} K_1 &= \lambda m_1 \mu_1 [1 + \alpha (d_1 + f_1 + \frac{\theta}{\tau} + \sum_{i=1}^M p_i f_{i1})] \\ K_2 &= [\lambda m_2 + \alpha \left\{ \sum_{i=1}^M p_i (\lambda m_2 f_{i1} + \lambda^2 m_1^2 f_{i2}) \right. \\ &\quad + (\lambda m_2 f_1 + \lambda^2 m_1^2 f_2) + (\lambda m_2 d_1 + \lambda^2 m_1^2 d_2) \\ &\quad + 2\lambda^2 m_1^2 \left[\left(\sum_{i=1}^M (f_{i1} p_i f_{i1} + d_1 p_i f_{i1} + d_1 f_{i1}) \right) \right. \\ &\quad \left. \left. + \frac{\theta}{\tau} \left(\frac{1}{\tau} + \sum_{i=1}^M p_i f_{i1} + f_1 + d_1 \right) \right] + \frac{\theta \lambda m_2}{\tau} \right\} \mu_1 \\ &\quad \left. + \lambda^2 m_1^2 (1 + \alpha (\sum_{i=1}^M p_i f_{i1} + \frac{\theta}{\tau} + f_1 + d_1))^2 \right] \mu_2 \end{aligned}$$

Distribution of number of customers in the system size at departure epoch

Let the n^{th} total service completion epoch be t_n . Then the sequence $N_n = N(t_n^+)$ form a Markov chain which is embedded Markov renewal process of the continuous time Markov process $X(t)$ as $t \rightarrow \infty$.

Define the limiting probability

$$\pi_j = \lim_{n \rightarrow \infty} P\{X_n = j\}, \quad j \geq 0$$

$$= B_0 \int_0^{\infty} P_j(x) \mu(x) dx, \quad j \geq 0, \quad \text{where } B_0 \text{ is the normalising constant.}$$

Let $\pi(z)$ be the probability generating function of π_j , $j \geq 0$

$$\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j$$

$$= \frac{B_0 \lambda I_0 (1-C(z)) A^*(\lambda) B^*(G(\lambda - \lambda C(z)))}{(C(z) + (1-C(z)) A^*(\lambda)) B^*(G(\lambda - \lambda C(z))) - z}$$

Using the normalising condition $\pi(1) = 1$, we get

$$B_0 = \frac{1}{\lambda m_1} \text{ and hence}$$

$$\begin{aligned} \pi(z) &= \frac{I_0 (1-C(z)) A^*(\lambda) B^*(G(\lambda - \lambda C(z)))}{m_1 [(C(z) + (1-C(z)) A^*(\lambda)) B^*(G(\lambda - \lambda C(z))) - z]} \\ &= \frac{1-C(z)}{m_1 (1-z)} P_S(z) \\ &= H(z) P_S(z) \end{aligned}$$

where $H(z)$ is the probability generating function of the number of customers placed before a tagged customer in an admission batch in which the tagged customer arrives.

8. Particular cases

Case (i)

If $r_i = 0$, $1 \leq i \leq M$ and $\theta = 0$, the model under study becomes $M^x/G/1$ retrial queue with breakdown and delayed repair. In this case, the probability that the system is empty is obtained as

$$I_0 = [1 - \lambda m_1 \mu_i (1 + \alpha (f_i + d_i)) - m_1 (1 - A^*(\lambda))] / A^*(\lambda)$$

The probability generating function of the number of customers in the orbit is

$$H(z) = \frac{I_0 A^*(\lambda) (1-z)}{[c(z) + (1-c(z)) A^*(\lambda)] B^*(G(\lambda - \lambda c(z))) - z}$$

The probability generating function of the number of customers in the system is

$$K(z) = \frac{I_0 A^*(\lambda) (1-z) B^*(G(\lambda - \lambda c(z)))}{[c(z) + (1-c(z)) A^*(\lambda)] B^*(G(\lambda - \lambda c(z))) - z}$$

Where $G(x) = x + \alpha - \alpha D^*(x) F^*(x)$

The above results agree with the results of Choudhury and Ke [9] with no vacation.

Case (ii)

If $\alpha = 0$, that is no breakdown, the system reduces to batch arrival retrial queue. In this case, the probability that the system is empty is

$$I_0 = [1 - \lambda m_1 \mu_i - m_1 (1 - A^*(\lambda))] / A^*(\lambda)$$

The probability generating function of the number of customers in the orbit is

$$H(z) = \frac{I_0 A^*(\lambda) (1-z)}{[c(z) + (1-c(z)) A^*(\lambda)] B^*(\lambda - \lambda c(z)) - z}$$

The probability generating function of the number of customers in the system is

$$K(z) = \frac{I_0 A^*(\lambda) (1-z) B^*(\lambda - \lambda c(z))}{[c(z) + (1-c(z)) A^*(\lambda)] B^*(\lambda - \lambda c(z)) - z}$$

The above results agree with the results of Kumar et al. [11] with no vacation.

9. Numerical results

For the purpose of numerical illustration, we assume that the retrial time, service time, delay time, essential repair time and optional repair time follow (a) Exponential distribution (b) Erlang-2 distribution (c) Hyper Exponential distribution with respective rates $\eta, \mu, \gamma, \beta_0, \beta_i$ ($1 \leq i \leq M$). The effect of the arrival rate λ , service rate μ and the failure rate α on the performance measures I_0 - the probability that the system is empty, L_S - the expected number of customers in the system, A - the availability of the server and F - the failure frequency of the server are plotted for the default parameters $\lambda = 1$, $\alpha = 0.5$, $M = 4$, $q = 0.4$, $p_1 = 0.2$, $p_2 = 0.2$, $p_3 = 0.1$, $p_4 = 0.1$, $\theta = 0.7$, $\tau = 0.5$, $\eta = 5$, $\mu = 27$, $\gamma = 5$, $\beta_0 = 6$, $\beta_1 = 5$, $\beta_2 = 4$, $\beta_3 = 4$, $\beta_4 = 5$, $C_1 = 0.5$, $C_2 = 0.5$ satisfying arbitrary condition.

Figure 1 (a) to (c) exhibits the trend for I_0 with respect to the parameters λ , μ and α . As expected I_0 decreases with increase in λ and α and increases with increase in μ . Figure 2 (a) to (c) summarizes the behaviour of L_S against the parameters λ , μ and α for selected distributions. For exponential and hyper exponential L_S increases gradually with λ and α and decreases with μ . However for Erlang, L_S sharply increases with λ and α and decreases with μ . Figure 3 presents the availability of the server versus λ , μ and α and shows that availability is an increasing function of μ and decreasing function of λ and α . Figure 4 display the variation of the failure frequency F with respect to λ , μ and α . As expected F increases with increase in λ and α and decreases with increase in μ .

(a) Probability of empty system for varying values of λ

(a) Mean system size for varying values of λ

(b) Probability of empty system for varying values of μ

(b) Mean System Size for varying values of μ

(c) Probability of empty system for varying values of α

(c) Mean System Size for varying values of α

Figure 1 Probability of empty system for varying values of λ , μ and α

Figure 2 Mean system size for varying values of λ , μ and α

(a) Availability for varying values of λ

(b) Availability for varying values of μ

(c) Availability for varying values of α

Figure 3 Availability for varying values of λ , μ and α

(a) Failure Frequency for varying values of λ

(b) Failure Frequency for varying values of μ

(c) Failure frequency for varying values of α

Figure 4 Failure frequency for varying values of λ , μ and α

10. Conclusion

Batch arrival retrial queueing system with delay time and multi optional repair is analysed using a supplementary variable technique. Probability generating functions of the server state and performance measures in steady state is derived and numerical results are obtained. The model can be discussed under fuzzy environment.

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Conflicts of interest

The authors have no conflicts of interest to declare.

References

- [1] Artalejo JR. A classified bibliography of research on retrial queues: progress in 1990-1999. TOP. 1999;7(2):187-211.
- [2] Artalejo JR. Accessible bibliography on retrial queues. Mathematical and Computer Modelling. 1999;30(3):1-6.
- [3] Kulkarni VG, Choi BD. Retrial queues with server subject to breakdowns and repairs. Queueing Systems. 1990; 7(2):191-208.
- [4] Yang T, Li H. The M/G/1 retrial queue with the server subject to starting failures. Queueing Systems. 1994;16(1):83-96.

- [5] Wang J, Cao J, Li Q. Reliability analysis of the retrial queue with server breakdowns and repairs. *Queueing Systems*. 2001;38(4):363-80.
- [6] Jain M, Bhargava C. Unreliable server M/G/1 queueing system with Bernoulli feedback, repeated attempts, modified vacation, phase repair and discouragement. *Journal of King Abdulaziz University: Engineering Sciences*. 2009;20(2):45-77.
- [7] Khalaf RF, Madan KC, Lukas CA. An M [X]/G/1 queue with Bernoulli schedule, general vacation times, random breakdowns, general delay times and general repair times. *Applied Mathematical Sciences*. 2011;5(1):35-51.
- [8] Choudhury G, Tadj L, Ke JC. A two-phase service system with Bernoulli vacation schedule, setup time and N-policy for an unreliable server with delaying repair. *Quality Technology & Quantitative Management*. 2011;8(3):271-84.
- [9] Choudhury G, Ke JC. A batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair. *Applied Mathematical Modelling*. 2012;36(1):255-69.
- [10] Jain M, Sharma GC, Sharma R. Working vacation queue with service interruption and multi optional repair. *International Journal of Information and Management Sciences*. 2011:157-76.
- [11] Kumar BK, Arivudainambi D, Vijayakumar A. On the M (x)/G/1 retrial queue with Bernoulli schedules and general retrial times. *Asia-Pacific Journal of Operational Research*. 2002;19(2):177-94.

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