GENERALIZED BETA-EXPONENTIAL WEIBULL DISTRIBUTION AND ITS APPLICATIONS

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Abstract

In this article, we investigate a distribution called the generalized beta-exponential Weibull distribution. Beta exponential x family of link function which is generated from family of generalized $T - X$ distributions is used in generating the new distribution. Its density and hazard functions have different shapes and contains special case of distributions that have been proposed in literature such as beta-Weibull, beta exponential, exponentiated-Weibull and exponentiated-exponential distribution. Various properties of the distribution were obtained namely; moments, generating function, Renyi entropy and quantile function. Estimation of model parameters through maximum likelihood estimation method and observed information matrix are derived. Thereafter, the proposed distribution is illustrated with applications to two different real data sets. Lastly, the distribution clearly shown that is better fitted to the two data sets than other distributions.

1. Introduction

Generalized beta distributions have been widely developed by several authors and researchers in literature which is a way of introducing shape parameters to govern both skewness and vary tail weight. These distributions are more flexible, robust and fitting for specific real life data sets. In generalized distributions/models, the baseline distribution can be obtained from the proposed distribution/model when setting one or more parameters of the distribution to one (1). Morais et al. [20] extended generalized logistics to beta generalized logistic distribution by introducing two extra shape parameters to provide greater flexibility in modelling observed data. Zea et al. [33] studied and introduced beta exponentiated Pareto distribution with the hope that it will be useful and applied in economics. Several beta generalized type distributions were developed, see, for example, Cordeiro et al. [9], Singla et al. [32], Mahmoudi [19], Cordeiro and Castro [8], Barreto-Souza et al. [5] and Pescim et al. [29].

Meanwhile, exponential and Weibull distributions are continuous probability distributions widely and most commonly used for analyzing and modelling life time phenomenon and waiting time events. Research has been carried out by researchers on various types of exponential and Weibull distributions such as Weibull exponential distribution by

Oguntunde et al. [25], exponentiated Weibull distribution by Kim et al. [16], Ashour and Afify [4], Elshathat [10] and Nassar and Eissa [23, 24], generalized exponential distribution include Okoli et al. [26], Olapade [27], Gupta and Kundu [12] to mention a few. Then, in the case of exponentiated $T - X$ distribution, Alzaghal et al. [3] and Alzaatreh et al. [2] but Alzaatreh proposed a new method for generating many new distributions using $T - X$ family of distribution. Therefore, motivated by the works of Hamdy and Mahmoud [13] extended a Weibull-exponential distribution which was generated from family of generalized $T - X$ distribution called generalized Weibull-exponential distribution.

The main aim of this article is to introduce a distribution called the generalized beta-exponential Weibull distribution with beta exponential x family of link function which is generated from family of generalized $T - X$ distributions used in generating the new distribution. In the proposed distribution, some special cases were generated when one or more parameters is equate to either 1 or 2. Some of its properties also obtained include; survival, hazard rate, reverse hazard rate function, moments, generating function, Renyi entropy and quantile function.

Here, let X be a random variable taken from Weibull distribution with probability density function (pdf) and cumulative distribution function (cdf), respectively, $f(x) = \gamma\theta x^{\theta-1}e^{-\gamma x^\theta}$ $x > 0$ and $F(x) = 1 - e^{-\gamma x^\theta}$. Then, let T be a random variable taken from beta exponential distribution with pdf $f(t) = \frac{\lambda e^{-\lambda\beta x}(1 - e^{-\lambda x})^{\alpha-1}}{B(\alpha, \beta)}$. Therefore, the pdf of the beta exponential-Weibull distribution is

$$f(x) = \frac{\lambda f(x)}{B(\alpha, \beta)} (1 - F(x))^{\lambda\beta-1} \left(1 - (1 - F(x))^\lambda\right)^{\alpha-1} \quad (1)$$

$$= \frac{\lambda\gamma\theta x^{\theta-1}e^{-\gamma x^\theta}}{B(\alpha, \beta)} \left(1 - (1 - e^{-\gamma x^\theta})\right)^{\lambda\beta-1} \left(1 - \left(1 - (1 - e^{-\gamma x^\theta})\right)^\lambda\right)^{\alpha-1}, \quad (2)$$

$x > 0$, $\alpha, \beta, \lambda, \gamma$ & $\theta > 0$. See Alzaatreh et al. [2] and Hamdy and Mahmoud [13].

The rest of the paper is divided as follows: The generalized beta exponential-Weibull (GBEW) distribution is defined in Section 2. Various properties such as moments, generating function, quantile function, Renyi entropy, skewness, kurtosis and order statistics are discussed in Section 3. In Section 4, estimation of model parameters and observed information matrix are derived. The new distribution is illustrated with an application to real life data set and comparison of the result for the GBEW distribution and other known distribution is done. Section 5 contains the conclusion.

2. Material and Method

2.1. The generalized beta exponential-Weibull (GBEW) distribution

Alzaghal et al. [3] investigated and presented the pdf of exponentiated $T - X$ distribution as follows:

$$f(x) = \frac{c\lambda}{B(\alpha, \beta)} f(x) F^{c-1}(x) (1 - F^c(x))^{\lambda\beta-1} \left(1 - (1 - F^c(x))^\lambda\right)^{\alpha-1}, \quad (3)$$

$$f(x) = \frac{c\lambda\gamma\theta x^{\theta-1} e^{-\gamma x^\theta} \left(1 - e^{-\gamma x^\theta}\right)^{c-1}}{B(\alpha, \beta)} \left(1 - \left(1 - e^{-\gamma x^\theta}\right)^c\right)^{\lambda\beta-1} \times \left(1 - \left(1 - \left(1 - e^{-\gamma x^\theta}\right)^c\right)^\lambda\right)^{\alpha-1}, \quad (4)$$

where $(\alpha, \beta, c$ and $\gamma)$ and (λ, θ) are shape and scale parameters, and Equation (4) becomes the density of the GBEW distribution.

From the proposed distribution (GBEWD), some special cases emanated when one or more parameters is equated to 1 or 2 and some of them are known distributions in literature as shown and stated in Table 1 below:

Table 1. Summary of the Special cases of GBEW distribution

Special Case	α	β	c	γ	λ	θ	Author
BW Distribution	--	--	1	--	1	--	Famoye et al. [11] and Lee et al. [17]
BE Distribution	--	--	1	--	1	1	Nadarajah and Kotz [22]
EE Distribution	1	1	--	--	--	1	Gupta and Kundu [12]
EW Distribution	--	1	--	--	1	--	Abo-Kasem [30] and Choudhury [7]
ER Distribution	1	1	1	--	1	2	Mahmoud and Ghazal [18]
EER Distribution	1	1	--	--	--	2	
BR Distribution	--	--	1	--	1	2	

However, the corresponding cdf of GBEW distribution is obtained as follows:

$$F(x) = P(X \leq x) = \int_0^x f(x) dx = \int_0^x \frac{c\lambda\gamma\theta x^{\theta-1} e^{-\gamma x^\theta} \left(1 - e^{-\gamma x^\theta}\right)^{c-1}}{B(\alpha, \beta)} \times \left(1 - \left(1 - e^{-\gamma x^\theta}\right)^c\right)^{\lambda\beta-1} \left(1 - \left(1 - \left(1 - e^{-\gamma x^\theta}\right)^c\right)^\lambda\right)^{\alpha-1} dx. \quad (5)$$

Let

$$U = \left(1 - \left(1 - e^{-\gamma x^\theta}\right)^c\right)^\lambda, \quad du = c\lambda\gamma\theta x^{\theta-1} e^{-\gamma x^\theta} \left(1 - e^{-\gamma x^\theta}\right)^{c-1} dx,$$

$$F(x) = \frac{1}{B(\alpha, \beta)} \int_0^{F(x)} U^{\beta-1} (1 - U)^{\alpha-1} du,$$

$$F(x) = \frac{B_{F(x)}(\alpha, \beta)}{B(\alpha, \beta)} = I_{\left[1 - \left(1 - e^{-\gamma x^\theta}\right)^c\right]^\lambda}(\alpha, \beta). \quad (6)$$

Also, survival function (Sur), hazard rate function (hz), reverse rate function (rhz), and cumulative hazard rate function (chz) are given below respectively.

$$Sur_{(u; \alpha, \beta)} = 1 - I_{F(u)}(\alpha, \beta) = \frac{B(\alpha, \beta) - I_{\left[1 - \left(1 - e^{-\gamma x^\theta}\right)^c\right]^\lambda}(\alpha, \beta)}{B(\alpha, \beta)}, \quad (7)$$

$$hz_{(u; \alpha, \beta)} = \frac{f(u; \alpha, \beta)}{1 - I_{F(u)}(\alpha, \beta)} = \frac{f(u) F(u)^{\beta-1} [1 - F(u)]^{\alpha-1}}{B(\alpha, \beta) - I_{\left[1 - \left(1 - e^{-\gamma x^\theta}\right)^c\right]^\lambda}(\alpha, \beta)}, \quad (8)$$

$$rhz_{(u; \alpha, \beta)} = \frac{f(u; \alpha, \beta)}{I_{F(u)}(\alpha, \beta)} = \frac{f(u) [F(u)]^{\beta-1} [1 - F(u)]^{\alpha-1}}{I_{\left[1 - \left(1 - e^{-\gamma x^\theta}\right)^c\right]^\lambda}(\alpha, \beta)}, \quad (9)$$

and

$$\begin{aligned} chz_{(u; \alpha, \beta)} &= -\log[Sur(u; \alpha, \beta)] \\ &= -\log \left[\frac{B(\alpha, \beta) - I_{\left[1 - \left(1 - e^{-\gamma x^\theta}\right)^c\right]^\lambda}(\alpha, \beta)}{B(\alpha, \beta)} \right]. \end{aligned} \quad (10)$$

See Handique and Chakraborty [14].

The shapes of the pdf and hazard function of GBEW distribution are plotted below:

(i)

(ii)

Figure 1. Plots of the GBEW densities in (i), where $(a, b, c, d, e, f) = (\alpha, \beta, c, \lambda, \gamma, \theta)$ for various values, i.e., $(3.5, 0.7, 2.4, 0.9, 0.2, 4.5)$, $(3.0, 0.7, 2.4, 0.9, 0.2, 4.0)$ and hazard function in (ii) $(a, b, c, d, e, f) = (\alpha, \beta, c, \lambda, \gamma, \theta)$ for different values, i.e., $(4.5, 0.7, 2.4, 0.9, 0.2, 1.5)$, $(2.5, 0.5, 2.4, 0.9, 0.2, 1.5)$ etc.

3. Moment Generating Function

The moment generating function (mgf) of GBEW distribution is expressed as follows:

$$M_x(s) = E(e^{sx}) = \int_{-\infty}^{\infty} e^{sx} f(x; \alpha, \beta, c, \lambda, \gamma, \theta) dx.$$

This can be written in terms of exponentiated-Weibull distribution

$$= \frac{1}{B(\alpha, \beta)} \int_{-\infty}^{\infty} e^{sx} \frac{(-1)^{j+1}}{(j+\beta)} \binom{\alpha-1}{j} \frac{d}{dx} [Sur(x; c, \lambda, \gamma, \theta)]^{j+\beta} dx.$$

By calculating the integral, we have

$$= \frac{1}{B(\alpha, \beta)(j+\beta)} \sum_{j=0}^{\alpha-1} (-1)^{j+1} \binom{\alpha-1}{j} \int_{-\infty}^{\infty} e^{sx} \frac{d}{dx} [Sur(x; c, \lambda, \gamma, \theta)]^{j+\beta} dx, \quad (11)$$

where $\int_{-\infty}^{\infty} e^{sx} \frac{d}{dx} [Sur(x; c, \lambda, \gamma, \theta)]^{j+\beta} dx$ is the moment generating function (mgf) of the exponentiated Weibull distribution Pal et al. [28] and is given as:

$$E(Z^s) = c\lambda^{-s} \Gamma\left(\frac{s}{\gamma} + 1\right) \sum_{i=0}^{c-1} \binom{c-1}{i} (-1)^i (i+1)^{\frac{s}{\gamma}+1}.$$

This implies that (11) can also be written as:

$$\begin{aligned} M_x(s) &= \frac{1}{B(\alpha, \beta)(j+\beta)} \sum_{j=0}^{\alpha-1} (-1)^{j+1} \binom{\alpha-1}{j} c\lambda^{-s} \Gamma\left(\frac{s}{\gamma} + 1\right) \sum_{i=0}^{c-1} \binom{c-1}{i} (-1)^i (i+1)^{\frac{s}{\gamma}+1} \\ &= \frac{c\lambda^{-s}}{B(\alpha, \beta)(j+\beta)} \Gamma\left(\frac{s}{\gamma} + 1\right) \sum_{i=0}^{c-1} \sum_{j=0}^{\alpha-1} (-1)^{(i+j)+1} \binom{c-1}{i} \binom{\alpha-1}{j} (i+1)^{\frac{s}{\gamma}+1}. \end{aligned} \quad (12)$$

The s -th moment of GBEW distribution is written from (12) as:

$$\mu'_s = E(X^s) = \frac{c\lambda^{-s}}{B(\alpha, \beta)(j + \beta)} \Gamma\left(\frac{s}{\gamma} + 1\right) \sum_{i=0}^{c-1} \sum_{j=0}^{\alpha-1} (-1)^{(i+j)+1} \binom{c-1}{i} \binom{\alpha-1}{j} (i+1)^{\frac{s}{\gamma}+1}. \quad (13)$$

Setting $\alpha = \beta = 1$ in (13) gives the s -th moment of the baseline distribution by Pal et al. [28].

$$\mu'_s = E(X^s) = c\lambda^{-s} \Gamma\left(\frac{s}{\gamma} + 1\right) \sum_{i=0}^{c-1} \binom{c-1}{i} (-1)^i (i+1)^{\frac{s}{\gamma}+1}.$$

3.1. Renyi entropy

The Renyi entropy is defined as:

$$I_R = (\delta) = \frac{1}{(1 - \delta)} \log \left(\int_{-\infty}^{\infty} f(x)^\delta dx \right),$$

where $\delta > 0$ and $\delta \neq 1$ by using binomial expression in (4) as it was used in Handique and Chakraborty [14] expressed as:

$$\begin{aligned} f(x) &= (x; \alpha, \beta, c, \lambda, \gamma, \theta)^\delta \\ &= \left[[B(\alpha, \beta)]^{-1} c\lambda\gamma\theta f(x) F(x)^{c-1} [1 - F(x)^c]^{\lambda\beta-1} \right]^\delta \left[1 - (1 - F(x)^c)^\lambda \right]^{(\alpha-1)\delta} \\ &= \left[\frac{c\lambda\gamma\theta x^{\theta-1} e^{-\gamma x^\theta} (1 - e^{-\gamma x^\theta})^{c-1}}{B(\alpha, \beta)} \left[1 - (1 - e^{-\gamma x^\theta})^c \right]^{\lambda\beta-1} \right]^\delta \\ &\quad \times \left[1 - \left(1 - (1 - e^{-\gamma x^\theta})^c \right)^\lambda \right]^{(\alpha-1)\delta} \end{aligned}$$

$$= \frac{1}{B(\alpha, \beta)^\delta} \left[c\lambda\gamma\theta f(x)F(x)^{c-1} [1 - F(x)^c]^{\lambda\beta-1} \right]^\delta \\ \times \sum_{a=0}^{\alpha-1} \binom{(\alpha-1)^\delta}{a} (-1)^a [1 - F(x)^c]^{\lambda a}.$$

Therefore, the Renyi entropy of X is obtained as

$$I_R(\delta) = \frac{1}{(1-\delta)} \log \left[\sum_{a=0}^{(\alpha-1)^\delta} R_a \int_0^\infty \left[c\lambda\gamma\theta f(x)F(x)^{c-1} [1 - F(x)^c]^{\lambda\beta-1} \right]^\delta [1 - F(x)^c]^{\beta a} dx \right], \\ I_R(\delta) = \frac{1}{(1-\delta)} \\ \log \left[\sum_{a=0}^{(\alpha-1)^\delta} R_a \int_0^\infty \left[c\lambda\gamma\theta x^{\theta-1} e^{-\gamma x^\theta} \left(1 - e^{-\gamma x^\theta} \right)^{c-1} \left[1 - \left(1 - e^{-\gamma x^\theta} \right)^c \right]^{\lambda\beta-1} \right]^\delta \right. \\ \left. \times \left[1 - \left(1 - \left(1 - e^{-\gamma x^\theta} \right)^c \right)^\lambda \right]^{\beta a} dx \right] \\ = \frac{1}{1-\delta} \log \left[\sum_{a=0}^{(\alpha-1)^\delta} R_a \int_0^\infty [f(x; c, \lambda, \beta)]^\delta [Sur(x; c, \lambda)]^a dx \right], \quad (14)$$

where $R_a = \frac{1}{B(\alpha, \beta)^\delta} \binom{(\alpha-1)^\delta}{a} (-1)^a$, $f(x; c, \lambda, \gamma, \theta)$ and $Sur(x; c, \lambda, \gamma, \theta)$

are the pdf and survival function of the exponentiated-Weibull distribution, and (14) becomes the Renyi entropy of the GBEW distribution.

3.2. Quantile function

The quantile function of X associating to (8) is given as

$$Q(u) = F^{-1}(u) = \frac{\beta^{-1}}{\left(I_u^{-1}(\alpha, \beta)^{\frac{1}{c\lambda}}\right)^{\frac{1}{\theta}}}, \quad 0 < u < 1, \quad (15)$$

where $I_u^{-1}(\alpha, \beta)$ is the inverse of the incomplete beta function and the function $I_u^{-1}(\alpha, \beta)$ is expressed as a power series

$$I_u^{-1}(\alpha, \beta) = \sum_{j=1}^{\infty} q_i [\alpha, B(\alpha, \beta)u]^{\frac{1}{c\lambda}},$$

where $q_i = 1$ and the remaining coefficients satisfy the following recursion:

$$q_i = \frac{1}{i^2 + (\alpha - 2)i + (1 - \alpha)} \left\{ (1 - \varphi_i, 2 \sum_{r=2}^{i-1} q_m q_i + 1 - n[m(1 - \alpha)(i - m) - m(m - 1)] + \sum_{m=1}^{i-1} \sum_{n=1}^{i-m} q_m q_n q_i + 1 - m - n[m(m - \alpha) + n(\alpha + \beta - 2)(i + 1 - m - n)] \right\},$$

see Zea et al. [33]. Where $\varphi_i, 2 = 1$ if $i = 2$ and $\varphi_i, 2 = 0$ if $i \neq 2$.

Following Zea et al. [33], the importance and motivation to use quantile-based measures are robust and less sensitive to outliers and exist even for distributions without moments. The skewness and kurtosis of the GBEW distribution based on the Bowley's skewness which based on quantiles Kenney and Keeping [15]:

$$SK(B) = \frac{Q(0.75) - 2Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)}.$$

And the Moors [21] kurtosis based on octiles:

$$KUR(M) = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(0.75) - Q(0.25)},$$

where $Q(.)$ stands for the quantile function define in (15).

Here, we followed Alizadeh et al. [1] and Handique and Chakraborty [14] method of limiting behaviour, we have

The asymptotic of Equations (4) and (8) as $x \rightarrow \infty$ are given by

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{c\lambda\gamma\theta^\infty{}^{\theta-1}e^{-\gamma^\infty\theta}\left(1-e^{-\gamma^\infty\theta}\right)^{c-1}}{B(\alpha, \beta)}\left(1-\left(1-e^{-\gamma^\infty\theta}\right)^c\right)^{\lambda\beta-1}$$

$$\times \left(1-\left(1-\left(1-e^{-\gamma^\infty\theta}\right)^c\right)^\lambda\right)^{\alpha-1} = \lim_{x \rightarrow \infty} f(x) = 0, \quad \text{as } x \rightarrow \infty,$$

$$\lim_{x \rightarrow \infty} hz(x) = \lim_{x \rightarrow \infty} \frac{c\lambda\gamma\theta^\infty{}^{\theta-1}e^{-\gamma^\infty\theta}\left(1-e^{-\gamma^\infty\theta}\right)^{c-1}}{B(\alpha, \beta) - I_{\left[1-\left(1-e^{-\gamma^\infty\theta}\right)^c\right]^\lambda}(\alpha, \beta)}\left(1-\left(1-e^{-\gamma^\infty\theta}\right)^c\right)^{\lambda\beta-1}$$

$$\times \left(1-\left(1-\left(1-e^{-\gamma^\infty\theta}\right)^c\right)^\lambda\right)^{\alpha-1} = \lim_{x \rightarrow \infty} hz(x) = 0, \quad \text{as } x \rightarrow \infty.$$

The asymptotic of Equations (4) or (5) and (10) as $x \rightarrow 0$ are given by

$$\lim_{f(x) \rightarrow 0} = \lim_{f(x) \rightarrow 0} \frac{c\lambda\gamma\theta^0{}^{\theta-1}e^{-\gamma^0\theta}\left(1-e^{-\gamma^0\theta}\right)^{c-1}}{B(\alpha, \beta)}\left(1-\left(1-e^{-\gamma^0\theta}\right)^c\right)^{\lambda\beta-1}$$

$$\times \left(1-\left(1-\left(1-e^{-\gamma^0\theta}\right)^c\right)^\lambda\right)^{\alpha-1} = \lim_{x \rightarrow 0} f(x) = 0, \quad \text{as } f(x) \rightarrow 0,$$

$$\lim_{f(x) \rightarrow 0} hz(x) = \lim_{f(x) \rightarrow 0} \frac{c\lambda\gamma\theta^0{}^{\theta-1}e^{-\gamma^0\theta}\left(1-e^{-\gamma^0\theta}\right)^{c-1}}{B(\alpha, \beta) - I_{\left[1-\left(1-e^{-\gamma^0\theta}\right)^c\right]^\lambda}(\alpha, \beta)}\left(1-\left(1-e^{-\gamma^0\theta}\right)^c\right)^{\lambda\beta-1}$$

$$\times \left(1-\left(1-\left(1-e^{-\gamma^0\theta}\right)^c\right)^\lambda\right)^{\alpha-1} = \lim_{f(x) \rightarrow 0} hz(x) = 0, \quad \text{as } f(x) \rightarrow 0.$$

4. Estimation of Parameter

The parameters of GBEW distribution is estimated through maximum likelihood. Here, let θ be a vector of parameters, Cordeiro and de Castro [8] gave the log-likelihood for $\theta = (\alpha, \beta, c, \lambda, \xi)$, where $\xi = (\gamma, \theta)$ as

$$\begin{aligned} l(\theta) = & n \log(c\lambda) + \sum_{i=1}^n \log[f(x; \xi)] - n \log[B(\alpha, \beta)] \\ & + (c-1) \sum_{i=1}^n \log[F(x; \xi)] + (\lambda\beta - 1) \sum_{i=1}^n \log[1 - F(x; \xi)^c] \\ & + (\alpha - 1) \sum_{i=1}^n \log[1 - [1 - F(x; \xi)^c]^\lambda], \end{aligned} \quad (16)$$

where $f(x; \xi) = \gamma\theta x^{\theta-1} e^{-\gamma x^\theta}$ and $F(x; \xi) = [1 - e^{-\gamma x^\theta}]$.

Taking the partial derivatives of the log-likelihood function with respect to $\alpha, \beta, c, \lambda, \xi$ components can be obtained as follows:

$$\frac{\partial l(\theta)}{\partial \alpha} = -n\varphi(\alpha) + n\varphi(\alpha + \beta) + \sum_{i=1}^n \log[1 - [1 - F(x; \xi)^c]^\lambda], \quad (17)$$

$$\frac{\partial l(\theta)}{\partial \beta} = -n\varphi(\beta) + n\varphi(\alpha + \beta) + \lambda \sum_{i=1}^n \log[1 - F(x; \xi)^c], \quad (18)$$

$$\begin{aligned} \frac{\partial l(\theta)}{\partial c} = & \frac{n}{c} + \sum_{i=1}^n \log[F(x; \xi)] + (1 - \lambda\beta) \sum_{i=1}^n \frac{F(x; \xi)^c \log[F(x; \xi)]}{1 - F(x; \xi)} \\ & + (\alpha - 1) \sum_{i=1}^n \frac{\lambda [1 - F(x; \xi)^c]^{\lambda-1} \log[F(x; \xi)]}{1 - [1 - F(x; \xi)^c]^\lambda}, \end{aligned} \quad (19)$$

$$\frac{\partial l(\theta)}{\partial \lambda} = \frac{n}{\lambda} + \beta \sum_{i=1}^n \log[1 - F(x; \xi)^c] + (1 - \alpha) \sum_{i=1}^n \frac{[1 - F(x; \xi)^c]^\lambda \log[F(x; \xi)^c]}{1 - [1 - F(x; \xi)^c]^\lambda}, \quad (20)$$

$$\begin{aligned} \frac{\partial l(\theta)}{\partial \gamma} &= \frac{n\theta}{\gamma} - \frac{\theta \gamma^{\theta-1} f(x; \xi)}{f(x; \xi)} + (c-1) \sum_{i=1}^n \frac{\theta \gamma^{\theta-1} F(x; \xi)}{F(x; \xi)} \\ &+ (1 - \lambda\beta) \sum_{i=1}^n \frac{cF(x; \xi)^{c-1} \theta \gamma^{\theta-1} F(x; \xi)}{1 - F(x; \xi)^c} \\ &+ (\alpha - 1) \sum_{i=1}^n \frac{\lambda [1 - F(x; \xi)^c]^{\lambda-1} cF(x; \xi)^{c-1} \theta \gamma^{\theta-1} F(x; \xi)}{1 - [1 - F(x; \xi)^c]^\lambda}. \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{\partial l(\theta)}{\partial \theta} &= \frac{n}{\theta} + \gamma^\theta \sum_{i=1}^n \frac{\gamma^\theta F(x; \xi)}{F(x; \xi)} + (c-1) \gamma^\theta \sum_{i=1}^n \frac{\gamma^\theta F(x; \xi)}{F(x; \xi)} \\ &+ (1 - \lambda\beta) \gamma^\theta \sum_{i=1}^n \frac{cF(x; \xi)^{c-1} F(x; \xi)}{1 - F(x; \xi)^c} \\ &+ (\alpha - 1) \gamma^\theta \sum_{i=1}^n \frac{\lambda [1 - F(x; \xi)^c]^{\lambda-1} cF(x; \xi)^{c-1} \gamma^\theta F(x; \xi)}{1 - [1 - F(x; \xi)^c]^\lambda}. \end{aligned} \quad (22)$$

Then, the asymptotic variance-covariance matrix for mles of the parameters of GBEW distribution is derived by taking second partial derivatives as follows:

$$\frac{\partial^2 l(\theta)}{\partial \alpha^2} = -n\psi'(\alpha) - n\psi'(\alpha + \beta),$$

$$\frac{\partial^2 l(\theta)}{\partial \beta^2} = -n\psi'(\beta) - n\psi'(\alpha + \beta),$$

$$\begin{aligned}
\frac{\partial^2 l(\theta)}{\partial c^2} &= \frac{n}{c^2} + (1 - \lambda\beta) \sum_{i=1}^n \\
&\times \left[\frac{\left[1 - e^{-\gamma x_i^\theta}\right]^{2c} \log\left(1 - e^{-\gamma x_i^\theta}\right)^2}{\left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^2} + \frac{\left(1 - e^{-\gamma x_i^\theta}\right)^c \log\left(1 - e^{-\gamma x_i^\theta}\right)^2}{1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c} \right] \\
&+ (1 - \alpha) \sum_{i=1}^n \frac{\lambda^2 \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{2(\lambda-1)} \left(1 - e^{-\gamma x_i^\theta}\right)^{2c} \log\left[\left(1 - e^{-\gamma x_i^\theta}\right)\right]^2}{\left[1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right)^\lambda\right]^2} \\
&+ (1 - \alpha) \sum_{i=1}^n \frac{\lambda(\lambda-1) \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{\lambda-2} \left(1 - e^{-\gamma x_i^\theta}\right)^{2c} \log\left[\left(1 - e^{-\gamma x_i^\theta}\right)\right]^2}{1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right)^\lambda} \\
&- (1 - \alpha) \sum_{i=1}^n \frac{\lambda \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{\lambda-1} \left(1 - e^{-\gamma x_i^\theta}\right)^c \log\left[\left(1 - e^{-\gamma x_i^\theta}\right)\right]^2}{1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right)^\lambda},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l(\theta)}{\partial \lambda^2} &= \frac{n}{\lambda} + (1 - \alpha) \sum_{i=1}^n \frac{\left[1 - (1 - e^{-\gamma x_i^\theta})^c \right]^{2\lambda} \log \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^2}{\left[1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right)^\lambda \right]^2} \\
&\quad + (1 - \alpha) \sum_{i=1}^n \frac{\left[1 - (1 - e^{-\gamma x_i^\theta})^c \right]^\lambda \log \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^2}{1 - \left[1 - 1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^2}, \\
\frac{\partial^2 l(\theta)}{\partial \gamma^2} &= \frac{n}{\gamma^2} + (c - 1) \sum_{i=1}^n \left(\frac{e^{-2\gamma x_i^\theta} \theta x_i^{\theta-1}}{\left[\left(1 - e^{-\gamma x_i^\theta} \right) \right]^2} - \frac{e^{-\gamma x_i^\theta} \theta x_i^{\theta-1}}{1 - e^{-\gamma x_i^\theta}} \right) \\
&\quad + (1 - \lambda\beta) \sum_{i=1}^n \frac{c^2 \left(1 - e^{-\gamma x_i^\theta} \right)^{2(c-1)} e^{-\gamma x_i^\theta} \theta x_i^{\theta-1}}{\left[1 - 1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^2} \\
&\quad + (1 - \lambda\beta) \sum_{i=1}^n \frac{c(c-1) \left(1 - e^{-\gamma x_i^\theta} \right)^{(c-2)} e^{-\gamma x_i^\theta} \theta x_i^{\theta-1}}{1 - 1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c} \\
&\quad - (1 - \lambda\beta) \sum_{i=1}^n \frac{c \left(1 - e^{-\gamma x_i^\theta} \right)^{(c-1)} e^{-\gamma x_i^\theta} \theta x_i^{\theta-1}}{1 - 1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c}
\end{aligned}$$

$$\begin{aligned}
& + (1 - \alpha) \sum_{i=1}^n \frac{c^2 \lambda^2 \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^{2(\lambda-1)} \left(1 - e^{-\gamma x_i^\theta} \right)^{2(c-1)} e^{-\gamma x_i^\theta} \theta x_i^{\theta-1}}{\left[1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right)^\lambda \right]^2} \\
& + (1 - \alpha) \sum_{i=1}^n \frac{c^2 \lambda (\lambda - 1) \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^{\lambda-2} \left(1 - e^{-\gamma x_i^\theta} \right)^{2(c-1)} e^{-\gamma x_i^\theta} \theta x_i^{\theta-1}}{1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right)^\lambda} \\
& - (1 - \alpha) \sum_{i=1}^n \frac{c(c-1) \lambda \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^{\lambda-1} \left(1 - e^{-\gamma x_i^\theta} \right)^{c-2} e^{-\gamma x_i^\theta} \theta x_i^{\theta-1}}{1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right)^\lambda} \\
& + (1 - \alpha) \sum_{i=1}^n \frac{c \lambda \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^{\lambda-1} \left(1 - e^{-\gamma x_i^\theta} \right)^{c-1} e^{-\gamma x_i^\theta} \theta x_i^{\theta-1}}{1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right)^\lambda}, \\
\frac{\partial^2 l(\theta)}{\partial \theta^2} &= \frac{n}{\theta^2} + \frac{e^{-\theta x_i^{\theta-1}} e^{-\gamma x_i^\theta} \theta x_i^{\theta-1}}{e^{-\gamma x_i^\theta}} + (c-1) \sum_{i=1}^n \frac{e^{-2x_i^\theta} \theta x_i^{\theta-1}}{\left(1 - e^{-\gamma x_i^\theta} \right)} - \frac{e^{-x_i^\theta} \theta x_i^{\theta-1}}{1 - e^{-\gamma x_i^\theta}} \\
& + (1 - \lambda \beta) \sum_{i=1}^n \frac{c^2 \left(1 - e^{-\gamma x_i^\theta} \right)^{2(c-1)} e^{-x_i^\theta} \theta x_i^{\theta-1}}{\left[1 - 1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^2}
\end{aligned}$$

$$\begin{aligned}
& + (1 - \lambda\beta) \sum_{i=1}^n \frac{c(c-1) \left(1 - e^{-\gamma x_i^\theta}\right)^{(c-2)} e^{-x_i^\theta} \theta^2 x_i^{\theta-2}}{1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c} \\
& - (1 - \lambda\beta) \sum_{i=1}^n \frac{c \left(1 - e^{-\gamma x_i^\theta}\right)^{(c-1)} e^{-x_i^\theta} \theta^2 x_i^{\theta-2}}{1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c} \\
& + (1 - \alpha) \sum_{i=1}^n \frac{c^2 \lambda^2 \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{2(\lambda-1)} \left(1 - e^{-\gamma x_i^\theta}\right)^{2(c-1)} e^{-x_i^\theta} \theta^2 x_i^{\theta-2}}{\left[1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right)^\lambda\right]^2} \\
& + (1 - \alpha) \sum_{i=1}^n \frac{c^2 \lambda (\lambda-1) \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{\lambda-2} \left(1 - e^{-\gamma x_i^\theta}\right)^{2(c-1)} e^{-x_i^\theta} \theta^2 x_i^{\theta-2}}{1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right)^\lambda} \\
& - (1 - \alpha) \sum_{i=1}^n \frac{c(c-1) \lambda \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{\lambda-1} \left(1 - e^{-\gamma x_i^\theta}\right)^{c-2} e^{-x_i^\theta} \theta^2 x_i^{\theta-2}}{1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right)^\lambda} \\
& + (1 - \alpha) \sum_{i=1}^n \frac{c \lambda \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{\lambda-1} \left(1 - e^{-\gamma x_i^\theta}\right)^{c-1} e^{-x_i^\theta} \theta^2 x_i^{\theta-2}}{1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right)^\lambda},
\end{aligned}$$

$$\frac{\partial^2 l(\theta)}{\partial \alpha \partial \beta} = n \psi'(\alpha + \beta),$$

$$\frac{\partial^2 l(\theta)}{\partial \alpha \partial c} = \sum_{i=1}^n \frac{\lambda \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^{\lambda-1} \left(1 - e^{-\gamma x_i^\theta} \right)^c \log \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right) \right]}{1 - \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^\lambda},$$

$$\frac{\partial^2 l(\theta)}{\partial \alpha \partial \lambda} = \sum_{i=1}^n - \frac{\left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^{\lambda-1} \log \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]}{1 - \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^\lambda},$$

$$\frac{\partial^2 l(\theta)}{\partial \alpha \partial \gamma} = \sum_{i=1}^n \frac{c \lambda \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^\lambda \left(1 - e^{-\gamma x_i^\theta} \right)^c e^{-\gamma x_i^\theta} x_i^\theta}{1 - \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^\lambda},$$

$$\frac{\partial^2 l(\theta)}{\partial \alpha \partial \theta} = \sum_{i=1}^n \frac{c \lambda \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^\lambda \left(1 - e^{-\gamma x_i^\theta} \right)^c e^{-\gamma x_i^\theta}}{1 - \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right]^\lambda},$$

$$\frac{\partial^2 l(\theta)}{\partial \beta \partial \alpha} = n \psi'(\alpha + \beta), \quad \frac{\partial^2 l(\theta)}{\partial \beta \partial c} = \lambda \sum_{i=1}^n - \frac{\left(1 - e^{-\gamma x_i^\theta} \right)^c \log \left(1 - e^{-\gamma x_i^\theta} \right)}{1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c},$$

$$\frac{\partial^2 l(\theta)}{\partial \beta \partial \lambda} = \sum_{i=1}^n \log \left[1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c \right], \quad \frac{\partial^2 l(\theta)}{\partial \beta \partial \gamma} = \lambda \sum_{i=1}^n - \frac{c \gamma \left(1 - e^{-\gamma x_i^\theta} \right)^{c-1} e^{-\gamma x_i^\theta}}{1 - \left(1 - e^{-\gamma x_i^\theta} \right)^c},$$

$$\begin{aligned}
\frac{\partial^2 l(\theta)}{\partial \beta \partial \lambda} &= -\beta \sum_{i=1}^n - \frac{\left(1 - e^{-\gamma x_i^\theta}\right)^c \log\left(1 - e^{-\gamma x_i^\theta}\right)}{1 - \left(1 - e^{-\gamma x_i^\theta}\right)} \\
&\quad + (\alpha - 1) \sum_{i=1}^n \frac{\left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{\lambda-1} \left(1 - e^{-\gamma x_i^\theta}\right)^c \log\left(1 - e^{-\gamma x_i^\theta}\right)}{1 - \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^\lambda} \\
&\quad + \lambda(\alpha - 1) \sum_{i=1}^n \frac{\left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{2\lambda-1} \left(1 - e^{-\gamma x_i^\theta}\right)^c \log\left(1 - e^{-\gamma x_i^\theta}\right) \log\left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]}{\left[1 - \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^\lambda\right]^2} \\
&\quad + \lambda(\alpha - 1) \sum_{i=1}^n \frac{\left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{\lambda-1} \left(1 - e^{-\gamma x_i^\theta}\right)^c \log\left(1 - e^{-\gamma x_i^\theta}\right) \log\left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]}{1 - \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^\lambda}, \\
\frac{\partial^2 l(\theta)}{\partial c \partial \gamma} &= \sum_{i=1}^n \frac{\gamma e^{-\gamma x_i^\theta} x_i^\theta}{1 - e^{-\gamma x_i^\theta}} + (1 - \beta\lambda) \sum_{i=1}^n \frac{c \left(1 - e^{-\gamma x_i^\theta}\right)^{c-1} e^{-\gamma x_i^\theta} x_i^\theta}{1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c} \\
&\quad + (1 - \beta\lambda) \sum_{i=1}^n \frac{c \left(1 - e^{-\gamma x_i^\theta}\right)^{2c-1} e^{-\gamma x_i^\theta} \log\left(1 - e^{-\gamma x_i^\theta}\right) x_i^\theta}{\left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^2}
\end{aligned}$$

$$\begin{aligned}
& + (1 - \beta\lambda) \sum_{i=1}^n \frac{c \left(1 - e^{-\gamma x_i^\theta}\right)^{c-1} e^{-\gamma x_i^\theta} \log\left(1 - e^{-\gamma x_i^\theta}\right) x_i^\theta}{1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c} \\
& + (\alpha - 1) \sum_{i=1}^n \frac{c\lambda \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{\lambda-1} \left(1 - e^{-\gamma x_i^\theta}\right)^{c-1} e^{-\gamma x_i^\theta} x_i^\theta}{1 - \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^\lambda} \\
& + (\alpha - 1) \sum_{i=1}^n \frac{c\lambda^2 \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{2(\lambda-1)} \left(1 - e^{-\gamma x_i^\theta}\right)^{2c-1} e^{-\gamma x_i^\theta} x_i^\theta \log\left(1 - e^{-\gamma x_i^\theta}\right)}{\left[1 - \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^\lambda\right]^2} \\
& + (\alpha - 1) \sum_{i=1}^n \frac{c\lambda(\lambda - 1) \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{\lambda-2} \left(1 - e^{-\gamma x_i^\theta}\right)^{2c-1} e^{-\gamma x_i^\theta} x_i^\theta \log\left(1 - e^{-\gamma x_i^\theta}\right)}{1 - \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^\lambda} \\
& + (\alpha - 1) \sum_{i=1}^n \frac{c\lambda \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^{\lambda-1} \left(1 - e^{-\gamma x_i^\theta}\right)^{c-1} e^{-\gamma x_i^\theta} x_i^\theta \log\left(1 - e^{-\gamma x_i^\theta}\right)}{1 - \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c\right]^\lambda},
\end{aligned}$$

$$\frac{\partial^2 l(\theta)}{\partial c \partial \theta} = \sum_{i=1}^n \frac{\gamma e^{-\gamma x_i^\theta}}{1 - e^{-\gamma x_i^\theta}} + (1 - \beta \lambda) \sum_{i=1}^n \frac{c \left(1 - e^{-\gamma x_i^\theta}\right)^{c-1} e^{-\gamma x_i^\theta}}{1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c},$$

$$\frac{\partial^2 l(\theta)}{\partial \lambda \partial \gamma} = -\beta \sum_{i=1}^n \frac{c \left(1 - e^{-\gamma x_i^\theta}\right)^{c-1} e^{-\gamma x_i^\theta} x_i^\theta}{1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c}, \quad \frac{\partial^2 l(\theta)}{\partial \lambda \partial \theta} = -\beta \sum_{i=1}^n \frac{c \left(1 - e^{-\gamma x_i^\theta}\right)^{c-1} e^{-\gamma x_i^\theta}}{1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c},$$

and lastly,

$$\begin{aligned} \frac{\partial^2 l(\theta)}{\partial \gamma \partial \theta} &= \frac{\gamma e^{-\gamma x_i^\theta}}{1 - e^{-\gamma x_i^\theta}} + (c - 1) \sum_{i=1}^n \frac{\gamma e^{-\gamma x_i^\theta} \log \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right) \right] x_i^\theta}{1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c} \\ &\quad + (\lambda \beta - 1) \sum_{i=1}^n \frac{\left(\gamma e^{-\gamma x_i^\theta} \right)^{c-1} \log \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right) \right]^2 x_i^\theta}{\left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c \right]^2} \\ &\quad + (\alpha - 1) \sum_{i=1}^n \frac{\gamma \left[1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c \right]^{\lambda-1} \log \left[1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c \right) \right]^\lambda x_i^\theta}{\left[1 - \left(1 - \left(1 - e^{-\gamma x_i^\theta}\right)^c \right)^\lambda \right]^2}. \end{aligned}$$

Hence, the asymptotic variance covariance matrix for mles of the parameters of GBEW $(\alpha, \beta, c, \lambda, \gamma, \theta)$ distribution is estimated as given below:

$$l^{-1}(\hat{\theta}) = \begin{bmatrix} v(\hat{\alpha}) & cv(\hat{\alpha}, \hat{\beta}) & cv(\hat{\alpha}, \hat{c}) & cv(\hat{\alpha}, \hat{\lambda}) & cv(\hat{\alpha}, \hat{\gamma}) & cv(\hat{\alpha}, \hat{\theta}) \\ cv(\hat{\beta}, \hat{\alpha}) & v(\hat{\beta}) & cv(\hat{\beta}, \hat{c}) & cv(\hat{\beta}, \hat{\lambda}) & cv(\hat{\beta}, \hat{\gamma}) & cv(\hat{\beta}, \hat{\theta}) \\ cv(\hat{c}, \hat{\alpha}) & cv(\hat{c}, \hat{\beta}) & v(\hat{c}) & cv(\hat{c}, \hat{\lambda}) & cv(\hat{c}, \hat{\gamma}) & cv(\hat{c}, \hat{\theta}) \\ cv(\hat{\lambda}, \hat{\alpha}) & cv(\hat{\lambda}, \hat{\beta}) & cv(\hat{\lambda}, \hat{c}) & v(\hat{\lambda}) & cv(\hat{\lambda}, \hat{\gamma}) & cv(\hat{\lambda}, \hat{\theta}) \\ cv(\hat{\gamma}, \hat{\alpha}) & cv(\hat{\gamma}, \hat{\beta}) & cv(\hat{\gamma}, \hat{c}) & cv(\hat{\gamma}, \hat{\lambda}) & v(\hat{\gamma}) & cv(\hat{\gamma}, \hat{\theta}) \\ cv(\hat{\theta}, \hat{\alpha}) & cv(\hat{\theta}, \hat{\beta}) & cv(\hat{\theta}, \hat{c}) & cv(\hat{\theta}, \hat{\lambda}) & cv(\hat{\theta}, \hat{\gamma}) & v(\hat{\theta}) \end{bmatrix}. \quad (23)$$

Therefore, the elements of the information matrix $l^{-1}(\hat{\theta}) = \left[\frac{\partial^2 l(\theta)}{\partial \theta_i \partial \theta_j} \right]_{\theta=\hat{\theta}}$

have been derived in the above using second partial derivatives.

4.1. Analysis of data

4.1.1. Data set 1

The data set 1 is on the strengths of 1.5cm glass fibres. The data was originally obtained by workers at the UK National Physical Laboratory and it has been used by Smith and Naylor [31] and Bourguignon et al. [6]. In the first application, GBEW is compared with other generalized models such as the Beta Weibull (BW) (Famoye et al. [11] and Lee et al. [16]), Beta Exponential (BE) (Nadarajah and Kotz [22]), Exponentiated Exponential (EE) (Gupta and Kundu [12]), Exponential Weibull (EW) (Salem and Abo-Kasem [30] and Choudhury [7]), Exponentiated Rayleigh (ER) (Mahmoud and Ghazal [18]), Exponentiated Exponential Rayleigh (EER), and Beta Rayleigh (BR).

The values of the maximum likelihood estimates (MLEs) and their corresponding standard errors (in parentheses) of the model parameters for the first data set are given in Table 2. Then, in Table 3, we compare

the fits of the GBEW, BW, BE, EE, EW, ER, EER, and BR distributions. The values in Table 3 now indicate that the generalized beta exponential Weibull (GBEW) model has the lowest values for the log-likelihood, AIC, CAIC, HQIC, and BIC statistics (for the first data set) among the fitted models. Hence, the GBEW model is the best fitted model among other competing models. While, Figure 2 are the plots of the observed Histogram, estimated pdfs and cdfs for the distributions for data set 1 (strengths of 1.5cm glass fibres data).

Table 2. MLEs of the parameters and their standard errors (in parentheses) for strengths of 1.5cm glass fibres data

Model/Parameter	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\theta}$
GBEW ($\alpha, \beta, c, \gamma, \lambda, \theta$)	0.0045 (0.0008)	4.9183 (0.0013)	1.8114 (3.1050)	2.8749 (3.0930)	2.8671 (3.0920)	3.2318 (3.0930)
BW ($\alpha, \beta, 1, \gamma, 1, \theta$)	0.0144 (0.0017)	4.9660 (1.0021)	1	2.9830 (0.0220)	1	3.0770 (0.0100)
BE ($\alpha, \beta, 1, \gamma, 1, 1$)	0.0119 (0.2011)	4.9401 (1.2390)	1	2.9352 (0.3422)	1	1
EE ($1, 1, c, \gamma, \lambda, 1$)	1	1	0.5653 (0.1144)	2.5570 (0.0004)	0.5310 (0.0025)	1
EW ($\alpha, 1, c, \gamma, 1, \theta$)	0.0185 (0.0043)	1	1.9110 (0.1900)	2.9950 (0.0000)	1	3.0300 (0.0000)
ER ($1, 1, 1, \gamma, 1, 2$)	1	1	1	2.9310 (0.0000)	1	2
EER ($1, 1, c, \gamma, \lambda, 2$)	1	1	1.9050 (0.0000)	2.8750 (0.0000)	2.8960 (0.0000)	2
BR ($\alpha, \beta, 1, \gamma, 1, 2$)	0.0028 (NA)	4.9520 (NA)	1	2.9670 (0.0001)	1	2

Table 3. Goodness-of-fit statistics for strengths of 1.5cm glass fibres data

Model	-2Loglik	AIC	CAIC	HQIC	BIC
GBEW	890.2217	1792.443	1793.943	1797.501	1805.302
BW	909.2020	1826.401	1827.094	1829.776	1834.977
EW	912.2010	1832.402	1833.092	1835.774	1840.975
EER	914.3140	1834.628	1835.035	1837.157	1841.057
BR	915.1140	1836.228	1836.635	1838.757	1842.657
BE	916.1310	1838.262	1838.669	1840.791	1844.691
EE	918.1010	1842.202	1842.609	1844.731	1848.631
ER	972.0010	1946.002	1946.068	1946.845	1948.145

Shape of the observed histogram, estimated pdfs and cdfs of the compared distributions with data set 1 in (a) and (b) graphs below:

(a) Estimated PDFs

(b) Estimated CDFs

Figure 2. Consist graphs of the observed Histogram, estimated pdfs and cdfs of the GBEW distribution and compared distributions in different colours with data set 1 (the strengths of 1.5cm glass fibres).

4.1.2. Data set 2

The data set two for the second application was used by Handique and Chakraborty [14]. It consists of survival times (in years) for forty-six (46) group of patients given chemotherapy treatment. Also, in the second application, GBEW is compared with other generalized models listed in the first application.

Therefore, the values of MLEs and their corresponding standard errors (in parentheses) of the model parameters for the second data set are given in Table 4. Meanwhile, in Table 5, we compare the fits of the GBEW, BW, BE, EE, EW, ER, EER, and BR distributions. The values in Table 5 show that the generalized beta exponential Weibull (GBEW) model has the lowest values for the log-likelihood, AIC, CAIC, HQIC, and BIC statistics (for the first data set) among the fitted models. Hence, the GBEW model is the better fitted model among other competing models. While, Figure 3 depicts the plots of the observed histogram, estimated pdfs and cdfs for the distributions for data set 1 (survival times (in years) data).

Table 4. MLEs of the parameters and their standard errors (in parentheses) for survival times (in years) data

Model/Parameter	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\theta}$
GBEW ($\alpha, \beta, c, \gamma, \lambda, \theta$)	4.9750	0.0002	1.5560	2.4750	0.7493	0.6406
	(0.0118)	(1.0022)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
BW ($\alpha, \beta, 1, \gamma, 1, \theta$)	4.9730	3.8240	1	2.4650	1	0.6866
	(0.0927)	(0.0921)		(0.0010)		(0.0000)
BE ($\alpha, \beta, 1, \gamma, 1, 1$)	4.9930	0.2340	1	2.4900	1	1
	(0.0051)	(0.3931)		(0.1641)		
EE (1, 1, $c, \gamma, \lambda, 1$)	1	1	1.3890	2.5570	0.5310	1
			(0.1031)	(0.0000)	(0.0000)	
EW ($\alpha, 1, c, \gamma, 1, \theta$)	5.0490	1	1.0840	2.3770	1	0.7217
	(0.0003)		(0.0030)	(0.0002)		(0.0001)
ER (1, 1, 1, $\gamma, 1, 2$)	1	1	1	2.5880	1	2
				(0.0000)		
EER (1, 1, $c, \gamma, \lambda, 2$)	1	1	1.5090	2.5883	0.5737	2
			(0.0000)	(NA)	(0.0000)	
BR ($\alpha, \beta, 1, \gamma, 1, 2$)	4.9950	0.0070	1	2.5020	1	2
	(NA)	(NA)		(0.0001)		

Table 5. Goodness-of-fit statistics for survival times (in years) data

Model	- 2Loglik	AIC	CAIC	HQIC	BIC
GBEW	1833.826	3679.652	3681.806	3683.762	3690.624
BW	1858.800	3845.000	3727.036	3728.340	3732.915
BE	1891.100	3788.200	3789.277	3790.255	3793.686
EE	1901.220	3808.440	3809.011	3810.495	3813.926
EW	1911.320	3826.640	3826.919	3828.010	3830.297
BR	1914.540	3835.080	3835.651	3837.135	3840.566
EER	1919.500	3845.000	3845.571	3847.055	3850.486
ER	1941.010	3884.020	3884.111	3884.705	3885.849

Shape of the estimated pdfs and cdfs of the compared distributions with data set 2 in (c) and (d) graphs below:

(c) Estimated PDFs

(d) Estimated CDFs

Figure 3. Contain plots of the observed Histogram, estimated pdfs and cdfs of the GBEW distribution and compared distributions in different colours with data set 2 (survival times (in years) for forty-six (46) group of patients given chemotherapy treatments).

5. Conclusion

We investigate a distribution namely the generalized beta-exponential Weibull distribution by adding two shape parameters, which generalizes the family of $T - X$ distribution called generalized beta exponential. Both density and hazard functions have different shapes and contains special case of distributions that have been proposed in literature such as beta-Weibull, beta exponential, exponentiated-Weibull and exponentiated-exponential distribution. Various properties of the distribution were obtained namely; moments, generating function, order statistics, Renyi entropy and Quantile function. Estimation of model parameters through maximum likelihood estimation method and observed information matrix are derived. Thereafter, the proposed distribution is illustrated with applications to two different real data sets. Finally, the GBEW distribution clearly shown that is better fitted to the two data sets than other competing distributions.

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